Extracting semi-Dyck words from fsa using the CYK algorithm

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November 30, 2018

Outline

Motivation: Chomsky-Schützenberger parsing

- ChoSchü theorem [CS63]: decompose context-free language into
  - reg. language R
  - alph. string homomorphism h
  - semi-Dyck language D

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  - goal: extract semi-Dyck words from reg. language $R \cap h^{-1}(w)$

Motivation: existing algorithm to extract Dyck words [Hul11]

Require: finite state automaton $\mathcal{A} = (Q, \Sigma \cup \overline{\Sigma}, q_{\mathsf{init}}, q_{\mathsf{fin}}, T)$
Ensure: enumerate words in $\mathrm{L}(\mathcal{A}) \cap \mathrm{D}(\Sigma)$

procedure $\textsc{extractDyck}(\mathcal{A})$
  $A, C := \{v \mid (p, \sigma, q), (q, \overline{\sigma}, r) \in T\}, \emptyset$
  for $(p, v, q) \in A$ **do**
    $A \setminus= \{(p, v, q)\}; C \cup= \{(p, v, q)\}$
    if $(p, q) = (q_{\mathsf{init}}, q_{\mathsf{fin}})$ **then yield** v
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- dynamic programming: store intermediate results (backlinks) for state
- backlinks are equivalent to reduct grammar [BPS61]

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- $\mathcal{A} = (Q, \Sigma \cup \overline{\Sigma}, q_{\text{init}}, q_{\text{fin}}, T)$ is n -centered
  - surjective function $f\colon Q \to \{0, \ldots, n\}$:
    $$(p, \sigma, q) \in T \Rightarrow \begin{cases} f(p) = f(r) - 1 & \text{if } (q, \overline{\sigma}, r) \in T \\ f(p) = f(q) & \text{otherwise} \end{cases}$$
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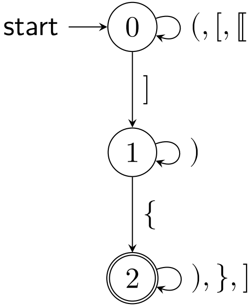

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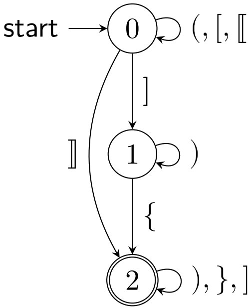

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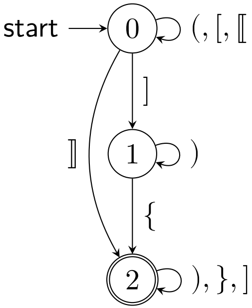
  - $\hat{n}$ smallest number s.t. $\mathcal{A}$ is $(\leq \hat{n})$-centered $\Rightarrow$ f is surjective



Closure properties

L is $(\leq \ell)$-centered, M is $(\leq m)$-centered reg. language over Σ, for $\ell, m \in \mathbb{N} \cup \{\infty\}$

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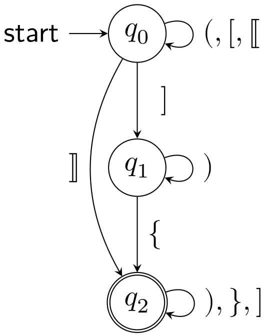
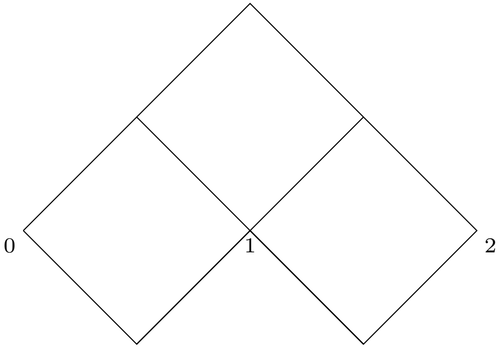
CYK algorithm for extraction of semi-Dyck words: example

- n-CYK algorithm applicable for $(\leq n)$-centered automata





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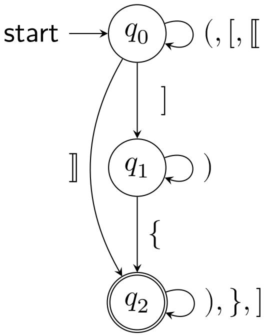

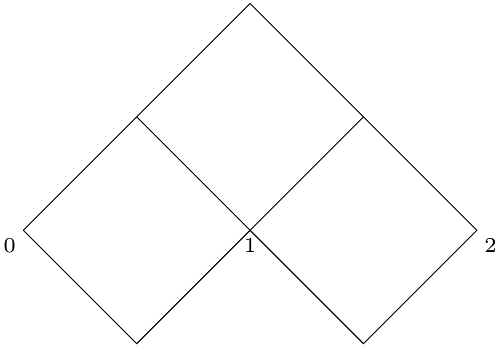

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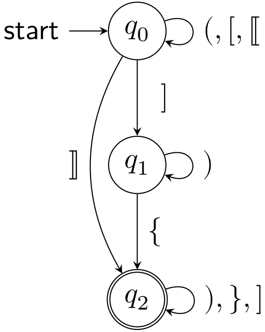

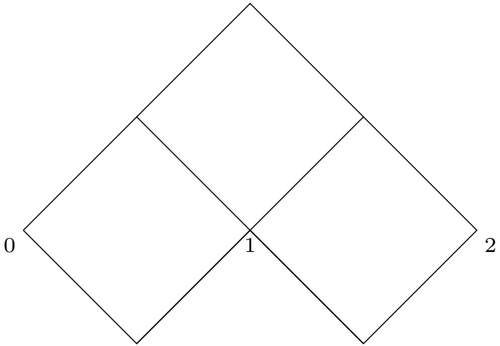

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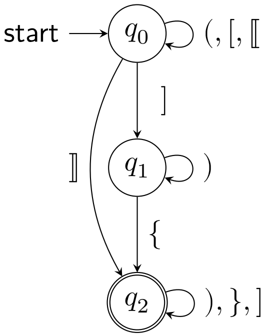

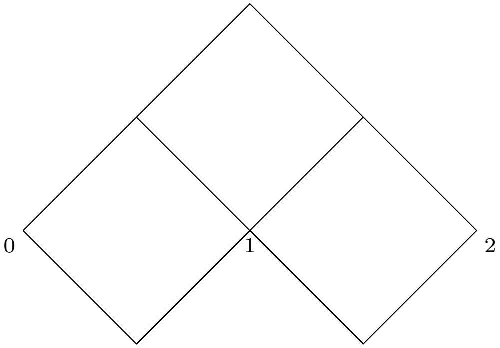

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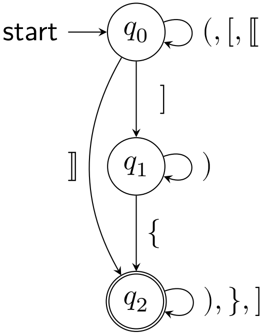

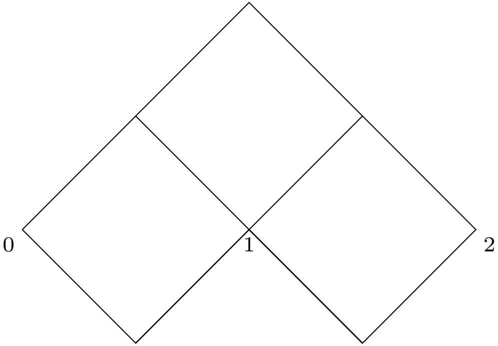

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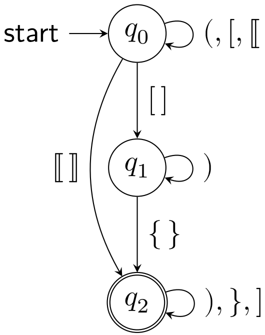

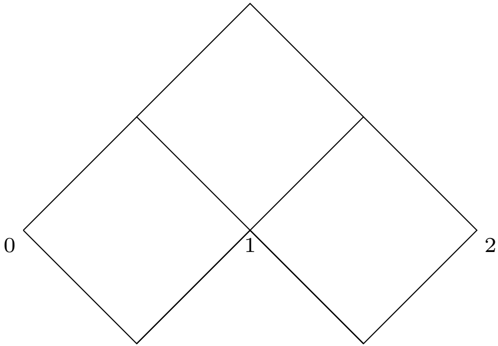

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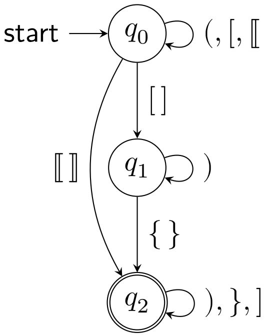

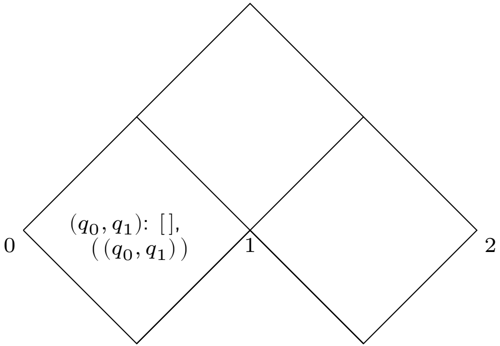

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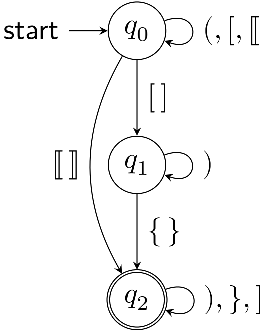

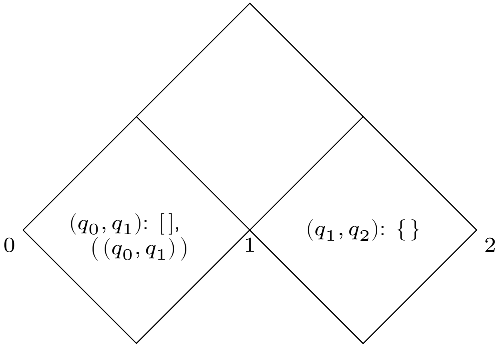

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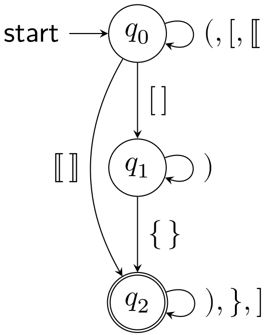

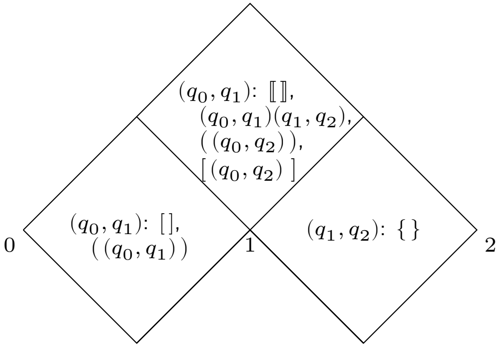

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Ensure: enumerates elements of $\mathcal{L}(\mathcal{A}) \cap D(\Sigma)$

procedure EXTRACTDYCK($\mathcal{A}$)
    $\mathcal{A}' :=$ NORMALFORM($\mathcal{A}$) ▷ combine transitions $(o, \sigma, p), (p, \overline{\sigma}, q)$ to $(o, \sigma\overline{\sigma}, q)$
    $C :=$ CYK($\mathcal{A}'$)
    ENUMERATE($C, q_{\text{init}}, q_{\text{fin}}$) ▷ c.f. Huang and Chiang [HC05]
function CYK($\mathcal{A}$)
    for $r \in \{1, \ldots, n\}$ **do**
        for $l \in \{0, \ldots, n-1\}$ **do**
            $S_{l,l+r} := \{(p, q) \mid (p, \sigma\overline{\sigma}, q) \in T, f(p) = l, f(q) = l + r\}$
            for $m \in \{1, \ldots, r-1\}$ **do**
                $S_{l,l+r} \cup= \{(o, q) \mid (o, p) \in S_{l,m}, (p, q) \in S_{m,l+r}\}$
            $S_{l,l+r} \cup= \bigcup_{(p,q) \in S_{l,l+r}} \mathrm{R}_{\mathcal{A}}(p, q)$ ▷ transitively reachable (o, r) via $(o, \sigma, p), (q, \overline{\sigma}, r) \in T$
    return $(S_{i,j} \mid i \in \{0, \ldots, n-1\}, j \in \{i+1, \ldots, n\})$

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  - parse multiple words at same time
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