Task 18 (concatenation and Kleene star for recognizable tree languages)

Let $\Sigma$ be a ranked alphabet.

(a) Show that $\text{REC}(\Sigma)$ is closed under top concatenation without using the fact that it is closed under tree concatenation.

(b) Why can we not use the closure of $\text{REC}(\Sigma)$ under tree concatenation to prove the closure under Kleene star?

Prove or refute the following two statements:

(c) For every $\alpha \in \Sigma^{(0)}$, the binary operation $\cdot_\alpha$ is associative. Assume that $\cdot_\alpha$ distributes over $\cup$.

(d) $(L_1 \cdot_\alpha L_2) \cdot_\beta L_3 = L_1 \cdot_\alpha (L_2 \cdot_\beta L_3)$ for arbitrary $L_1, L_2, L_3 \in \text{REC}(\Sigma)$ and $\alpha, \beta \in \Sigma^{(0)}$.

Let $\Delta = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$ be a ranked alphabet.

(e) Using the construction from the lecture, show that $\{\sigma(\alpha, \beta)\}^* \cdot_\beta \{\alpha\} \in \text{REC}(\Sigma)$. 