Some notes on data structures used in search algorithms

Freitagsseminar

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Outline

1. Pushdowns
2. Log-domain
3. Double-ended priority queues

Pushdowns

- **Usage:** e.g. in pushdown automata; represent backlinks in graph search

[1] R. E. Tarjan (Apr. 1985). "Amortized Computational Complexity". *SIAM Journal on Algebraic Discrete Methods* 6.2, pp. 306–318. DOI: 10.1137/0606031

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- **Complexities:**

	growable array
pop, peek	$\mathcal{O}(1)$
push	$\mathcal{O}(1)$ (amortised[1])

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examples	vector (C++),	List (Haskell),
	Vec (Rust)	Pushdown (Rustomata)

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```haskell
data [a]
= []
| a : [a]
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- current element is always unique

```rust
pub fn pop(self) -> Result<(Self, A), Self> {
  match self {
    Pushdown::Empty              => Err(Pushdown::Empty            ),
    Pushdown::Cons{val,below} => Ok((below.deref().clone(), val)),
  }
}
```

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$$z = \ln\left(\exp(x) + \exp(y)\right) \qquad \text{(3 transcendentals)}$$
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&= \ln\left(\exp(x) + \exp(x) \cdot \frac{\exp(y)}{\exp(x)}\right) \\
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\end{aligned}$$

- implementation:

```
173 LogDomain(x + (y - x).exp().ln_1p())
```

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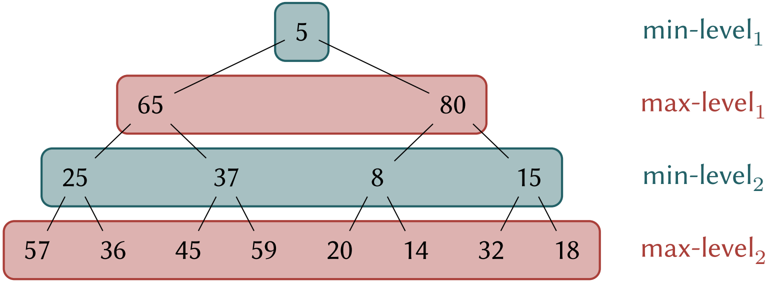
- **Idea:** shuffle a max-heap into a min-heap

Double-ended priority queues – min-max-heap

- data structure:



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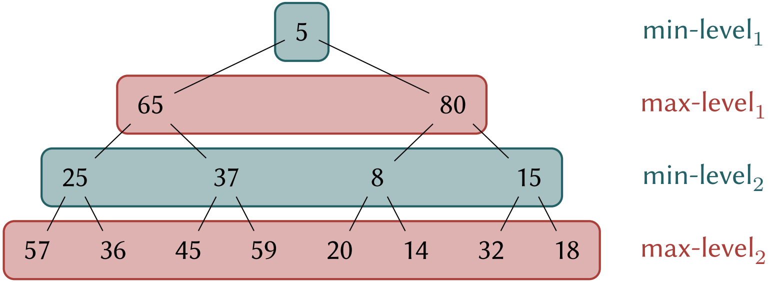
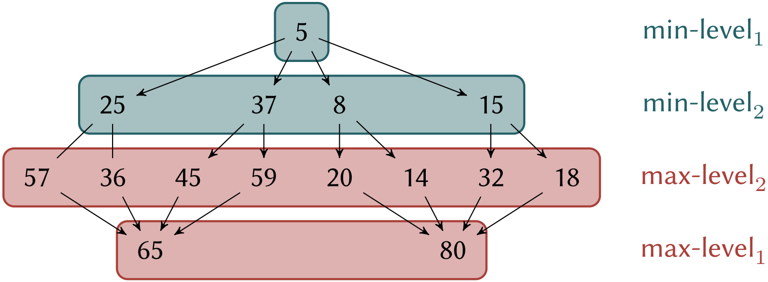


- Hasse diagram:



References

M. D. Atkinson, J.-R. Sack, B. Santoro, and T. Strothotte (1986). "Min-max heaps and generalized priority queues". *Communications of the ACM*. DOI: 10.1145/6617.6621.

D. H. Larkin, S. Sen, and R. E. Tarjan (2013). "A Back-to-Basics Empirical Study of Priority Queues". DOI: 10.1137/1.9781611973198.7.

Professur GDP (2018a). *Log-domain in Rust (referenced version).* URL: https://github.com/tud-fop/rust-log-domain/blob/9568c5b7db6992fc11151829f236bc91337b182a/src/lib.rs.

Professur GDP (2018b). *Pushdowns in rustomata (referenced version).* URL: https://github.com/tud-fop/rustomata/blob/d8cb2798688ea1345735b08236441097e0757788/src/util/push_down.rs.

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