The Chomsky-Schützenberger Theorem for Quantitative Context-Free Languages

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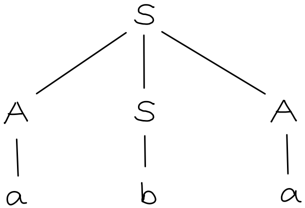
context-free $\rightsquigarrow$ CF

$$G: \quad \rho_1: \quad S \to A\,S\,A$$

$$\rho_2: \quad S \to b$$

$$\rho_3: \quad A \to a$$



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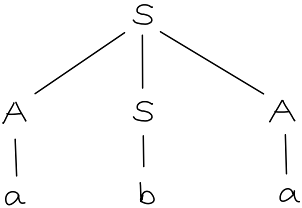

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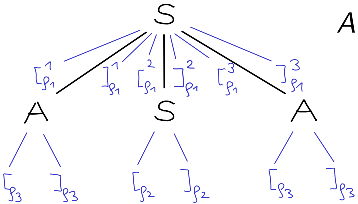

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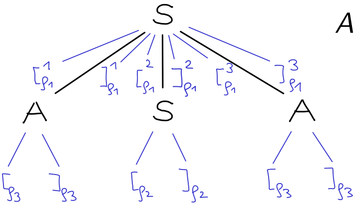

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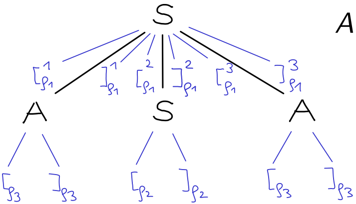

Dyck-language

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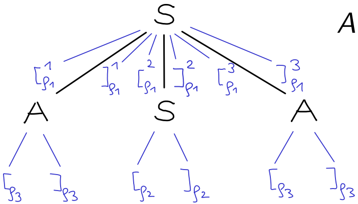

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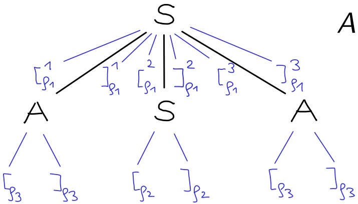

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check local properties: $\quad D(Y) \cap R$

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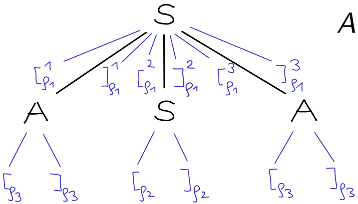

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alphabetic morphism:

$$h: Y \cup \overline{Y} \to \Sigma \cup \{\varepsilon\}$$

$$h(y) = \begin{cases} b & \text{if } y = [_{\rho_2} \\ a & \text{if } y = [_{\rho_3} \\ \varepsilon & \text{otherwise} \end{cases}$$

Theorem [Chomsky, Schützenberger 63]

Let $L \subseteq \Sigma^*$.

If $L = L(G)$ for some CF grammar G,

then there are

- an alphabet Y,
- a recognizable language R over $Y \cup \overline{Y}$, and
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such that $L = h'(D(Y) \cap R)$.

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Moreover, for every $w \in L(G)$:

$$|D_G(w)| = |\{u \in D(Y) \cap R \mid h'(u) = w\}|$$

(degree of ambiguity)

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$L : \Sigma^* \to \mathbb{N}; \quad w \mapsto |D_G(w)|$

weighted languages (formal power series) $L : \Sigma^* \to K$

$(K, +, \cdot, 0, 1)$ semiring

- $(K, +, 0)$ commutative monoid
- $(K, \cdot, 1)$ monoid
- $\cdot$ distributes over $+$
- 0 is absorbing for $\cdot$, i.e., $a \cdot 0 = 0 \cdot a = 0$

Examples:

1. natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$
2. Boolean semiring $(\{\text{true}, \text{false}\}, \vee, \wedge, \text{false}, \text{true})$
3. distributive bounded lattices $(L, \vee, \wedge, 0, 1)$
4. fields

weighted CF languages over semiring $(K, +, \cdot, 0, 1)$:

- CF grammar with weights $G = (N, \Sigma, Z, P, \mathrm{wt})$

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- weighted language of G:

$$\|G\| : \Sigma^* \to K \ , \quad w \mapsto \sum_{d \in D_G(w)} \mathrm{wt}(d)$$

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- weighted CF grammar:
  - CF grammar with weights and
  - K is complete <u>or</u> $D_G(w)$ is finite for every $w \in \Sigma^*$
- weighted language of a weighted CF grammar G:

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$$\|G\| : \Sigma^* \to K \quad , \quad w \mapsto \sum_{d \in D_G(w)} \mathrm{wt}(d)$$

- $L : \Sigma^* \to K$ is a weighted CF language:
  $\exists$ weighted CF grammar G over semiring K: $\quad L = \|G\|$.

Theorem [Chomsky, Schützenberger 63] (revisited)

Let $L : \Sigma^* \to \mathbb{N}$ and $(\mathbb{N}, +, \cdot, 0, 1)$ natural number semiring.

If $L = L(G)$ for weighted CF grammar G with $\mathrm{wt}(p) = 1$ for every $p \in P$,

then there are

- an alphabet Y,
- a recognizable language R over $Y \cup \overline{Y}$, and
- an alphabetic morphism $h : Y \cup \overline{Y} \to \Sigma \cup \{\varepsilon\}$

such that $L = h'(\mathrm{char}(D(Y) \cap R))$

Theorem [Salomaa, Soittola 78]

Let $L : \Sigma^* \to K$ and $(K, +, \cdot, 0, 1)$ commutative semiring.

The following are equivalent:

1. L is weighted CF language
2. there are
   - an alphabet Y,
   - a recognizable language R over $Y \cup \overline{Y}$, and
   - an alphabetic morphism $h : Y \cup \overline{Y} \to K^{\Sigma \cup \{\varepsilon\}}$

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quantitative languages $L : \Sigma^* \to K$ "average"

[Chatterjee, Doyen, Henzinger 10]

$(K, +, \mathrm{val}, 0)$ valuation monoid [Droste, Meinecke 10]

- $(K, +, 0)$ commutative monoid
- $\mathrm{val} : K^+ \to K$, $\mathrm{val}(a) = a$, $\mathrm{val}(\ldots 0 \ldots) = 0$

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1. $(\mathbb{R} \cup \{-\infty\}, \sup, \mathrm{avg}, -\infty)$ $\mathrm{avg}(a_1 \dots a_n) = \frac{1}{n} \cdot \sum_{i=1}^{n} a_i$

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2. strong bimonoid $(K, +, \cdot, 0, 1)$
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   - 0 absorbing for $\cdot$

e.g., semirings, bounded lattices

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$\mathrm{avg}(\dots + \infty \dots) = \mathrm{avg}(\dots \ \dots)$

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In general, val is

- not commutative
- not associative
- not distributive over $+$

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quantitative CF languages over unital val. monoid $(K, +, \mathrm{val}, 0, 1)$:

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  $\mathrm{wt} : P \to K$

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  $$\mathrm{wt}(\rho_1 \dots \rho_n) = \mathrm{val}(\mathrm{wt}(\rho_1) \dots \mathrm{wt}(\rho_n))$$

- weighted CF grammar:
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- quantitative language of a weighted CF grammar G:

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- $L : \Sigma^* \to K$ is a weighted CF language:
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- $L : \Sigma^* \to K$ is a quantitative CF language:
  $\exists$ weighted CF gr. G over unital val. monoid K: $\quad L = \|G\|$.

... in a very similar way define:

weighted pushdown automaton over unital valuation monoid

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Theorem [Droste, Vogler 13]

Let $L : \Sigma^* \to K$ and $(K, +, \mathrm{val}, 0, 1)$ unital val. monoid.
The following are equivalent:

1. L is quantitative CF language
2. L is quantitative behaviour of a weighted PDA.

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The following are equivalent:

1. L is quantitative CF language
2. there are
   - an alphabet Δ,
   - an unambiguous CF grammar G over Δ, and
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   such that $L = h'(L(G))$

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The following are equivalent:

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4. there are
   - an alphabet Y,
   - a deterministically recognizable series $r \in K^{(Y \cup \overline{Y})^*}$, and
   - an alphabetic morphism $h : Y \cup \overline{Y} \to \Sigma \cup \{\varepsilon\}$

   such that $L = h'(D(Y) \cap r)$

$$(D(Y) \cap r) \in K^{(Y \cup \overline{Y})^*} \quad w \mapsto \begin{cases} (r, w) & \text{if } w \in D(Y) \\ 0 & \text{otherwise} \end{cases}$$