Formale Baumsprachen

Task 21 (pumping lemma for Rec*)*

Prove the following lemma.

Lemma. Let Σ be a ranked alphabet and $L \in \mathrm{Rec}(\Sigma)$. Then there is a $p \in \mathbb{N}$ such that for every $\xi \in L$, the following implication holds:

If $\mathrm{height}(\xi) \geq p$, then there are $u, v \in \mathrm{C}_{\Sigma,1}$ and $w \in \mathrm{T}_{\Sigma}$ such that

(i) $\xi = u[v[w]]$,

(ii) $\mathrm{height}(v[w]) \leq p$,

(iii) $\mathrm{height}(v) \geq 2$, and

(iv) for every $n \in \mathbb{N}$, $u[v^n[w]] \in L$.

Task 22 (semirings)

Which of the following ring-like algebras are semirings?

(a) $([0,1], \max, \cdot, 0, 1)$,

(b) $(\mathbb{N}, \ominus, \cdot, 0, 1)$,

(c) $(\mathbb{Z}, -, \cdot, 0, 1)$,

(d) $(\mathbb{Z}, +, \cdot, 0, 1)$,

(e) $(\mathbb{Z}, \cdot, +, 1, 0)$,

(f) $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$,

(g) $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \max, \min, 0, \infty)$,

(h) $(\mathcal{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*)$,

(i) $(\mathcal{P}(\Sigma^*), \cap, \circ, \Sigma^*, \{\varepsilon\})$, and

(j) $(\mathcal{P}(\mathrm{T}_{\Sigma}), \cup, \cdot_{\alpha}, \emptyset, \{\alpha\})$

where $a \ominus b = |a - b|$ for every $a, b \in \mathbb{N}$, $\circ$ is language concatenation, and $\cdot_{\alpha}$ is tree concatenation for every $\alpha \in \Sigma^{(0)}$.

Task 23 (isomorphism between $\mathcal{P}(A)$ and $\mathbb{B}^A$)

Show that the semirings $(\mathcal{P}(A), \cup, \cap, \emptyset, A)$ and $(\mathbb{B}^A, \tilde{\vee}, \tilde{\wedge}, \tilde{0}, \tilde{1})$ as isomorphic for every set A.

Task 24 (semiring homomorphisms)

A semiring $(S, +, \cdot, 0, 1)$ is called *zero-sum free* iff $a + b = 0$ implies $a = 0$ and $b = 0$ for every $a, b \in S$. S is called *zero-divisor free* iff $a \cdot b = 0$ implies $a = 0$ or $b = 0$ for every $a, b \in S$. Prove the following lemma.

Lemma. Let $(S, +, \cdot, 0, 1)$ be a semiring and $h \colon S \to \mathbb{B}$ be a mapping such that for every $a \in S$:

$$h(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise.} \end{cases}$$

h is a semiring homomorphism iff S is zero-sum free and zero-divisor free.

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