Underlying Principles and Recurring Ideas of Formal Grammars

after a paper by Alexander Okhotin*

talk by Tobias Denkinger

Freitagsseminar 2018-04-20

* [Okhotin 18]

Overview

1. Grammar types
2. Parse trees
3. Language equations
4. Parsing
5. Characterisation results

Grammar types I

- ordinary grammar (aka context-free grammar) [Chomsky 56]

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characterised by 3 elements:

- constituents (CON) — here: strings
- operations on constituents (C-OP) — here: concatenation $\cdot$
- logical operations (L-OP) — here: disjunction $+$

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- Boolean grammar [Okhotin 04, 13]: L-OP: +, &, negation

Grammar types III

- M-component grammars [Vijay-Shanker+87, Seki+91]

  $B_0(\alpha_{1,1}, \ldots, \alpha_{m_0}) \leftarrow B_1(x_{1,1}, \ldots, x_{1,m_1}) \cdots B_k(x_{k,1}, \ldots, x_{k,m_k})$

  CON: ($\leq M$)-tuples of strings $[m_i \leq M]$

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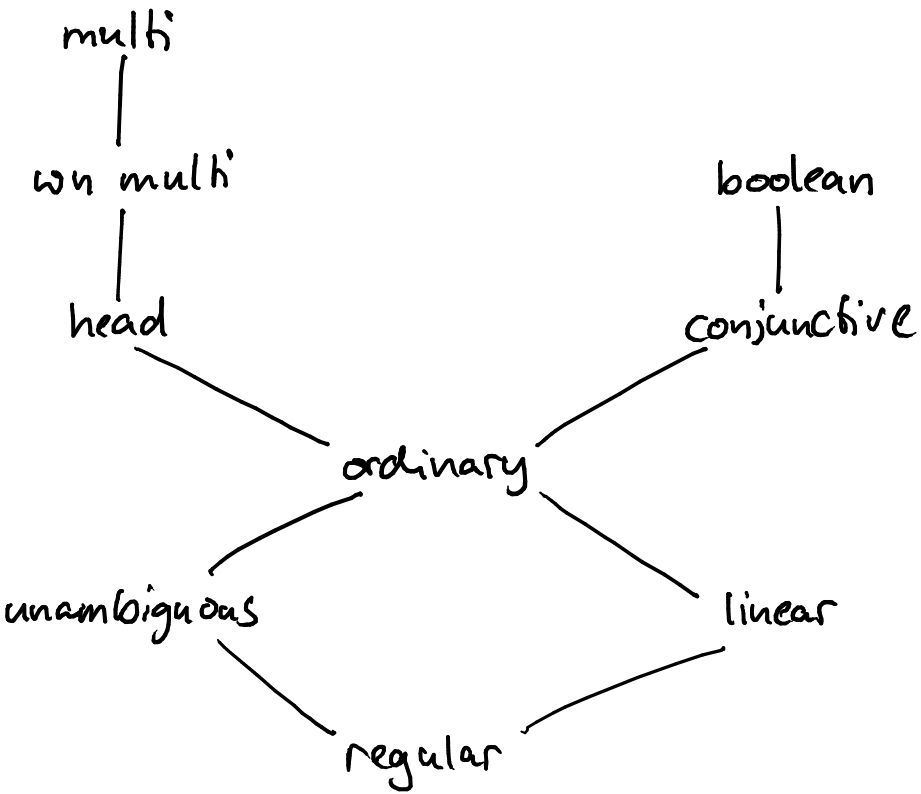
- head grammars = well-nested 2-component grammars

Hierarchy of grammar formalisms (by inclusion)



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$$\dfrac{\dfrac{\dfrac{}{A(\varepsilon)}}{A(a)} \quad \dfrac{\dfrac{}{B(\varepsilon)}}{B(bc)} \quad \dfrac{\dfrac{}{D(\varepsilon)}}{D(ab)} \quad \dfrac{\dfrac{}{C(\varepsilon)}}{C(c)}}{S(abc)}\,(S \to AB \,\&\, DC)$$

Parse trees II

- multi-component

$$\frac{A(t,u) \qquad B(v,w)}{S(tvuw)} \left(S(x_1 y_1 x_2 y_2) \rightarrow A(x_1, x_2)\ B(y_1, y_2)\right)$$

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$$\dfrac{\dfrac{\dfrac{}{A(\varepsilon,\varepsilon)}}{A(a,c)} \qquad \dfrac{\dfrac{}{B(\varepsilon,\varepsilon)}}{B(b,d)}}{\dfrac{A(aa,cc) \qquad B(b,d)}{S(aabccd)}\ (S(x_1y_1x_2y_2) \rightarrow \ldots)}$$

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- conjunctive/Boolean grammars

→ & becomes $\cap$, $\neg S$ becomes $\Sigma^* \setminus S$

Parsing

- use dynamic programming algorithm (CKY-like)

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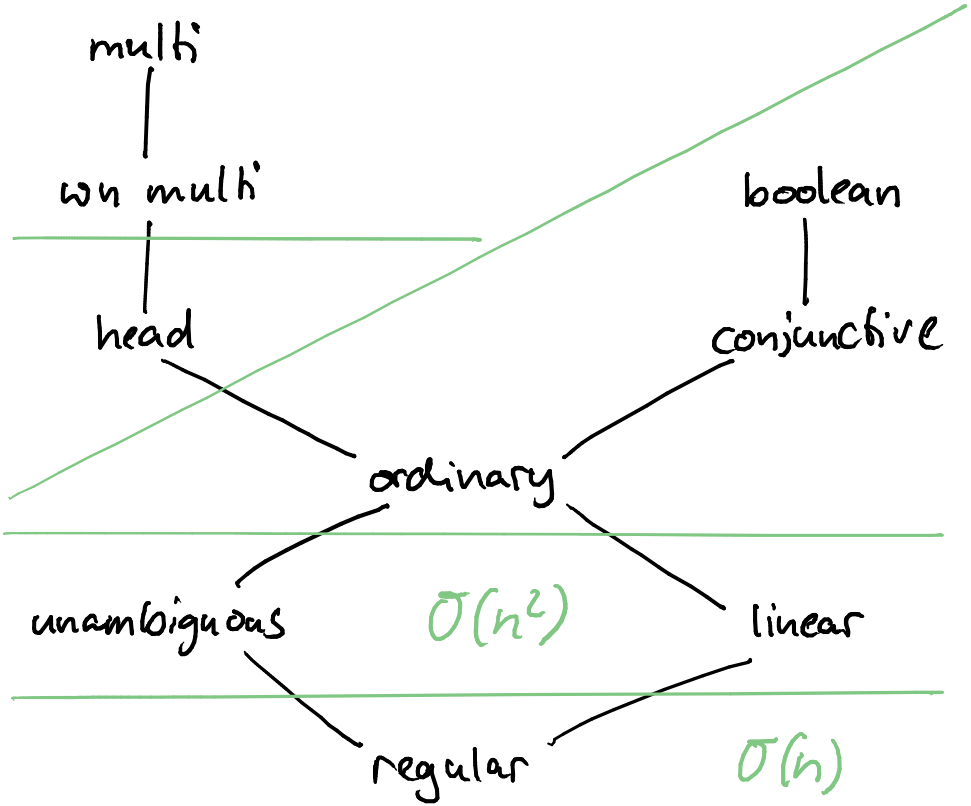

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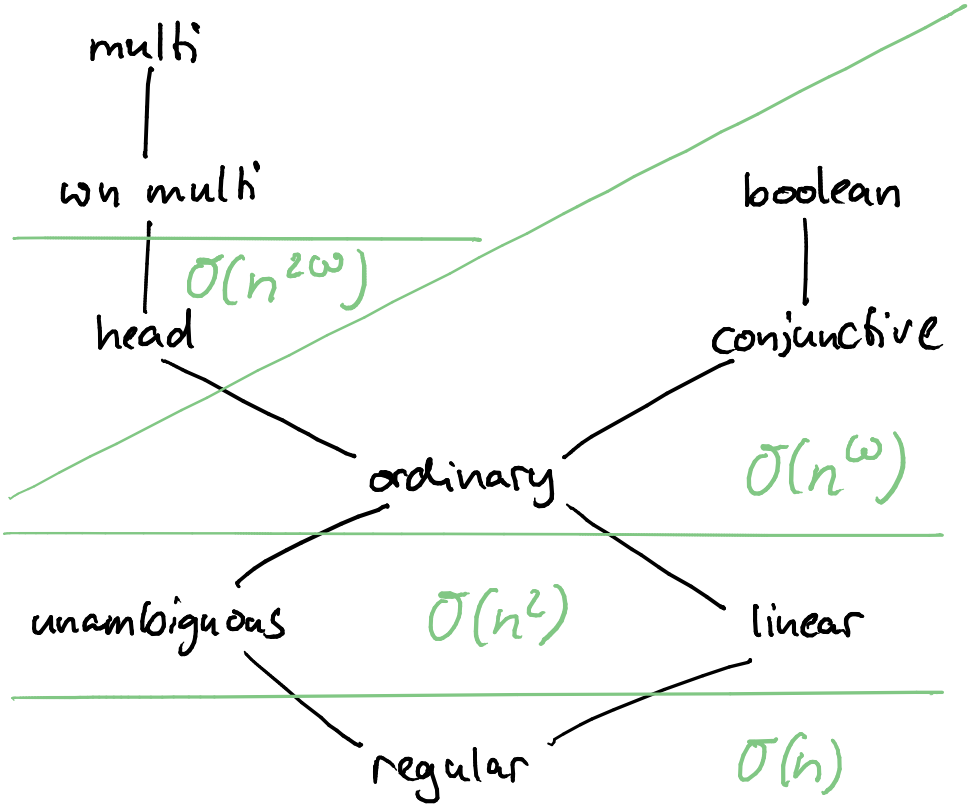

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- two $n \times n$ matrices can be multiplied in time $O(n^{\omega})$ [Valiant 75]

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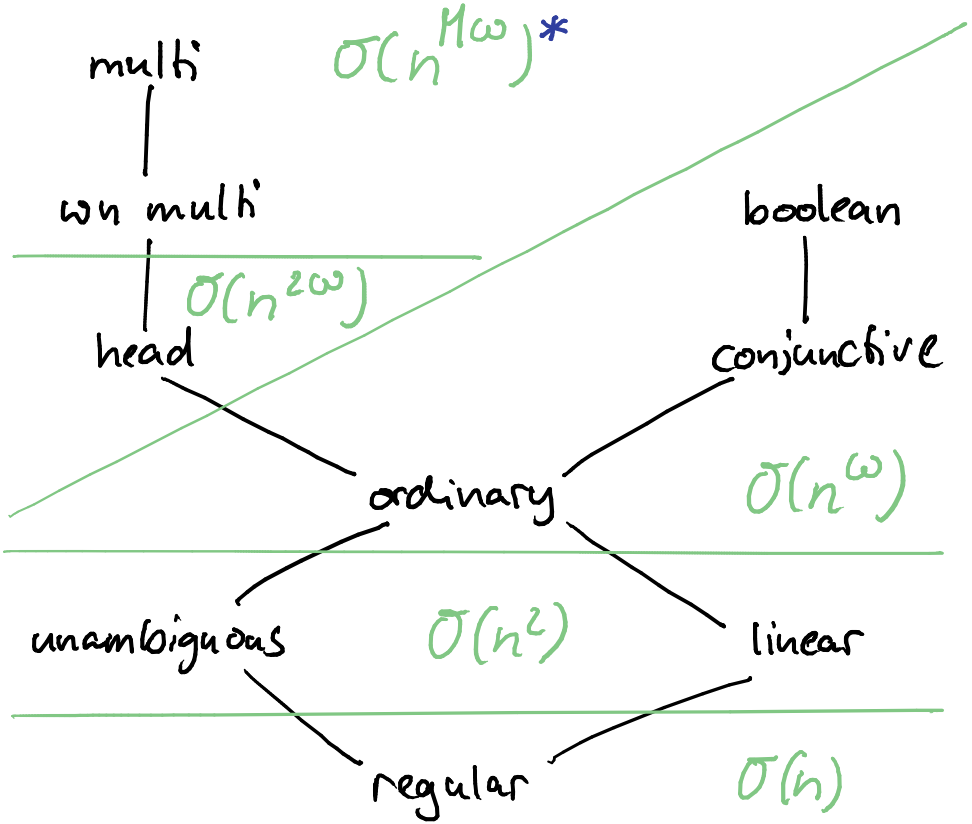

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- homomorphic characterisation (Chomsky-Schützenberger results)

  find $\mathcal{Z}$ s.t. $\forall G\ \exists h, R, D$ with

  $L(G) = h(R \cap D)$ $\quad$ $R \in REG$, $h \in HOM$,

  D is a $\mathcal{Z}$-Dyck language

References I (in order of appearance)

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