Formale Übersetzungsmodelle

Task 18 (h-TOP = HOM and r-TOP = REL)

(a) Prove by construction that h-TOP = HOM.

(b) Prove by construction that r-TOP = REL.

Hint: Define relatedness for a top-down tree homomorphism (relabeling) and a bottom-up tree homomorphism (relabeling). Show that the respective transducers induce the same tree transformation if they are related (Lemma). Use the Lemma to obtain the equivalence of the respective classes.

Solution for Task 18

(a) **Definition.** Let $T = (\{*\}, \Sigma, \Delta, \{*\}, R_T)$ be a hom. td-tt and $B = (\{*\}, \Sigma, \Delta, \{*\}, R_B)$ be a hom. bu-tt. We call T and B *related* if

$$\begin{aligned} & *(\sigma(x_1, \dots, x_k)) \to t[\,*(x_{i_1}), \dots, *(x_{i_n})] \in R_T \\ \iff & \sigma(\,*(x_1), \dots, *(x_k)) \to *(t[x_{i_1}, \dots, x_{i_n}]) \in R_B \end{aligned}$$

Lemma. T and B are related iff $\tau(T) = \tau(B)$.

Proof. We show $*(s) \Rightarrow_T^* t \iff s \Rightarrow_B^* *(t)$ for all $s \in T_\Sigma$ and $t \in T_\Delta$ by structural induction on s. Let $s = \sigma(s_1, \dots, s_k)$ and $t = t'[t_{i_1}, \dots, t_{i_n}]$ for some $\sigma \in \Sigma$, $s_1, \dots, s_k \in T_\Sigma$, $t' \in T_\Delta(X_k)$, $t_1, \dots, t_k \in T_\Delta$, and $i_1, \dots, i_n \in [k]$.

$$\begin{aligned} & *(s) \Rightarrow_T^* t \\ \iff & *(s) \Rightarrow_T t'[*(s_{i_1}), \dots, *(s_{i_n})] \Rightarrow_T^* t'[t_{i_1}, *(s_{i_2}), \dots, *(s_{i_n})] \Rightarrow_T^* \cdots \Rightarrow_T^* t \\ \iff & \exists *(\sigma(x_1, \dots, x_k)) \to t'[*(x_{i_1}), \dots, *(x_{i_n})] \in R_T, \quad \forall i \in [k]\colon *(s_i) \Rightarrow_T^* t_i \\ \iff & \exists *(\sigma(x_1, \dots, x_k)) \to t'[*(x_{i_1}), \dots, *(x_{i_n})] \in R_T, \quad \forall i \in [k]\colon s_i \Rightarrow_B^* *(t_i) \qquad \text{(IH)} \\ \iff & \exists \sigma(*(x_1), \dots, *(x_k)) \to *(t'[x_{i_1}, \dots, x_{i_n}]) \in R_B, \quad \forall i \in [k]\colon s_i \Rightarrow_B^* *(t_i) \qquad \text{(related)} \\ \iff & s \Rightarrow_B^* \sigma(*(t_1), s_2, \dots, s_k) \Rightarrow_B^* \cdots \Rightarrow_B^* \sigma(*(t_1), \dots, *(t_k)) \Rightarrow_B *(t) \\ \iff & s \Rightarrow_B^* *(t) \end{aligned}$$

□

Theorem. h-TOP = HOM

Proof. We show $\tau \in h\text{-TOP} \iff \tau \in \text{HOM}$.

$$\begin{aligned} \tau \in h\text{-TOP} \iff & \exists \text{ top-down tree homomorphism } T\colon \tau = \tau(T) && \text{(Definition } h\text{-TOP)} \\ \iff & \exists \text{ bottom-up tree homomorphism } B\colon \tau = \tau(B) && \text{(Lemma)} \\ \iff & \tau \in \text{HOM} && \text{(Definition HOM)} \end{aligned}$$

□

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(b) **Definitino.** Let $T = (\{*\}, \Sigma, \Delta, \{*\}, R_T)$ be a rel. td-tt and $B = (\{*\}, \Sigma, \Delta, \{*\}, R_B)$ be a rel. bu-tt. We call T and B *related* if

$$\begin{aligned} & *(\sigma(x_1, \dots, x_k)) \to \delta(*(x_1), \dots, *(x_k)) \in R_T \\ \iff & \sigma(*(x_1), \dots, *(x_k)) \to *(\delta(x_1, \dots, x_k)) \in R_B \end{aligned}$$

Lemma. T and B are related iff $\tau(T) = \tau(B)$.

Proof. We show $*(s) \Rightarrow_T^* t \iff s \Rightarrow_B^* *(t)$ for all $s \in T_\Sigma$ and $t \in T_\Delta$ by structural induction on s. Let $s = \sigma(s_1, \dots, s_k)$ and $t = \delta(t_1, \dots, t_k)$ for some $\sigma \in \Sigma$, $s_1, \dots, s_k \in T_\Sigma$, $\delta \in \Delta$, and $t_1, \dots, t_k \in T_\Delta$.

$$\begin{aligned} & *(s) \Rightarrow_T^* t \\ \iff & *(s) \Rightarrow_T \delta(*(s_1), \dots, *(s_k)) \Rightarrow_T^* \delta(t_1, *(s_2), \dots, *(s_k)) \Rightarrow_T^* \cdots \Rightarrow_T^* t \\ \iff & \exists *(\sigma(x_1, \dots, x_k)) \to \delta(*(x_1), \dots, *(x_k)) \in R_T, \quad \forall i \in [k]\colon *(s_i) \Rightarrow_T^* t_i \\ \iff & \exists *(\sigma(x_1, \dots, x_k)) \to \delta(*(x_1), \dots, *(x_k)) \in R_T, \quad \forall i \in [k]\colon s_i \Rightarrow_B^* *(t_i) && \text{(IH)} \\ \iff & \exists \sigma(*(x_1), \dots, *(x_k)) \to *(\delta(x_1, \dots, x_k)) \in R_B, \quad \forall i \in [k]\colon s_i \Rightarrow_B^* *(t_i) && \text{(related)} \\ \iff & s \Rightarrow_B^* \sigma(*(t_1), s_2, \dots, s_k) \Rightarrow_B^* \cdots \Rightarrow_B^* \sigma(*(t_1), \dots, *(t_k)) \Rightarrow_B *(t) \\ \iff & s \Rightarrow_B^* *(t) \end{aligned}$$

□

Theorem. $r\text{-TOP} = \text{REL}$

Proof. We show $\tau \in r\text{-TOP} \iff \tau \in \text{REL}$.

$$\begin{aligned} \tau \in r\text{-TOP} & \iff \exists \text{ top-down tree relabeling } T\colon \tau = \tau(T) && \text{(Definition } r\text{-TOP)} \\ & \iff \exists \text{ bottom-up tree relabeling } B\colon \tau = \tau(B) && \text{(Lemma)} \\ & \iff \tau \in \text{REL} && \text{(Definition REL)} \end{aligned}$$

□

Task 19 (l-TOP $\subsetneq$ l-BOT)

Consider the linear td-tt $T = (\{q_0, q_1\}, \Sigma, \Delta, \{q_0\}, R)$ where

$$\begin{aligned} R = \{\ & q_0(\sigma(x_1, x_2)) \to \gamma'(q_1(x_1)), \quad q_0(\sigma(x_1, x_2)) \to \sigma(q_0(x_1), q_0(x_2)), \\ & q_0(\sigma(x_1, x_2)) \to \gamma'(q_1(x_2)), \quad q_1(\sigma(x_1, x_2)) \to \sigma(q_1(x_1), q_1(x_2)), \\ & q_1(\sigma(x_1, x_2)) \to \gamma'(q_1(x_1)), \quad q_0(\gamma(x_1)) \to \gamma(q_0(x_1)), \\ & q_1(\sigma(x_1, x_2)) \to \gamma'(q_1(x_2)), \quad q_1(\gamma(x_1)) \to \gamma(q_0(x_1)), \quad q_0(\alpha) \to \alpha\ \} \end{aligned}$$

Give a linear bu-tt B such that $\tau(T) = \tau(B)$.

Solution for Task 19

Define the bu-tt $B = (\{q_0, q_1, e\}, \Sigma, \Delta, \{q_0\}, R')$ where

$$\begin{aligned} R' = \{\ & \sigma(q_1(x_1), e(x_2)) \to q_0(\gamma'(x_1)), \quad \sigma(q_0(x_1), q_0(x_2)) \to q_0(\sigma(x_1, x_2)), \\ & \sigma(e(x_1), q_1(x_2)) \to q_0(\gamma'(x_2)), \quad \sigma(q_1(x_1), q_1(x_2)) \to q_1(\sigma(x_1, x_2)), \\ & \sigma(q_1(x_1), e(x_2)) \to q_1(\gamma'(x_1)), \quad \gamma(q_0(x_1)) \to q_0(\gamma(x_1)), \\ & \sigma(e(x_1), q_1(x_2)) \to q_1(\gamma'(x_2)), \quad \gamma(q_0(x_1)) \to q_1(\gamma(x_1)), \quad \alpha \to q_0(\alpha)\ \} \\ \cup \{\ & \sigma(e(x_1), e(x_2)) \to e(\alpha), \quad \gamma(e(x_1)) \to e(\alpha), \quad \alpha \to e(\alpha)\ \} \end{aligned}$$

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