Pure and O-Substitution*

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Abstract

The basic properties of distributivity and deletion of pure and o-substitution are investigated. The obtained results are applied to show preservation of recognizability in a number of surprising cases. It is proved that linear and recognizable tree series are closed under o-substitution provided that the underlying semiring is commutative, continuous, and additively idempotent. It is known that, in general, pure substitution does not preserve recognizability (not even for linear target tree series), but it is shown that recognizable linear probability distributions (represented as tree series) are closed under pure substitution.

1 Introduction

Tree series substitution is an operation on *tree series* (*i. e.*, mappings that associate to each tree some coefficient of a semiring) that corresponds to substitution on tree languages. We distinguish two main types of substitution on tree languages: IO and OI substitution. An IO substitution replaces all occurrences of some variable by the same tree whereas an OI substitution may choose one tree for each occurrence of a variable. In this contribution we consider IO substitutions.

An IO tree series substitution (pure or o-substitution) is the main operation used to the define the semantics of *tree series transducers* [7, 10], which were introduced as a joint generalization of tree transducers [23, 24] and weighted tree automata [1, 17]. It remains an open problem to identify suitable classes of tree series transducers that preserve *recognizable tree series* (*i. e.*, a tree series that can be computed by a weighted tree automaton). The only positive result in this direction is [18, Corollary 14] where the author shows that nondeleting and linear (top-down or bottom-up) tree series transducers preserve recognizable tree series.

In this contribution we investigate whether pure and o-substitution [10] (both IO tree series substitutions) preserve recognizable tree series. This can be seen as a first step towards a result that states that certain tree series transducers preserve recognizability because pure and o-substitution operation occupies such a central place in the definition of the semantics of tree series transducers. Tree series transducers recently found applications in machine translation [15], where preservation of recognizability is a central question.

Let us now discuss pure and o-substitution informally. To this end, we use the semiring $(\mathbb{R}_+, +, \cdot, 0, 1)$ of nonnegative real numbers. On the tree level pure and o-substitution just perform IO substitution of trees. It remains to show how the coefficents are obtained. Let us interpret the coefficient (in the interval $[0, 1]$) as a probability. In o-substitution the probability (*e. g.*, reliability), that is associated with an output tree, is taken to the nth power, if the output tree is used in n copies (is copied n times into some other tree). In this approach, an output tree stands for a composite, and the probability associated with the output tree reflects, *e. g.*, the reliability of this particular composite. When we combine composites into a new composite, then we obtain the

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reliability of the composite by a simple multiplication of the reliabilities of its components; each component taken as often as needed to assemble the composite (under the assumption that each component is critical for the correct functioning of the composite).

On the other hand pure substitution represents a computational approach; *i. e.*, the output trees represent values of computations, and the coefficient associated to an output tree can be viewed as the cost of computing this value. When we combine output trees, we simply multiply their coefficients to obtain the coefficient of the combined output tree. This is done irrespective of the number of uses of an output tree; *i. e.*, an output tree may be copied without penalty, which represents the computational approach in the sense that a value is available and can be reused without recomputation (call-by-value or eager evaluation).

Let us note that, in general, infinite sums may occur in pure and o-substitution. We adopt a set of axioms from [14] and show our results for any infinitary summation that fulfils these axioms. The axioms we use are much weaker than the axioms usually imposed (*e. g.*, in [18, 10]). In particular, we do not demand that every infinitary sum is well-defined. This immediately yields that not every pure or o-substitution is well-defined (*i. e.*, the result of a pure or o-substitution is sometimes undefined).

We approach the problem of proving preservation of recognizability by first proving several properties (like distributivity) of the substitutions. In particular, we investigate how deletion is handled by pure and o-substitution. Our main result then shows that the result of a pure or o-substitution of recognizable tree series is a well-defined recognizable tree series provided that the *target* tree series (*i. e.*, the tree series into which we substitute) is linear and certain sums of coefficients are well-defined and equal to 1 (the neutral element of the multiplication). Let us clarify the additional condition for pure substitution first. For every source tree series (*i. e.*, the tree series which we substitute into the target tree series) we add all of its coefficients and this sum must be well-defined and equal to 1. For o-substitution we instead add as many 1's as there are nonzero coefficients in a source tree series. Again this sum must be well-defined and equal to 1.

From our main result we can easily derive several corollaries. For example, we show that o-substitution preserves recognizable tree series in semirings that are commutative, idempotent, and continuous [8], whenever the target tree series is linear. For pure substitution we show that recognizable linear probability distributions are closed under substitution. In many contexts (*e. g.*, natural language processing) only the relations of the coefficients to one another are interesting. If the sum of all coefficients is well-defined (using real numbers and the usual notions of convergence), then we can scale the series so that the coefficient sum becomes 1. This process retains all factors between individual coefficients. Our mentioned result can then be applied after the scaling.

This report is structured as follows. In the second section we recall the required notions and notations. In Section 3 we discuss distributivity and deletion. Our main results are derived and illustrated in Section 4.

2 Preliminaries

We use $\mathbb{N}$ and $\mathbb{N}_+$ to represent the nonnegative and positive integers, respectively. Further let $[k]$ be an abbreviation for $\{n \in \mathbb{N} \mid 1 \leqslant n \leqslant k\}$. Given sets I and J and a subset $I_j \subseteq I$ for every $j \in J$, the family $(I_j)_{j \in J}$ is a *partition* of I if $I = \bigcup_{j \in J} I_j$ and $I_{j_1} \cap I_{j_2} = \emptyset$ for every $j_1, j_2 \in J$ with $j_1 \neq j_2$. A set Δ that is nonempty and finite is also called an *alphabet*.

A *ranked alphabet* is an alphabet Δ with a mapping $\mathrm{rk}_\Delta \colon \Delta \longrightarrow \mathbb{N}$. We use Δ_k to represent $\{\delta \in \Delta \mid \mathrm{rk}_\Delta(\delta) = k\}$. Moreover, we use the set $\mathrm{Z} = \{z_i \mid i \in \mathbb{N}_+\}$ of *variables* and $\mathrm{Z}_k = \{z_i \mid i \in [k]\}$. Given a ranked alphabet Δ and $V \subseteq \mathrm{Z}$, the set of *Δ-trees indexed by V*, denoted by $T_\Delta(V)$, is inductively defined to be the smallest set T such that (i) $V \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\delta \in \Delta_k$, and $t_1, \ldots, t_k \in T$ also $\delta(t_1, \ldots, t_k) \in T$. Since we generally assume that $\Delta \cap \mathrm{Z} = \emptyset$, we write α instead of $\alpha()$ whenever $\alpha \in \Delta_0$. Moreover, we also write T_Δ to denote $T_\Delta(\emptyset)$.

For every $t \in T_\Delta(\mathrm{Z})$, we denote by $|t|_z$ the number of occurrences of $z \in \mathrm{Z}$ in t. Given $t_1, \ldots, t_n \in T_\Delta(\mathrm{Z})$, the expression $t[t_1, \ldots, t_n]$ denotes the result of substituting in t every z_i by t_i

for every $i \in [n]$. Let $V \subseteq \mathrm{Z}$. We say that $t \in T_\Delta(\mathrm{Z})$ is *linear* and *nondeleting* in V, if every $z \in V$ occurs at most once and at least once in t, respectively. Moreover, we write $\mathrm{var}(t)$ for the set $\{i \in \mathbb{N}_+ \mid |t|_{z_i} \geq 1\}$.

A *(commutative) semiring* is an algebraic structure $\mathcal{A} = (A, +, \cdot, 0, 1)$ consisting of two commutative monoids $(A, +, 0)$ and $(A, \cdot, 1)$ such that $\cdot$ distributes over $+$ and 0 is absorbing with respect to $\cdot$. As usual we use $\sum_{i\in I} a_i$ for sums of families $(a_i)_{i\in I}$ of $a_i \in A$ where for only finitely many $i \in I$ we have $a_i \neq 0$. A semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$ is called *additively idempotent* if $1 + 1 = 1$.

A set I is called countable, if its cardinality is smaller or equal to the cardinality of $\mathbb{N}$. Let $\mathfrak{A} \subseteq \bigcup_{I \text{ countable}} A^I$ be a collection of families. An *infinitary sum operation* $\sum$ is a mapping $\sum\colon \mathfrak{A} \longrightarrow A$. We use the usual $\sum_{i\in I} a_i$ instead of $\sum(a_i)_{i\in I}$ for every countable set I and family $(a_i)_{i\in I} \in A^I$. Moreover, we say that $\sum_{i\in I} a_i$ is *well-defined* if $(a_i)_{i\in I} \in \mathfrak{A}$. In the sequel we always assume an infinitary sum operation $\sum$ such that the following axioms (*cf.*, [14]) are fulfilled:

- all finite sums are well-defined; *i. e.*, $\bigcup_{I \text{ finite}} A^I \subseteq \mathfrak{A}$,
- for every $I = \{j\}$ and $(a_i)_{i\in I} \in A^I$
$$\sum_{i\in\{j\}} a_i = a_j \tag{U}$$
- for every $I = \{j_1, j_2\}$ with $j_1 \neq j_2$ and $(a_i)_{i\in I} \in A^I$
$$\sum_{i\in\{j_1,j_2\}} a_i = a_{j_1} + a_{j_2} \tag{B}$$
- for every countable set I and $(a_i)_{i\in I} \in A^I$ and every countable set J and partition $(I_j)_{j\in J}$ of I we have that (i) the left hand side of (GP) is well-defined if and only if the right hand side is well-defined and (ii) the following equality holds:
$$\sum_{i\in I} a_i = \sum_{j\in J}\Big(\sum_{i\in I_j} a_i\Big) \tag{GP}$$
- for all countable sets I and J and every $(a_i)_{i\in I} \in A^I$ and $(a_j)_{j\in J} \in A^J$ such that the sums in the right hand side of (D) are well-defined, the left hand side of (D) is well-defined and
$$\sum_{i\in I, j\in J} (a_i \cdot a_j) = \Big(\sum_{i\in I} a_i\Big) \cdot \Big(\sum_{j\in J} a_j\Big) \quad . \tag{D}$$

Proposition 2.1 (see [14, Theorem IV.2.4]) Let I and J be countable sets with $J \subseteq I$ and $(a_i)_{i\in I} \in A^I$. If $a_i = 0$ for every $i \in I \setminus J$, then (i) $\sum_{i\in I} a_i$ is well-defined if and only if $\sum_{j\in J} a_i$ is well-defined and (ii) $\sum_{i\in I} a_i = \sum_{j\in J} a_i$.

Proof. The proof of that statement only uses axioms (U) and (GP). □

An infinitary summation $\sum$ is termed $\aleph_0$-*complete* if $\mathfrak{A} = \bigcup_{I \text{ countable}} A^I$. Whenever we speak of an $\aleph_0$-complete semiring, we silently assume that an $\aleph_0$-complete infinitary sum operation $\sum$ (obeying the above laws) is given.

A semiring is *naturally ordered*, whenever $\sqsubseteq\, \subseteq A^2$, defined by $a \sqsubseteq b$ if and only if there exists a $c \in A$ such that $a + c = b$, constitutes a partial order on A. Let $\mathcal{A}$ be $\aleph_0$-complete and naturally ordered. We say that $\mathcal{A}$ is *continuous*, if for every countable set I and $(a_i)_{i\in I} \in A^I$ the following supremum (with respect to the natural order $\sqsubseteq$) exists and

$$\sum_{i\in I} a_i = \sup\{\sum_{i\in F} a_i \mid F \subseteq I, F \text{ finite}\} \quad .$$

The Boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ with the usual disjunction $\vee$ and conjunction $\wedge$ is an example of a continuous semiring (where supremum is the infinitary sum operation).

Let S be a set and $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a semiring. A *(formal) power series* ψ is a mapping $\psi\colon S \longrightarrow A$. Given $s \in S$, we denote $\psi(s)$ also by (ψ, s) and write the series as $\sum_{s \in S} (\psi, s)\, s$. The *support* of ψ is $\mathrm{supp}(\psi) = \{s \in S \mid (\psi, s) \neq 0\}$. We sometimes simply write the series ψ as $\sum_{s \in \mathrm{supp}(\psi)} (\psi, s)\, s$.

Power series with finite support are called *polynomials*. We denote the set of all power series by $\mathcal{A}\langle\langle S\rangle\rangle$ and the set of polynomials by $\mathcal{A}\langle S\rangle$. The polynomial with empty support is denoted by $\widetilde{0}$. Power series $\psi, \psi' \in \mathcal{A}\langle\langle S\rangle\rangle$ are added componentwise; *i. e.*, $(\psi + \psi', s) = (\psi, s) + (\psi', s)$ for every $s \in S$, and we multiply ψ with a coefficient $a \in A$ componentwise; *i. e.*, $(a \cdot \psi, s) = a \cdot (\psi, s)$ for every $s \in S$. The summation extends to semirings with an infinitary sum operation $\sum$ as follows: Let I be a countable set and $(\psi_i)_{i \in I} \in \mathcal{A}\langle\langle S\rangle\rangle^I$. Precisely when $\sum_{i \in I} (\psi_i, s)$ is well-defined for every $s \in S$, then $\sum_{i \in I} \psi_i$ is well-defined and $(\sum_{i \in I} \psi_i, s) = \sum_{i \in I} (\psi_i, s)$ for every $s \in S$.

The previous definition shows how cumbersome the special treatment of well-definedness is. In the sequel, we adopt a style that treats "undefined" as a value (distinct to every other value); *i. e.*, we would define the above summation simply as $(\sum_{i \in I} \psi_i, s) = \sum_{i \in I} (\psi_i, s)$ for every $s \in S$.

In this report, we only consider power series in which the set S is a set of trees. Such power series are also called *tree series*. Let Δ be a ranked alphabet. A tree series $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ is said to be *linear* and *nondeleting* in $V \subseteq \mathrm{Z}$, if every $u \in \mathrm{supp}(\psi)$ is linear and nondeleting in V, respectively. Finally, we denote $\bigcup_{u \in \mathrm{supp}(\psi)} \mathrm{var}(u)$ by $\mathrm{var}(\psi)$.

2.1 Tree Series Substitution

Let $\mathrm{sel}_\Delta\colon T_\Delta(\mathrm{Z}) \times \mathbb{N}_+ \times \{\varepsilon, \mathrm{o}\} \longrightarrow \mathbb{N}$ be defined for every $u \in T_\Delta(\mathrm{Z})$, $i \in \mathbb{N}_+$, and $\eta \in \{\varepsilon, \mathrm{o}\}$ by

$$\mathrm{sel}_\Delta(u, i, \eta) = \begin{cases} 1 & \text{if } \eta = \varepsilon, \\ |u|_{z_i} & \text{if } \eta = \mathrm{o}. \end{cases}$$

Since Δ is usually obvious from the context, we regularly omit it and just write sel.

Let $\eta \in \{\varepsilon, \mathrm{o}\}$, $I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$, and $\psi_i \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ for every $i \in I$. The *η-substitution of $(\psi_i)_{i \in I}$ into ψ* [10, Definitions 3.1 and 3.2], denoted by $\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in I}$, is defined for every $t \in T_\Delta(\mathrm{Z})$ by

$$(\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in I}, t) = \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} (\psi, u) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u, i, \eta)} .$$

Note that compared to [10] we have defined η-substitution also for non-contiguous blocks of variables. The ε-substitution is also called *pure substitution*. In an expression $\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in I}$ we call ψ the *target* and each ψ_i a *source*. If $I = [n]$ for some $n \in \mathbb{N}$, we occasionally write $\psi \underset{\eta}{\longleftarrow} (\psi_1, \dots, \psi_n)$ instead of $\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in [n]}$.

The following three properties of paramount importance from [10, Proposition 3.4] will be used without explicit mention.

1. If $I = \emptyset$, then $\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in I} = \psi$.

2. If $\psi = \widetilde{0}$, then $\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in I} = \widetilde{0}$.

3. If $\psi_i = \widetilde{0}$ for some $i \in I$, then $\psi \underset{\eta}{\longleftarrow} (\psi_i)_{i \in I} = \widetilde{0}$.

Note that independent of ψ and $(\psi_i)_{i \in I}$ the above η-substitutions are well-defined.

3 Basic properties of substitutions

In this section we investigate basic properties, namely distributivity and deletion, of η-substitutions. Distributivity is important in a number of results (*e. g.*, associativity, compositions of tree series transducers [20], and equivalence of rewrite and initial-algebra semantics of tree series transducers [11]). Deletion needs to be handled in results on associativity, which are usually required for composition results on tree series transducers [20]. Moreover, our results on preservation of recognizability also use the properties of deletion.

In contrast to the published results, we do not assume that all countable sums are well-defined. If we, for example, take the real number semiring together with the finite and the absolutely convergent series (and the obvious summation), then this infinitary summation is not $\aleph_0$-complete. The real number semiring has major applications, for example, in natural language processing [15].

For the rest of this report, let $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a commutative semiring with $0 \neq 1$ and an infinitary summation $\sum$. Moreover, let $\eta \in \{\varepsilon, \mathrm{o}\}$, $I \subseteq \mathbb{N}_+$ be a finite set, J a countable set, and J_i a countable set for every $i \in I$. Finally, let $b = 1$ if $\eta = \varepsilon$, and let $b = 0$ otherwise.

3.1 Distributivity

In this section we investigate η-substitution with respect to distributivity. If we consider only $\aleph_0$-complete semirings, then the results for pure substitution already appeared in [7], while the results for o-substitution were first published in [21]. We start with a straightforward observation that shows distributivity in the target tree series.

Proposition 3.1 (see [7, Proposition 2.9] and [10, Proposition 3.14])
For every $j \in J$ let $\psi_j \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$, and for every $i \in I$ let $\psi_i \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$. Then

$$\sum_{j \in J} \psi_j \xleftarrow[\eta]{} (\psi_i)_{i \in I} = \Big(\sum_{j \in J} \psi_j\Big) \xleftarrow[\eta]{} (\psi_i)_{i \in I} \quad ,$$

provided that the right hand side is well-defined (*i. e.*, the η-substitution and the sum are well-defined).

Proof. In the proof we use the axioms (U), (D), and (GP). Let $t \in T_\Delta(\mathrm{Z})$.

$$\begin{aligned}
&\Big(\Big(\sum_{j \in J} \psi_j\Big) \xleftarrow[\eta]{} (\psi_i)_{i \in I}, t\Big) \\
= \;& [\text{by definition of } \xleftarrow[\eta]{}\,] \\
&\sum_{\substack{u \in \mathrm{supp}(\sum_{j \in J} \psi_j), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} \Big(\sum_{j \in J} \psi_j, u\Big) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} \\
= \;& [\text{by Proposition 2.1}] \\
&\sum_{\substack{u \in T_\Delta(\mathrm{Z}), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} \Big(\sum_{j \in J} \psi_j, u\Big) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} \\
= \;& [\text{by definition of sum of series}] \\
&\sum_{\substack{u \in T_\Delta(\mathrm{Z}), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} \Big(\sum_{j \in J} (\psi_j, u)\Big) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} \\
= \;& [\text{by axiom (D)}] \\
&\sum_{\substack{u \in T_\Delta(\mathrm{Z}), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} \Big(\sum_{j \in J} (\psi_j, u) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)}\Big) \\
= \;& [\text{by axiom (GP); applied twice}] \\
&\sum_{j \in J} \Big(\sum_{\substack{u \in T_\Delta(\mathrm{Z}), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} (\psi_j, u) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)}\Big)
\end{aligned}$$

$$= \quad \text{[by Proposition 2.1]}$$

$$\sum_{j\in J}\Big(\sum_{\substack{u\in\mathrm{supp}(\psi_j),\\(\forall i\in I)\colon u_i\in\mathrm{supp}(\psi_i),\\ t=u[u_i]_{i\in I}}}(\psi_j,u)\cdot\prod_{i\in I}(\psi_i,u_i)^{\mathrm{sel}(u,i,\eta)}\Big)$$

$$= \quad \text{[by definition of } \xleftarrow[\eta]{} \text{]}$$

$$\sum_{j\in J}(\psi_j \xleftarrow[\eta]{} (\psi_i)_{i\in I}, t)$$

$$= \quad \text{[by definition of sum of series]}$$

$$\Big(\sum_{j\in J}\psi_j \xleftarrow[\eta]{} (\psi_i)_{i\in I}, t\Big) \qquad \square$$

If we recall the definition of o-substitution, we see that for the distributivity of o-substitution in source tree series we obviously need a law of the form $(\sum_{j\in J} a_j)^n = \sum_{j\in J} a_j^n$ where J is nonempty, $(a_j)_{j\in J} \in A^J$, and $n \in \mathbb{N}$.

Definition 3.2 Let $N \subseteq \mathbb{N}$. The semiring $\mathcal{A}$ is called N-Frobenius*,* if for every $n \in N$, nonempty and countable J, and family $(a_j)_{j\in J} \in A^J$ the equality $(\sum_{j\in J} a_j)^n = \sum_{j\in J} a_j^n$ holds, provided that both sides are well-defined. Semirings that are $\mathbb{N}$-Frobenius are also called Frobenius semirings.

Note that with the help of axioms (D) and (GP) we can show that whenever $(\sum_{j\in J} a_j)^n$ with $n \geq 1$ is well-defined then also $\sum_{j\in J} a_j^n$ is well-defined. We require the set J to be nonempty in the previous definition because $\sum_{j\in\emptyset} a_j^n = 0$ for every $n \in \mathbb{N}$ but $(\sum_{j\in\emptyset} a_j)^n = 1$ if $n = 0$. Thus only the trivial semiring with $0 = 1$ would be $\{0, 1\}$-Frobenius, if we would omit the nonemptiness condition for J in Definition 3.2. If a semiring is N-Frobenius, then it immediately follows that for every $n \in N$ the n-th power Frobenius mapping $f_n\colon A \longrightarrow A$ defined for every $a \in A$ by $f(a) = a^n$ is a semiring homomorphism.

Let us focus on finite sums for the moment (*i. e.*, $\sum_{j\in J} a_j$ is undefined if and only if J is infinite; note that this infinitary summation does not fulfil all the axioms) and show examples of N-Frobenius semirings for various $N \subseteq \mathbb{N}$.

- Every semiring is $\{1\}$-Frobenius.
- Every additively idempotent semiring is $\{0, 1\}$-Frobenius.
- Every *additively extremal* (*i. e.*, $a_1 + a_2 \in \{a_1, a_2\}$ for every $a_1, a_2 \in A$) semiring is Frobenius.
- Every additively idempotent, *multiplicatively cancellative* (*i. e.*, $a \cdot a_1 = a \cdot a_2$ implies that $a_1 = a_2$ for every $a, a_1, a_2 \in A$ with $a \neq 0$), and commutative semiring is Frobenius [13, Proposition 4.43].

In fact, a semiring is $\{0\}$-Frobenius if and only if it is additively idempotent. This observation is used in the following proposition.

Proposition 3.3 Let $(\psi_j)_{j\in J} \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle^J$ and $n \in \mathbb{N}$ be such that the semiring $\mathcal{A}$ is $\{n\}$-Frobenius. Then

$$\sum_{j\in J}\Big(\sum_{u\in\mathrm{supp}(\psi_j)}(\psi_j,u)^n\Big) = \sum_{u\in\mathrm{supp}(\sum_{j\in J}\psi_j)}\Big(\sum_{j\in J}\psi_j,u\Big)^n$$

provided that the right hand side and $\sum_{j\in J}(\psi_j, u)^n$ for every $u \in T_\Delta(\mathrm{Z})$ are well-defined.

Proof. In the proof we use the axioms (U) and (GP). We distinguish two cases. Let $n \neq 0$.

$$\sum_{u\in\mathrm{supp}(\sum_{j\in J}\psi_j)}\Big(\sum_{j\in J}\psi_j,u\Big)^n$$

$$= \quad \text{[by Proposition 2.1]}$$

$$\sum_{u\in T_\Delta(\mathrm{Z})} \Big(\sum_{j\in J} \psi_j, u\Big)^n$$

$$= \quad \text{[by definition of sum of series]}$$

$$\sum_{u\in T_\Delta(\mathrm{Z})} \Big(\sum_{j\in J} (\psi_j, u)\Big)^n$$

$$= \quad \text{[because } \mathcal{A} \text{ is } \{n\}\text{-Frobenius]}$$

$$\sum_{u\in T_\Delta(\mathrm{Z})} \Big(\sum_{j\in J} (\psi_j, u)^n\Big)$$

$$= \quad \text{[by axiom (GP); applied twice]}$$

$$\sum_{j\in J}\Big(\sum_{u\in T_\Delta(\mathrm{Z})} (\psi_j, u)^n\Big)$$

$$= \quad \text{[by Proposition 2.1]}$$

$$\sum_{j\in J}\Big(\sum_{u\in \mathrm{supp}(\psi_j)} (\psi_j, u)^n\Big)$$

Now let $n = 0$. Thus $\mathcal{A}$ is $\{0\}$-Frobenius and hence additively idempotent. Furthermore, $(\sum_{j\in J} \psi_j, u)^0$ and $\sum_{j\in J}(\psi_j, u)^0$ are well-defined. Hence

$$1 = \Big(\sum_{j\in J} \psi_j, u\Big)^0 = \sum_{j\in J} (\psi_j, u)^0 = \sum_{j\in J} 1 \enspace .$$

If J is countably infinite, then this yields that arbitrary nonempty sums of 1's are 1. In this case the statement of the proposition is easily verified. Now let us suppose that J is finite. Since $\mathcal{A}$ is additively idempotent, we also have that $\mathcal{A}$ is zero-sum free [13, p. 4]. We conclude that $\mathrm{supp}(\sum_{j\in J} \psi_j) = \bigcup_{j\in J} \mathrm{supp}(\psi_j)$ by [5, Section VI.3].

$$\sum_{u\in \mathrm{supp}(\sum_{j\in J} \psi_j)} \Big(\sum_{j\in J} \psi_j, u\Big)^0$$

$$= \quad \text{[by the above equalities]}$$

$$\sum_{u\in \bigcup_{j\in J} \mathrm{supp}(\psi_j)} \Big(\sum_{j\in J} (\psi_j, u)^0\Big)$$

$$= \quad \text{[because } \mathcal{A} \text{ is additively idempotent]}$$

$$\sum_{u\in \bigcup_{j\in J} \mathrm{supp}(\psi_j)} \Big(\sum_{\substack{j\in J,\\ (\psi_j,u)\neq 0}} (\psi_j, u)^0\Big)$$

$$= \quad \text{[by axiom (GP); applied twice]}$$

$$\sum_{j\in J}\Big(\sum_{u\in \mathrm{supp}(\psi_j)} (\psi_j, u)^0\Big)$$

□

Using the notion "N-Frobenius" we can prove distributivity of η-substitution (*cf.* [7, Proposition 2.9]), *e. g.*, for $\eta = \mathrm{o}$ and linear tree series over additively idempotent semirings.

Proposition 3.4 (see [7, Proposition 2.9]) Let $\psi \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$, and for every $i \in I$ and $j_i \in J_i$ let $\psi_{j_i} \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$. Moreover, let $\mathcal{A}$ be an N-Frobenius semiring for some $N \subseteq \mathbb{N}$. Provided that the right hand side of (1) is well-defined and $\mathrm{sel}(u, i, \eta) \in N$ for every $i \in I$ and $u \in \mathrm{supp}(\psi)$, then

$$\sum_{(\forall i\in I)\colon\, j_i\in J_i} \psi \xleftarrow[\eta]{} (\psi_{j_i})_{i\in I} = \psi \xleftarrow[\eta]{} \Big(\sum_{j_i\in J_i} \psi_{j_i}\Big)_{i\in I} \enspace . \qquad (1)$$

Proof. The result for $\eta = \varepsilon$ (in this case the restriction $\mathrm{sel}(u, i, \eta) \in N$ can be simplified to $1 \in N$ and the semiring $\mathcal{A}$ is always $\{1\}$-FROBENIUS) is stated for continuous semirings and proved for $\aleph_0$-complete semirings in [7, Proposition 2.9]. A similar statement for semirings that are not $\aleph_0$-complete with respect to $\sum$ is claimed in [9, Proposition 2.3], but unfortunately, it is wrong.

Clearly, the statement holds if $J_i = \emptyset$ for some $i \in I$. Thus we assume that $J_i \neq \emptyset$ for every $i \in I$. Let $t \in T_\Delta(\mathrm{Z})$.

$$\Big(\psi \xleftarrow[\eta]{} \Big(\sum_{j_i \in J_i} \psi_{j_i}\Big)_{i\in I}, t\Big)$$

$$= \quad [\text{by definition of } \xleftarrow[\eta]{}]$$

$$\sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\sum_{j_i \in J_i} \psi_{j_i}), \\ t = u[u_i]_{i\in I}}} (\psi, u) \cdot \prod_{i \in I} \Big(\sum_{j_i \in J_i} \psi_{j_i}, u_i\Big)^{\mathrm{sel}(u,i,\eta)}$$

$$= \quad [\text{by axioms (D) and (GP) and Proposition 3.3}]$$

$$\sum_{(\forall i \in I)\colon j_i \in J_i} \Big(\sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_{j_i}), \\ t = u[u_i]_{i\in I}}} (\psi, u) \cdot \prod_{i \in I} (\psi_{j_i}, u_i)^{\mathrm{sel}(u,i,\eta)}$$

$$= \quad [\text{by definition of } \xleftarrow[\eta]{}]$$

$$\sum_{(\forall i \in I)\colon j_i \in J_i} (\psi \xleftarrow[\eta]{} (\psi_{j_i})_{i\in I}, t)$$

$$= \quad [\text{by definition of sum of series}]$$

$$\Big(\sum_{(\forall i \in I)\colon j_i \in J_i} \psi \xleftarrow[\eta]{} (\psi_{j_i})_{i\in I}, t\Big)$$

□

As a corollary we obtain the stated distributivity result (for o-substitution) for linear target tree series in additively idempotent semirings.

Corollary 3.5 (of Proposition 3.4) Let $\mathcal{A}$ be additively idempotent, $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ be linear in Z_I. Moreover, let J_i be finite for every $i \in I$, and let $\psi_{j_i} \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $i \in I$ and $j_i \in J_i$.

$$\sum_{(\forall i \in I)\colon j_i \in J_i} \psi \xleftarrow[\mathrm{o}]{} (\psi_{j_i})_{i\in I} = \psi \xleftarrow[\mathrm{o}]{} \Big(\sum_{j_i \in J_i} \psi_{j_i}\Big)_{i\in I}$$

3.2 Handling deletion

In this section we study the effect of deletion. The obtained law is of paramount importance for associativity results, preservation of recognizibility, and compositions of tree series transformations [20]. Let $u \in T_\Delta(\mathrm{Z})$ and $u_i \in T_\Delta(\mathrm{Z})$ for every $i \in I$. We present a proposition that generalizes the result that $u[u_i]_{i\in I} = u[u_j]_{j\in J}$ for every $J \subseteq I$ such that $J \cap \mathrm{var}(u) = I \cap \mathrm{var}(u)$. Intuitively speaking, this asserts that u_i is irrelevant in $u[u_i]_{i\in I}$, if $i \notin \mathrm{var}(u)$. This generalizes nicely to tree languages $L, L_i \subseteq T_\Delta(\mathrm{Z})$; *i. e.*, $L[L_i]_{i\in I} = L[L_j]_{j\in J}$ for every $J \subseteq I$ such that $J \cap \mathrm{var}(L) = I \cap \mathrm{var}(L)$ and $L_i \neq \emptyset$ for every $i \in I \setminus J$. The additional restriction is derived from the fact that $L[L_i]_{i\in I} = \emptyset$ whenever $L_i = \emptyset$ for some $i \in I$.

Proposition 3.6 Let $J \subseteq I$ and $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ be such that $J \cap \mathrm{var}(\psi) = I \cap \mathrm{var}(\psi)$. Moreover, for every $i \in I$ let $\psi_i \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$. Then

$$\psi \xleftarrow[\eta]{} (\psi_i)_{i\in I} = \psi \xleftarrow[\eta]{} (\psi_j)_{j\in J} \;,$$

provided that:

(i) the left hand side is well-defined; and

(ii) $\sum_{u \in \mathrm{supp}(\psi_i)} (\psi_i, u)^b = 1$ for every $i \in I \setminus J$.

Proof. Let $t \in T_\Delta(\mathrm{Z})$.

$$
\begin{aligned}
& (\psi \xleftarrow[\eta]{} (\psi_i)_{i\in I}, t) \\
= \quad & [\text{by definition of } \xleftarrow[\eta]{}] \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I):\, u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i\in I}}} (\psi, u) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} \\
= \quad & [\text{because } i \notin \mathrm{var}(u) \subseteq \mathrm{var}(\psi) \text{ for every } i \in I \setminus J \text{ and thus } \mathrm{sel}(u,i,\eta) = b] \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I):\, u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_j]_{j\in J}}} (\psi, u) \cdot \prod_{j \in J} (\psi_j, u_j)^{\mathrm{sel}(u,j,\eta)} \cdot \prod_{i \in I \setminus J} (\psi_i, u_i)^b \\
= \quad & [\text{by axiom (D)}] \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall j \in J):\, u_j \in \mathrm{supp}(\psi_j), \\ t = u[u_j]_{j\in J}}} (\psi, u) \cdot \prod_{j \in J} (\psi_j, u_j)^{\mathrm{sel}(u,j,\eta)} \cdot \prod_{i \in I \setminus J} \Big(\sum_{u_i \in \mathrm{supp}(\psi_i)} (\psi_i, u_i)^b \Big) \\
= \quad & [\text{by condition (ii)}] \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall j \in J):\, u_j \in \mathrm{supp}(\psi_j), \\ t = u[u_j]_{j\in J}}} (\psi, u) \cdot \prod_{j \in J} (\psi_j, u_j)^{\mathrm{sel}(u,j,\eta)} \\
= \quad & [\text{by definition of } \xleftarrow[\eta]{}] \\
& (\psi \xleftarrow[\eta]{} (\psi_j)_{j\in J}, t)
\end{aligned}
$$

□

The condition $J \cap \mathrm{var}(\psi) = I \cap \mathrm{var}(\psi)$ asserts that J covers all those variables of ψ which are also covered by I; *i. e.*, $(I \setminus J) \cap \mathrm{var}(\psi) = \emptyset$. We have already seen that this restriction is necessary even for substitution on trees. Moreover, we have seen that, for the corresponding statement on tree languages, the condition $L_i \neq \emptyset$ is necessary for every $i \in I \setminus J$. This condition is hidden in $\sum_{u \in \mathrm{supp}(\psi_i)} (\psi_i, u)^b = 1$ for every $i \in I \setminus J$. Unless $0 = 1$ (which we generally excluded), this means that $\psi_i \neq \widetilde{0}$ for every $i \in I \setminus J$.

Let us consider the condition $\sum_{u \in \mathrm{supp}(\psi_i)} (\psi_i, u)^b = 1$ in some more detail. When $\mathrm{supp}(\psi_i)$ is infinite, we need the notion of a necessary summation [13].

Definition 3.7 (see [13, p. 251]) We call $\sum$ *necessary* if $\sum_{j\in J} a_j = \sum_{j \in J} c_j$ for all countable sets J and $(a_j)_{j\in J}, (c_j)_{j \in J} \in A^J$ such that:

1. $\sum_{j \in J} a_j$ and $\sum_{j\in J} c_j$ are well-defined; and
2. for each finite subset $J' \subseteq J$ there exists a finite set J'' with $\sum_{j \in J''} a_j = \sum_{j\in J''} c_j$ and $J' \subseteq J'' \subseteq J$.

Note that not every infinitary summation of an $\aleph_0$-complete semiring is necessary. Examples of infinitary summations that are not necessary can be found in [13, Example 22.18].

Proposition 3.8 Let $\mathcal{A}$ be continuous with respect to $\sum$. Then $\sum$ is necessary.

Proof. Clearly, $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$, so all sums with countably many summands are defined. It is well-known that the natural order $\sqsubseteq$ is a partial order if $\mathcal{A}$ is continuous. Let J be a countable index set and $(a_j)_{j\in J} \in A^J$ and $(c_j)_{j\in J} \in A^J$ be families. By definition of $\sqsubseteq$ we have $\sum_{j\in J'} a_j \sqsubseteq \sum_{j \in J} a_j$ for every $J' \subseteq J$.

Suppose that for every finite $J' \subseteq J$ there exists a finite subset J'', denoted by $S_{J'}$, such that $J' \subseteq J'' \subseteq J$ and $\sum_{j \in J''} a_j = \sum_{j \in J''} c_j$. It remains to show that $\sum_{j \in J} a_j = \sum_{j \in J} c_j$.

$$\begin{aligned}\sum_{j \in J} a_j &= \sup\{\sum_{j \in F} a_j \mid F \subseteq J, F \text{ finite}\} = \sup\{\sum_{j \in S_F} a_j \mid F \subseteq J, F \text{ finite}\} \\ &= \sup\{\sum_{j \in S_F} c_j \mid F \subseteq J, F \text{ finite}\} = \sup\{\sum_{j \in F} c_j \mid F \subseteq J, F \text{ finite}\} = \sum_{j \in J} c_j \qquad \square\end{aligned}$$

Note that every semiring that is $\aleph_0$-complete with respect to a necessary summation is naturally ordered [13, Proposition 22.29]. Further, in an additively idempotent semiring with necessary summation we have $\sum_{i \in I} a = a$ for every $a \in A$ and countable index set I such that $\sum_{i \in I} a$ is well-defined. Let us now come back to the discussion of Condition (ii) in Proposition 3.6. The next proposition lists some simple conditions; each of them implies Condition (ii) of Proposition 3.6.

Proposition 3.9 Let $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ with $\psi \neq \widetilde{0}$ be such that $\sum_{u \in \mathrm{supp}(\psi)} (\psi, u)^b$ is well-defined. Then $\sum_{u \in \mathrm{supp}(\psi)} (\psi, u)^b = 1$ if

- $b = 0$ (*i. e.*, $\eta = \mathrm{o}$) and
  - $\mathrm{supp}(\psi)$ is a singleton;
  - $\mathcal{A}$ is additively idempotent and ψ is polynomial; or
  - $\mathcal{A}$ is additively idempotent with necessary $\sum$.
- $b = 1$ (*i. e.*, $\eta = \varepsilon$) and
  - $\mathcal{A}$ is simple [13] (*i. e.*, $a + 1 = 1$ for every $a \in A$) and $(\psi, u) = 1$ for some $u \in \mathrm{supp}(\psi)$; or
  - ψ is boolean and any of the conditions of the case $b = 0$ applies.

4 Preservation of recognizability

In this section we consider the question whether η-substitution preserves recognizability. Let Δ be a ranked alphabet. It is known that substitution of the same tree u' for two occurrences of a variable z, in general, does not preserve recognizability; *i. e.*, already for $n \in \mathbb{N}_+$, recognizable tree languages $L_1, \ldots, L_n$ and $L = \{u\}$ with $u \in T_\Sigma(\mathrm{Z}_n)$ we have that $L[L_1, \ldots, L_n]$ is not necessarily recognizable (although $L_1, \ldots, L_n$ and $\{u\}$ are recognizable). However, IO substitution on tree languages preserves recognizable tree languages, if the target tree language is linear (see [6, Theorem 3.65] or [12, Theorem 4.16]); *i. e.*, for every $n \in \mathbb{N}$ and $L, L_1, \ldots, L_n \subseteq T_\Sigma(\mathrm{Z})$ such that L is linear in Z_n and $L, L_1, \ldots, L_n$ are recognizable also $L[L_1, \ldots, L_n]$ is recognizable.

First, let us clarify the notion of recognizable tree series [1, 17, 3]. We refer the reader to [2] for a detailed introduction and references to further models and results. We have chosen the automaton model called bu-w-fta-f (bottom-up finite-state weighted tree automaton with final weights) in [2, Section 4.1.3].

Definition 4.1 (see [2, Chapter 4]) A *bottom-up weighted tree automaton (over Δ and $\mathcal{A}$)* is a tuple $M = (Q, \Delta, \mathcal{A}, F, \mu)$ where Q is an alphabet of *states;* Δ is a ranked alphabet of *input symbols;* $\mathcal{A} = (A, +, \cdot, 0, 1)$ is a semiring; $F\colon Q \longrightarrow A$ is a *final weight distribution;* and $\mu = (\mu_k)_{k \in \mathbb{N}}$ with $\mu_k\colon \Delta_k \longrightarrow A^{Q \times Q^k}$ is a *tree representation.*

The *initial algebra semantics* of M [2, Section 4.1] is determined by the mapping $h_\mu\colon T_\Delta \longrightarrow A^Q$, which is defined by

$$h_\mu(\delta(u_1, \ldots, u_k))_q = \sum_{q_1, \ldots, q_k \in Q} \mu_k(\delta)_{q, q_1 \cdots q_k} \cdot \prod_{i \in [k]} h_\mu(u_i)_{q_i}$$

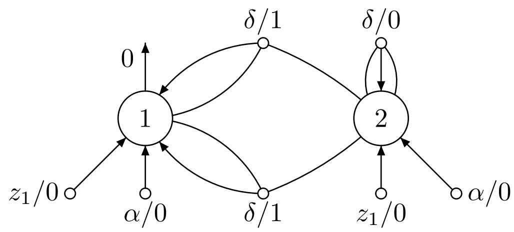

Figure 1: Bottom-up weighted tree automaton over $\mathbb{A}_\infty$.

for every $k \in \mathbb{N}$, $\delta \in \Delta_k$, $q \in Q$, and $u_1, \ldots, u_k \in T_\Delta$. The tree series *recognized by M,* denoted by $\|M\|$, is defined for every $u \in T_\Delta$ by

$$(\|M\|, u) = \sum_{q \in Q} F_q \cdot h_\mu(u)_q \enspace .$$

We use the method of [22] and [2, Example 3.1.2] to graphically represent weighted tree automata. Note that we write $\mu_0(\alpha)_q$ instead of $\mu_0(\alpha)_{q,\varepsilon}$ for every $\alpha \in \Delta_0$ and $q \in Q$. Let Δ be a ranked alphabet and $\mathcal{A}$ be a semiring. A tree series $\psi \in \mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle$ is termed *recognizable,* if there exists a bottom-up weighted tree automaton M over Δ and $\mathcal{A}$ such that $\psi = \|M\|$. The class of all recognizable tree series over Δ and $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Delta \rangle\!\rangle$. For every finite $I \subseteq \mathbb{N}_+$, we say that a tree series $\psi \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z}_I) \rangle\!\rangle$ is *recognizable,* if $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Gamma \rangle\!\rangle$ where $\Gamma_k = \Delta_k$ for every $k \in \mathbb{N}_+$ and $\Gamma_0 = \Delta_0 \cup \mathrm{Z}_I$; *i. e.*, the elements of Z_I are treated as new nullary symbols. Consequently, $\mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Delta(\mathrm{Z}_I) \rangle\!\rangle = \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Gamma \rangle\!\rangle$ denotes the class of all recognizable tree series over Δ, $\mathcal{A}$, and I.

Let us illustrate the previous definition. Let us use $\Delta = \{\delta^{(2)}, \alpha^{(0)}\}$ as ranked alphabet. We show that $\psi = \max_{u \in T_\Delta(\mathrm{Z}_1)} \mathrm{height}(u)\; u$ is a recognizable tree series using the arctic semiring $\mathbb{A}_\infty = (\mathbb{N} \cup \{-\infty, \infty\}, \max, +, \infty, 0)$ by presenting a bottom-up weighted tree automaton that recognizes ψ. Let $M = (Q, \Gamma, \mathbb{A}_\infty, F, \mu)$ be the bottom-up weighted tree automaton specified by $Q = \{1, 2\}$; $\Gamma = \{\delta^{(2)}, \alpha^{(0)}, z_1^{(0)}\}$; $F_1 = 0$ and $F_2 = -\infty$; and

$$\mu_0(\alpha)_1 = \mu_0(\alpha)_2 = \mu_0(z_1)_1 = \mu_0(z_1)_2 = 0$$
$$\mu_2(\delta)_{1,12} = \mu_2(\delta)_{1,21} = 1 \quad \text{and} \quad \mu_2(\delta)_{2,22} = 0 \enspace .$$

All remaining entries of $\mu_2(\delta)$ are assumed to be $-\infty$. The automaton is illustrated in Figure 1. We claim that $(\|M\|, u) = \mathrm{height}(u)$ for every $u \in T_\Gamma$. This claim can be proved by a straightforward induction.

In fact, for every ranked alphabet Δ and finite $I \subseteq \mathbb{N}_+$ we can give a bottom-up weighted tree automaton (over $\mathbb{A}_\infty$) recognizing $\max_{u \in T_\Delta(\mathrm{Z}_I)} \mathrm{height}(u)\; u$.

Now let us return to the question of preservation of recognizability. In [18, Corollary 14] it is proved that recognizability (of tree series) is preserved whenever the target tree series is nondeleting and linear and the semiring is commutative and continuous. This statement is proved for OI substitution in [18], but OI and η-substitution coincide whenever the target tree series is nondeleting and linear.

We should like to obtain a statement on preservation of recognizability in which the target tree series is only linear (and not necessarily nondeleting). Let us illustrate the main idea in a simplified setting. Let $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Delta(\mathrm{Z}_1) \rangle\!\rangle$ be linear in Z_1 and $\psi_1 \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Delta \rangle\!\rangle$. We want to show that $\psi \underset{\eta}{\longleftarrow} (\psi_1)$ is recognizable, thus we need to present a bottom-up weighted tree automaton $M' = (Q', \Delta, \mathcal{A}, F', \mu')$ that recognizes $\psi \underset{\eta}{\longleftarrow} (\psi_1)$. Let $M = (Q, \Gamma, \mathcal{A}, F, \mu)$ be a bottom-up weighted tree automaton recognizing ψ and $M_1 = (Q_1, \Delta, \mathcal{A}, F_1, \mu_1)$ be a bottom-up weighted tree automaton recognizing ψ_1. We employ a standard idea for the construction of M'. Roughly speaking, we take the disjoint union of M and M_1 and add transitions that nondeterministically change from M_1 to M. More precisely, for every $k \in \mathbb{N}_+$, $\delta \in \Delta_k$, $q \in Q$, and $q_1, \ldots, q_k \in Q_1$ we set

$$\mu'_k(\delta)_{q, q_1 \cdots q_k} = \sum_{p \in Q_1} \mu_0(z_1)_q \cdot (F_1)_p \cdot (\mu_1)_k(\delta)_{p, q_1 \cdots q_k} \enspace .$$

Informally speaking, for each state p of M_1 we take the weight of the transition $(\mu_1)_k(\delta)_{p,q_1\cdots q_k}$ of M_1, multiply the corresponding entry $(F_1)_p$ in the final weight distribution, and multiply the weight $\mu_0(z_1)_q$ for entering M (via z_1) in state q. Nullary symbols δ are treated similarly. We employ a proof method, which requires us to make the input alphabets Δ and Γ disjoint. This simplifies the proof because each tree then admits a unique decomposition into (at most one) part that needs to be processed by M_1 and a part that needs to be processed by M.

Theorem 4.2 Let $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Delta(\mathrm{Z}_I)\rangle\rangle$ be linear in Z_I. Moreover, let $\psi_i \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Delta\rangle\rangle$ for every $i \in I$. Then $\psi \xleftarrow[\eta]{} (\psi_i)_{i\in I}$ is well-defined and recognizable, provided that $\sum_{u\in \mathrm{supp}(\psi_i)}(\psi_i, u)^b = 1$ for every $i \in I$.

Proof. Let $\psi_i = \widetilde{0}$ for some $i \in I$. Then $\psi \xleftarrow[\eta]{} (\psi_i)_{i\in I} = \widetilde{0}$, which is clearly recognizable. Note that it is decidable in zero-sum free (*e. g.*, all $\aleph_0$-complete semirings) whether $\psi_i = \widetilde{0}$ [19]. Hence for the remainder of the proof we assume that $\psi_i \neq \widetilde{0}$ for all $i \in I$. For every $k \in \mathbb{N}_+$ let $\Gamma_k = \Delta_k$ and $\Gamma_0 = \Delta_0 \cup \mathrm{Z}_I$. Since $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Delta(\mathrm{Z}_I)\rangle\rangle$ and $\psi_i \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Delta\rangle\rangle$ for every $i \in I$, there exist bottom-up weighted tree automata $M = (Q, \Gamma, \mathcal{A}, F, \mu)$ and $M_i = (Q_i, \Delta, \mathcal{A}, F_i, \mu_i)$ such that $\|M\| = \psi$ and $\|M_i\| = \psi_i$ for every $i \in I$.

For every $i \in I$ let $\overline{\Delta}^i$ be the ranked alphabet given by $\overline{\Delta}^i_k = \{\overline{\delta}^i \mid \delta \in \Delta_k\}$ for every $k \in \mathbb{N}$. For every $i \in I$ we define the mapping $\mathrm{bar}_i \colon T_\Delta \longrightarrow T_{\overline{\Delta}^i}$ by

$$\mathrm{bar}_i(\delta(t_1,\dots,t_k)) = \overline{\delta}^i(\mathrm{bar}_i(t_1),\dots,\mathrm{bar}_i(t_k))$$

for every $k \in \mathbb{N}$, $\delta \in \Delta_k$, and $t_1,\dots,t_k \in T_\Delta$. Moreover, we extend bar_i to tree series as follows. We define the mapping $\mathrm{bar}_i \colon \mathcal{A}\langle\langle T_\Delta\rangle\rangle \longrightarrow \mathcal{A}\langle\langle T_{\overline{\Delta}^i}\rangle\rangle$, which relabels all δ-nodes by their corresponding i-overlined version, for every $\varphi \in \mathcal{A}\langle\langle T_\Delta\rangle\rangle$ by

$$\mathrm{bar}_i(\varphi) = \sum_{t\in T_\Delta} (\varphi, t)\ \mathrm{bar}_i(t) \enspace .$$

Without loss of generality, let us suppose that for every $i \in I$ we have that (i) Δ and $\overline{\Delta}^i$ are disjoint and (ii) Q and Q_i are disjoint. We let $\Delta'_k = \Delta_k \cup \bigcup_{i\in I}\overline{\Delta}^i_k$ for every $k \in \mathbb{N}$, and $Q' = Q \cup \bigcup_{i\in I} Q_i$. We construct a bottom-up weighted tree automaton M' recognizing $\psi \xleftarrow[\eta]{} (\mathrm{bar}_i(\psi_i))_{i\in I}$ as follows. Let $M' = (Q', \Delta', \mathcal{A}, F', \mu')$ where for every $i \in I$, $k \in \mathbb{N}$, $\delta \in \Delta_k$:

- $F'_q = F_q$ for every $q \in Q$ and $F'_p = 0$ for every $p \in \bigcup_{i\in I} Q_i$;
- $\mu'_k(\overline{\delta}^i)_{p,w} = (\mu_i)_k(\delta)_{p,w}$ for every $p \in Q_i$ and $w \in (Q_i)^k$;
- $\mu'_k(\delta)_{q,w} = \mu_k(\delta)_{q,w}$ for every $q \in Q$ and $w \in Q^k$; and
- $\mu'_k(\overline{\delta}^i)_{q,w} = \sum_{p\in Q_i} \mu_0(z_i)_q \cdot (F_i)_p \cdot (\mu_i)_k(\delta)_{p,w}$ for every $q \in Q$ and $w \in (Q_i)^k$.

All the remaining entries in μ' are set to 0.

Clearly, $h_{\mu'}(\mathrm{bar}_i(t))_p = h_{\mu_i}(t)_p$ for every $i \in I$, $t \in T_\Delta$, and $p \in Q_i$. Next we prove that for every $q \in Q$ and $t \in T_\Delta(\mathrm{Z}_I)$, which is linear in Z_I, and family $(u_i)_{i\in \mathrm{var}(t)} \in (T_\Delta)^{\mathrm{var}(t)}$ we have

$$h_{\mu'}(t[\mathrm{bar}_i(u_i)]_{i\in \mathrm{var}(t)})_q = h_\mu(t)_q \cdot \prod_{i\in \mathrm{var}(t)} \big(\|M_i\|, u_i\big) \enspace .$$

We prove this statement inductively, so let $t = z_j$ for some $j \in I$. Moreover, let $u_j = \delta(t_1,\dots,t_k)$ for some $k \in \mathbb{N}$, $\delta \in \Delta_k$, and $t_1,\dots,t_k \in T_\Delta$.

$$\begin{aligned}
& h_{\mu'}(z_j[\mathrm{bar}_i(u_i)]_{i\in \mathrm{var}(z_j)})_q \\
= \quad & [\text{by substitution and definition of } \mathrm{bar}_j] \\
& h_{\mu'}\big(\overline{\delta}^j(\mathrm{bar}_j(t_1),\dots,\mathrm{bar}_j(t_k))\big)_q \\
= \quad & [\text{by Definition 4.1}] \\
& \sum_{q_1,\dots,q_k\in Q'} \mu'_k(\overline{\delta}^j)_{q,q_1\cdots q_k} \cdot \prod_{i\in[k]} h_{\mu'}(\mathrm{bar}_j(t_i))_{q_i}
\end{aligned}$$

$$
\begin{aligned}
&= \quad [\text{by definition of } \mu' \text{ and } h_{\mu'}(\mathrm{bar}_j(t_i))_{q_i} = h_{\mu_j}(t_i)_{q_i}] \\
&\quad \sum_{q_1,\dots,q_k \in Q_j} \Big(\sum_{p \in Q_j} \mu_0(z_j)_q \cdot (F_j)_p \cdot (\mu_j)_k(\delta)_{p,q_1\cdots q_k} \Big) \cdot \prod_{i \in [k]} h_{\mu_j}(t_i)_{q_i} \\
&= \quad [\text{by distributivity and Definition 4.1}] \\
&\quad \sum_{p \in Q_j} h_\mu(z_j)_q \cdot (F_j)_p \cdot h_{\mu_j}(\delta(t_1,\dots,t_k))_p \\
&= \quad [\text{by distributivity and Definition 4.1}] \\
&\quad h_\mu(z_j)_q \cdot \prod_{i \in \mathrm{var}(z_j)} (\|M_i\|, u_i)
\end{aligned}
$$

Now, let $t = \delta(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\delta \in \Delta_k$, and $t_1, \dots, t_k \in T_\Delta(\mathrm{Z}_I)$.

$$
\begin{aligned}
&h_{\mu'}(\delta(t_1,\dots,t_k)[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)})_q \\
&= \quad [\text{by substitution}] \\
&\quad h_{\mu'}(\delta(t_1[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t_1)}, \dots, t_k[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t_k)}))_q \\
&= \quad [\text{by Definition 4.1}] \\
&\quad \sum_{q_1,\dots,q_k \in Q'} \mu'_k(\delta)_{q,q_1\cdots q_k} \cdot \prod_{j \in [k]} h_{\mu'}(t_j[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t_j)})_{q_j} \\
&= \quad [\text{by induction hypothesis and definition of } \mu'] \\
&\quad \sum_{q_1,\dots,q_k \in Q} \mu_k(\delta)_{q,q_1\cdots q_k} \cdot \prod_{j \in [k]} \Big(h_\mu(t_j)_{q_j} \cdot \prod_{i \in \mathrm{var}(t_j)} (\|M_i\|, u_i) \Big) \\
&= \quad [\text{by associativity and Definition 4.1}] \\
&\quad h_\mu(\delta(t_1,\dots,t_k))_q \cdot \prod_{j \in [k], i \in \mathrm{var}(t_j)} (\|M_i\|, u_i) \\
&= \quad [\text{because } t \text{ is linear in } \mathrm{Z}_I] \\
&\quad h_\mu(\delta(t_1,\dots,t_k))_q \cdot \prod_{i \in \mathrm{var}(t)} (\|M_i\|, u_i)
\end{aligned}
$$

This completes the proof of the auxiliary statement. Consequently,

$$
(\|M'\|, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}) = (\|M\|, t) \cdot \prod_{i \in \mathrm{var}(t)} (\|M_i\|, u_i) = (\psi, t) \cdot \prod_{i \in \mathrm{var}(t)} (\psi_i, u_i) \ .
$$

Using this result, we can show that $\psi' = \psi \xleftarrow[\eta]{} (\mathrm{bar}_i(\psi_i))_{i \in I}$ is recognizable. In fact, this is the tree series that is recognized by M'. Note that the semantics $\|M'\|$ is well-defined. We delay the proof of the mentioned equality because we first want to prove that $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ and $\psi' = \psi \xleftarrow[\eta]{} (\mathrm{bar}_i(\psi_i))_{i \in I}$ are well-defined. Let $t \in T_\Delta$ and $n = \mathrm{height}(t)$. We start from a finite sum, which is well-defined.

$$
\begin{aligned}
&\sum_{\substack{u \in \mathrm{supp}(\psi), \mathrm{height}(u) \le n \\ (\forall i \in \mathrm{var}(u)):\ u_i \in \mathrm{supp}(\psi_i), \\ \mathrm{height}(u_i) \le n \\ t = u[u_i]_{i \in \mathrm{var}(u)}}} (\psi, u) \cdot \prod_{i \in \mathrm{var}(u)} (\psi_i, u_i) \\
&= \quad [\text{by the condition: } \textstyle\sum_{u_i \in \mathrm{supp}(\psi_i)} (\psi_i, u_i)^b = 1] \\
&\quad \sum_{\substack{u \in \mathrm{supp}(\psi), \mathrm{height}(u) \le n \\ (\forall i \in \mathrm{var}(u)):\ u_i \in \mathrm{supp}(\psi_i), \\ \mathrm{height}(u_i) \le n \\ t = u[u_i]_{i \in \mathrm{var}(u)}}} (\psi, u) \cdot \prod_{i \in \mathrm{var}(u)} (\psi_i, u_i) \cdot \prod_{i \in I \setminus \mathrm{var}(u)} \Big(\sum_{u_i \in \mathrm{supp}(\psi_i)} (\psi_i, u_i)^b \Big)
\end{aligned}
$$

$$= \quad \text{[by axioms (GP) and (D)]}$$

$$\sum_{\substack{u \in \mathrm{supp}(\psi), \mathrm{height}(u) \le n \\ (\forall i \in \mathrm{var}(u))\colon u_i \in \mathrm{supp}(\psi_i), \mathrm{height}(u_i) \le n, \\ (\forall i \in I \setminus \mathrm{var}(u))\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in \mathrm{var}(u)}}} (\psi, u) \cdot \prod_{i \in \mathrm{var}(u)} (\psi_i, u_i) \cdot \prod_{i \in I \setminus \mathrm{var}(u)} (\psi_i, u_i)^b$$

$$= \quad \text{[by commutativity and because } u \text{ is linear in } \mathrm{Z}_I]$$

$$\sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i), \\ t = u[u_i]_{i \in I}}} (\psi, u) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)}$$

$$= \quad \text{[by definition of } \xleftarrow[\eta]{}]$$

$$(\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}, t)$$

Thus $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ is well-defined. In a similar manner we can prove that $\psi \xleftarrow[\eta]{} (\mathrm{bar}_i(\psi_i))_{i \in I}$ is well-defined. Now we return to the proof that M' recognizes $\psi \xleftarrow[\eta]{} (\mathrm{bar}_i(\psi_i))_{i \in I}$. Clearly, $(\|M'\|, u) = 0$ for every $u \in T_{\Delta'}$ such that $u \neq t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}$ for every $t \in T_\Delta(\mathrm{Z}_I)$ and $u_i \in T_\Delta$ for every $i \in \mathrm{var}(u)$.

$$\|M'\|$$

$$= \quad \text{[by the above observation]}$$

$$\sum_{\substack{t \in T_\Delta(\mathrm{Z}_I), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in T_\Delta}} \big(\|M'\|, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}\big)\, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}$$

$$= \quad \text{[by the auxiliary statement]}$$

$$\sum_{\substack{t \in T_\Delta(\mathrm{Z}_I), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in T_\Delta}} \Big((\psi, t) \cdot \prod_{i \in \mathrm{var}(t)} (\psi_i, u_i)\Big)\, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}$$

$$= \quad \text{[by axioms (U) and (GP) and definition of } \mathrm{bar}_i]$$

$$\sum_{\substack{t \in \mathrm{supp}(\psi), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in \mathrm{supp}(\mathrm{bar}_i(\psi_i))}} \Big((\psi, t) \cdot \prod_{i \in \mathrm{var}(t)} (\mathrm{bar}_i(\psi_i), u_i)\Big)\, t[u_i]_{i \in \mathrm{var}(t)}$$

$$= \quad \text{[by axiom (GP) and definition of } \xleftarrow[\eta]{} \text{ because } t \text{ is linear]}$$

$$\sum_{t \in \mathrm{supp}(\psi)} \Big(\big((\psi, t)\; t\big) \xleftarrow[\eta]{} \big(\mathrm{bar}_i(\psi_i)\big)_{i \in \mathrm{var}(t)}\Big)$$

$$= \quad \text{[by Proposition 3.6]}$$

$$\sum_{t \in \mathrm{supp}(\psi)} \Big(\big((\psi, t)\; t\big) \xleftarrow[\eta]{} \big(\mathrm{bar}_i(\psi_i)\big)_{i \in I}\Big)$$

$$= \quad \text{[by Proposition 3.1]}$$

$$\psi \xleftarrow[\eta]{} (\mathrm{bar}_i(\psi_i))_{i \in I}$$

Finally, we need to remove the annotation. We define the mapping $\mathrm{unbar}\colon T_{\Delta'}(\mathrm{Z}) \longrightarrow T_\Delta(\mathrm{Z})$ for every $z \in \mathrm{Z}$, $k \in \mathbb{N}$, $i \in I$, $\delta \in \Delta_k$, and $t_1, \ldots, t_k \in T_{\Delta'}(\mathrm{Z})$ by

$$\begin{aligned} \mathrm{unbar}(z) &= z \\ \mathrm{unbar}(\delta(t_1, \ldots, t_k)) &= \delta(\mathrm{unbar}(t_1), \ldots, \mathrm{unbar}(t_k)) \\ \mathrm{unbar}(\overline{\delta}^i(t_1, \ldots, t_k)) &= \delta(\mathrm{unbar}(t_1), \ldots, \mathrm{unbar}(t_k)) \ . \end{aligned}$$

We now lift unbar to tree series as follows. Let unbar: $\mathcal{A}\langle\langle T_{\Delta'}(\mathrm{X})\rangle\rangle \longrightarrow \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ be the mapping defined for every $\varphi \in \mathcal{A}\langle\langle T_{\Delta'}(\mathrm{Z})\rangle\rangle$ by

$$\mathrm{unbar}(\varphi) = \sum_{t \in T_{\Delta'}(\mathrm{Z})} (\varphi, t)\ \mathrm{unbar}(t) \ .$$

Clearly, $\mathrm{unbar}(\psi') = \psi \xleftarrow[\eta]{} (\psi_i)_{i\in I}$. Moreover, unbar can be realized by a nondeleting, linear tree transducer (with one state and with OI substitution) of [18] (because it is a relabeling homomorphism). Since ψ' is a recognizable tree series and nondeleting, linear tree transducers of [18] preserve recognizability [18, Corollary 14], also $\mathrm{unbar}(\psi')$ is recognizable, which proves the statement. The proof of [18, Corollary 14] additionally assumes a continuous semiring, but this assumption is not needed for the special case considered here. Alternatively the last step can be shown by referring to the closure of the class of recognizable tree series under linear and nondeleting tree homomorphisms [4]. □

Let us illustrate the previous theorem. The first example shows that the linearity restriction is necessary and the second example demonstrates a successful application of Theorem 4.2 using o-substitution.

Let $\Delta = \{\delta^{(2)}, \alpha^{(0)}\}$ and let

$$\psi_1 = \max_{u \in T_\Delta} \mathrm{height}(u)\, u \quad \text{and} \quad \psi = \max_{u \in T_\Delta(\mathrm{Z}_1)} \mathrm{height}(u)\, u$$

over the semiring $\mathbb{A}_\infty$, which is additively idempotent and continuous (see Proposition 3.9). However, ψ is not linear in Z_1. Nevertheless we apply the construction found in the proof of Theorem 4.2 (see Figure 1 for the bottom-up weighted tree automaton recognizing ψ) and obtain the bottom-up weighted tree automaton $M = (Q, \Gamma, \mathbb{A}_\infty, F, \mu)$ with

- $Q = \{1, 2, 3, 4\}$;
- $\Gamma_2 = \{\delta, \overline{\delta}\}$ and $\Gamma_0 = \{\alpha, \overline{\alpha}\}$ (we omit the 1 at the overlining);
- $F_1 = 0$ and $F_2 = F_3 = F_4 = -\infty$; and
- $\mu_0(\alpha)_1 = \mu_0(\alpha)_2 = \mu_0(\overline{\alpha})_1 = \mu_0(\overline{\alpha})_2 = \mu_0(\overline{\alpha})_3 = \mu_0(\overline{\alpha})_4 = 0$ and

  $$\mu_2(\delta)_{1,12} = \mu_2(\delta)_{1,21} = \mu_2(\overline{\delta})_{3,34} = \mu_2(\overline{\delta})_{3,43} = 1$$
  $$\mu_2(\delta)_{2,22} = \mu_2(\overline{\delta})_{4,44} = 0$$
  $$\mu_2(\overline{\delta})_{1,34} = \mu_2(\overline{\delta})_{1,43} = 1 \ .$$

  All remaining entries of $\mu_2(\delta)$ and $\mu_2(\overline{\delta})$ are $-\infty$.

The automaton M is displayed in Figure 2. However, M does not recognize $\psi \xleftarrow[\mathrm{o}]{} (\mathrm{bar}_1(\psi_1))$. To demonstrate this, let $u = \delta(\overline{\delta}(\overline{\alpha}, \overline{\alpha}), \overline{\delta}(\overline{\alpha}, \overline{\alpha}))$. Clearly, we observe that $(\|M\|, u) = -\infty$, but $(\psi \xleftarrow[\mathrm{o}]{} (\mathrm{bar}_1(\psi_1)), u) = 3$. The latter can be seen using the decomposition $u = \delta(\mathrm{z}_1, \mathrm{z}_1)[\overline{\delta}(\overline{\alpha}, \overline{\alpha})]$.

Corollary 4.3 (of Theorem 4.2 and Proposition 3.9) Let $\mathcal{A}$ be additively idempotent and continuous. Let $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Delta(\mathrm{Z}_I)\rangle\rangle$ be linear in Z_I, and let $\psi_i \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Delta\rangle\rangle$ for every $i \in I$. Then $\psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i\in I}$ is well-defined and recognizable.

Finally, we conclude the report by an application of Theorem 4.2 to show preservation of recognizability using pure substitution. In this section we consider the real number semiring $(\mathbb{R}_+, +, \cdot, 0, 1)$ with the set $\mathfrak{A}$ of absolutely convergent series and the standard infinitary summation $\sum : \mathfrak{A} \longrightarrow \mathbb{R}_+$. It is known [16] that this infinitary summation fulfils the presented axioms.

Let Δ be a ranked alphabet and $\psi \in \mathbb{R}_+\langle\langle T_\Delta\rangle\rangle$. Then the classical notion of a probability distribution coincides with Condition (ii) in Proposition 3.6. More formally, ψ is called a *probability distribution* if $\sum_{u \in T_\Delta} (\psi, u) = 1$ (*i. e.*, $(\psi, u)_{u \in T_\Delta}$ is an absolutely convergent series and its sum is 1). Theorem 4.2 shows that the recognizable linear probability distributions are closed under pure substitution.

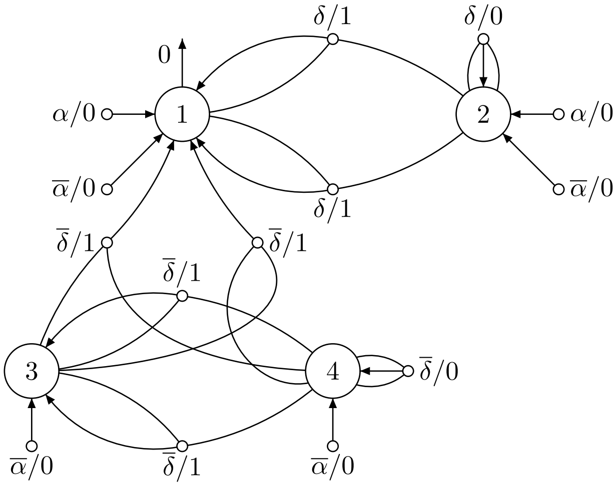

Figure 2: Bottom-up weighted tree automaton over $\mathbb{A}_\infty$.

Corollary 4.4 (of Theorem 4.2) Let $\psi \in (\mathbb{R}_+)^{\mathrm{rec}}\langle\langle T_\Delta(\mathrm{Z}_I)\rangle\rangle$ be a probability distribution that is linear in Z_I, and $\psi_i \in (\mathbb{R}_+)^{\mathrm{rec}}\langle\langle T_\Delta\rangle\rangle$ be a probability distribution for every $i \in I$. Then $\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}$ is a recognizable probability distribution.

Proof. By Theorem 4.2 we have that $\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}$ is well-defined and recognizable. It remains to prove that $\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}$ is a probability distribution.

$$
\begin{aligned}
& 1 \cdot \prod_{i\in I} 1 \\
= \quad & [\text{because } \psi \text{ and } \psi_i \text{ are probability distributions}] \\
& \Big(\sum_{u\in\mathrm{supp}(\psi)} (\psi, u)\Big) \cdot \prod_{i\in I}\Big(\sum_{u_i\in\mathrm{supp}(\psi_i)} (\psi_i, u_i)\Big) \\
= \quad & [\text{by axioms (GP) and (D)}] \\
& \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall i\in I):\, u_i\in\mathrm{supp}(\psi_i)}} (\psi, u) \cdot \prod_{i\in I} (\psi_i, u_i) \\
= \quad & [\text{by axiom (GP)}] \\
& \sum_{t\in T_\Delta}\Big(\sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall i\in I):\, u_i\in\mathrm{supp}(\psi_i),\\ t=u[u_i]_{i\in I}}} (\psi, u) \cdot \prod_{i\in I} (\psi_i, u_i)\Big) \\
= \quad & [\text{by axioms (U) and (GP) and definition of } \xleftarrow[\varepsilon]{}] \\
& \sum_{t\in\mathrm{supp}(\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I})} (\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}, t)
\end{aligned}
$$

□

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