prove (IH) by induction on $s$:

$$
\begin{aligned}
&\exists u \in T_\Sigma\colon\ q(s) \Rightarrow^*_{T_1} u,\ p(u) \Rightarrow^*_{T_2} t\\
\iff &\exists q(\sigma(x_1,\dots,x_k)) \to u'[q_1(x_{\pi(1)}),\dots,q_m(x_{\pi(m)})] \in R_1,\ \pi\colon [m] \to [k],\\
&u' \in C_\Delta(X_m),\ u_1,\dots,u_m \in T_\Delta\colon\\
&\quad q_1(s_{\pi(1)}) \Rightarrow^*_{T_1} u_1,\dots,q_m(s_{\pi(m)}) \Rightarrow^*_{T_1} u_m,\ p(u'[u_1,\dots,u_m]) \Rightarrow_{T_2} t\\
&&\text{(by decomposition)}\\
\iff &\exists q(\sigma(x_1,\dots,x_k)) \to u'[q_1(x_{\pi(1)}),\dots,q_m(x_{\pi(m)})] \in R_1,\ \pi\colon [m] \to [k],\\
&u' \in C_\Delta(X_m),\ u_1,\dots,u_m \in T_\Delta\colon\\
&\quad q_1(s_{\pi(1)}) \Rightarrow^*_{T_1} u_1,\dots,q_m(s_{\pi(m)}) \Rightarrow^*_{T_1} u_m,\\
&\quad \exists p_1,\dots,p_n \in P,\ t_1,\dots,t_n \in T_\Omega,\ t' \in C_\Omega(X_n),\ \rho\colon [n] \to [m]\colon\\
&\qquad p(u') \Rightarrow^*_{\hat{T}_2} t'[p_1(x_{\rho(1)}),\dots,p_n(x_{\rho(n)})],\ p_1(u_{\rho(1)}) \Rightarrow^*_{T_2} t_1,\dots,p_n(u_{\rho(n)}) \Rightarrow^*_{T_2} t_n\\
&&\text{(by $(*)$)}\\
\iff &\exists q(\sigma(x_1,\dots,x_k)) \to u'[q_1(x_{\pi(1)}),\dots,q_m(x_{\pi(m)})] \in R_1,\ \pi\colon [m] \to [k],\ u' \in C_\Delta(X_m),\\
&p_1,\dots,p_n \in P,\ t_1,\dots,t_n \in T_\Omega,\ t' \in C_\Omega(X_n),\ \rho\colon [n] \to [m]\colon\\
&\quad \langle q_{\rho(1)} p_1\rangle(s_{\pi(\rho(1))}) \Rightarrow^*_T t_1,\dots,\langle q_{\rho(n)} p_n\rangle(s_{\pi(\rho(n))}) \Rightarrow^*_T t_n,\\
&\quad p(u') \Rightarrow^*_{\hat{T}_2} t'[p_1(x_{\rho(1)}),\dots,p_n(x_{\rho(n)})] &\text{(by (IH))}\\
\iff &\exists \pi\colon [m] \to [k],\ \rho\colon [n] \to [m],\ u' \in C_\Delta(X_m),\ t' \in C_\Omega(X_n),\ t_1,\dots,t_n \in T_\Omega\colon\\
&\quad \langle q_{\rho(1)} p_1\rangle(s_{\pi(\rho(1))}) \Rightarrow^*_T t_1,\dots,\langle q_{\rho(n)} p_n\rangle(s_{\pi(\rho(n))}) \Rightarrow^*_T t_n,\\
&\quad \langle q,p\rangle(\sigma(x_1,\dots,x_k)) \to t'[\langle q_{\rho(1)},p_1\rangle(x_{\pi(\rho(1))}),\dots,\langle q_{\rho(n)},p_n\rangle(x_{\pi(\rho(n))})] \in R\\
&&\text{(by Construction)}\\
\iff &\langle q,p\rangle(s) \Rightarrow^*_T t'[t_1,\dots,t_n] = t
\end{aligned}
$$

In the following we show $(*)$: $(\Leftarrow)$ by decomposition. $(\Rightarrow)$ if $T_2$ is copying, i.e., $\exists i, i' : i \neq i', \rho(i) = \rho(i') = j$ then $T_1$ was deterministic: $u\rho(i) = u\rho(i') = u_j$ and $q_j = q_i = q_{i'}$; if $T_2$ is deleting, i.e., $\exists j\colon j \notin \mathsf{codom}(\rho)$, then $B_1$ was total: it did not block on $q_j(t_{\pi(\rho(j))})$. ☐

(c) Let $T_1' = (Q', \Sigma', \Delta', I_1', R_1')$ and $T_2' = (P', \Delta', \Omega', I_2', R_2')$ be td-tt where

- $\Sigma' = \{\alpha^{(0)}, \beta^{(0)}, \gamma^{(1)}\}$, $\Delta' = \{\alpha^{(0)}, \gamma^{(1)}\}$, and $\Omega' = \{\beta^{(0)}\}$ are ranked alphabets,
- $Q_1' = I_1' = \{*\}$, $P' = I_2' = \{q\}$,
- $R_1' = \{\ *(\alpha) \to \alpha,\ *(\gamma(x_1)) \to \gamma(*(x_1))\ \}$, and $R_2' = \{\ q(\gamma(x_1)) \to \beta\ \}$.

The removal of any rule would result in the conclusion to be true, therefore $T_1'$ and $T_2'$ are minimal with respect to the number of rules. Note that $\tau(T_1') \circ \tau(T_2') = \emptyset$.

(d) Let $T' = (Q' \times P', \Sigma', \Omega', I_1' \times I_2', R')$ be the bu-tt where $R' = \{\ \langle *, q\rangle(\gamma(x_1)) \to \alpha\ \}$.

(e) Let $(s,t) = (\gamma(\beta), \alpha)$. Then

1. $(s,t) \notin \tau(T_1') \circ \tau(T_2') = \emptyset$, and
2. $(s,t) \in \tau(T')$.