# Master Defense: Implementation and evaluation of $k$-best Chomsky-Schützenberger parsing for weighted multiple context-free grammars

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# CS parsing for context-free grammars [Hul09]

**Theorem (Chomsky-Schützenberger theorem for context-free languages [CS63])**

*Let $L$ be a language. T.f.a.e.*

1. *$L$ is a context-free language, and*
2. *there are a regular language $R$, a Dyck language $D$ and a string homomorphism $h$ such that $L = h(R \cap D)$.*

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**Input:** Grammar $G$, word $w$
**Output:** sequence $d_1 d_2 \ldots$ of ASTs for $w$ in $G$

construct $R$, $D$, $h$ with respect to $[\![G]\!]$
$R^w \leftarrow h^{-1}(w)$
$R^{local} \leftarrow R^w \cap R$
**return** $\big(\mathrm{enumerate} \; ; \; \mathrm{filter}(D) \; ; \; \mathrm{map}(\mathrm{toderiv})\big)(R^{local})$

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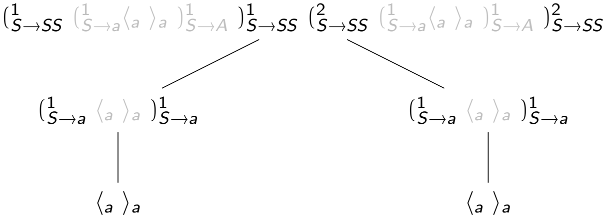
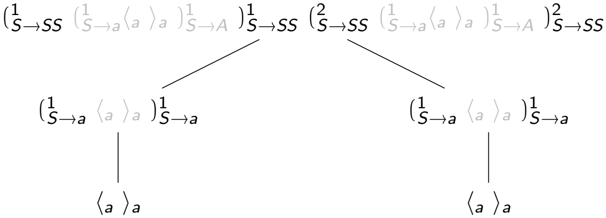
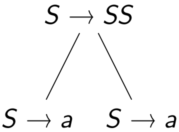


# CS parsing for context-free grammars [Hul09]: example

- example grammar $G$: $S \to SS \mid a$
- Dyck words are well-bracketed
- structure of parenthesis hierarchy relates to abstract syntax tree



# outline

weighted multiple context-free grammars

CS parsing for WMCFG

implementation

evaluation results

conclusions and future work

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# weighted multiple context-free grammars

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$$A \longrightarrow [ax_1^1, cx_1^2](A)$$

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- rules of the form

$$A \to [ax_1^1, cx_1^2](A)$$

- "CFG with tuples"
- composition is linear
- for each (weighted) MCFG, there is an equivalent (weighted) non-deleting MCFG [Sek+91; Den16]

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**Definition ($k$-best parsing problem for WMCFG)**

**Input** $W$-weighted MCFG $(G, \mu)$, word $w$ and total order $\trianglelefteq$ on $W$

**Output** sequence $d_1 \ldots d_k$ of ASTs for $w$ in $G$ s.t. $\mu(d_1) \trianglelefteq \ldots \trianglelefteq \mu(d_k)$ and $\nexists d \colon \mu(d_k) \ntrianglelefteq \mu(d)$

# CS theorem for WMCFG

**Theorem (CS theorem for CFL [CS63])**

*Let $L$ be a language. T.f.a.e.*

1. *$L$ is a CFL, and*
2. *there are a regular language $R$, a Dyck language $D$ and a string homomorphism $h$ such that $L = h(R \cap D)$.*

# CS theorem for WMCFG

**Theorem (CS theorem for weighted CFL [DV14])**

*Let $L$ be a language. T.f.a.e.*

1. *$L$ is a weighted CFL, and*
2. *there are a regular language $R$, a Dyck language $D$ and a weighted string homomorphism $h$ such that $L = h(R \cap D)$.*

# CS theorem for WMCFG

**Theorem (CS theorem for weighted multiple CFL [Den16])**

*Let $L$ be a language. T.f.a.e.*

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2. *there are a regular language $R$, a multiple Dyck language $D$ and a weighted string homomorphism $h$ such that $L = h(R \cap D)$.*

# $k$-best CS parsing algorithm for WMCFG

**Input:** $W$-weighted MCFG $(G, \mu)$, $w$, total order $\trianglelefteq$ on $W$, $k \in \mathbb{N}$
**Output:** sequence $d_1 \ldots d_k$ of ASTs for $w$ in $G$ s.t. $\mu(d_1) \trianglelefteq \ldots \trianglelefteq \mu(d_k)$ and $\nexists d\colon \mu(d_k) \ntrianglelefteq \mu(d)$

1: construct $\Delta$, $R \subseteq \Delta^*$, $D \subseteq \Delta^*$, $h\colon \Delta^* \to (\Sigma^* \to W)$ w.r.t. $[\![G, \mu]\!]$
2: $R^w \leftarrow \{v \in \Delta^* \mid h(v)(w) \neq 0\}$
3: $R^{local}\colon R \cap R^w \to W$ such that $R^{local}(v) = h(v)(w)$
4: **return** $\big(\mathrm{ordered}_{\trianglelefteq}\ ;\ \mathrm{filter}(D)\ ;\ \mathrm{map}(\mathrm{toderiv})\ ;\ \mathrm{take}(k)\big)(R^{local})$

# the regular language $R$

$$\rho_1 = S \to \langle \quad x_1^1 \quad x_2^1 \quad x_1^2 \quad x_2^2 \quad \rangle(A, B)$$

# the regular language $R$

$$\rho_1 = S \to \langle \quad [\![^1_{\rho_1} (^1_{\rho_1,1} \quad x^1_1 \quad )^1_{\rho_1,1} (^1_{\rho_1,2} \quad x^1_2 \quad )^1_{\rho_1,2} (^2_{\rho_1,1} \quad x^2_1 \quad )^2_{\rho_1,1} (^2_{\rho_1,2} \quad x^2_2 \quad )^2_{\rho_1,2} ]\!]^1_{\rho_1} \quad \rangle(A, B)$$

# the regular language $R$

$$S^1 \quad \overline{A^1} \quad \overline{B^1} \quad \overline{A^2} \quad \overline{B^2}$$

$$\rho_1 = S \to \langle \; \llbracket^1_{\rho_1} (^1_{\rho_1,1} \; x^1_1 \; )^1_{\rho_1,1} (^1_{\rho_1,2} \; x^1_2 \; )^1_{\rho_1,2} (^2_{\rho_1,1} \; x^2_1 \; )^2_{\rho_1,1} (^2_{\rho_1,2} \; x^2_2 \; )^2_{\rho_1,2} \rrbracket^1_{\rho_1} \; \rangle (A, B)$$

$$A^1 \quad B^1 \quad A^2 \quad B^2 \quad \overline{S^1}$$

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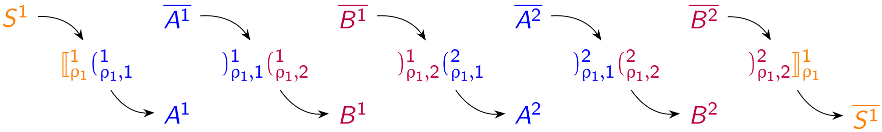

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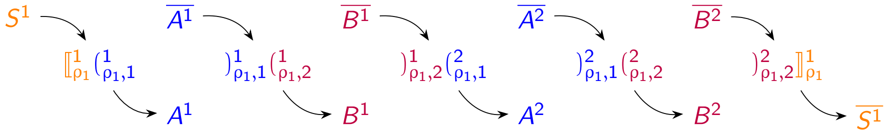
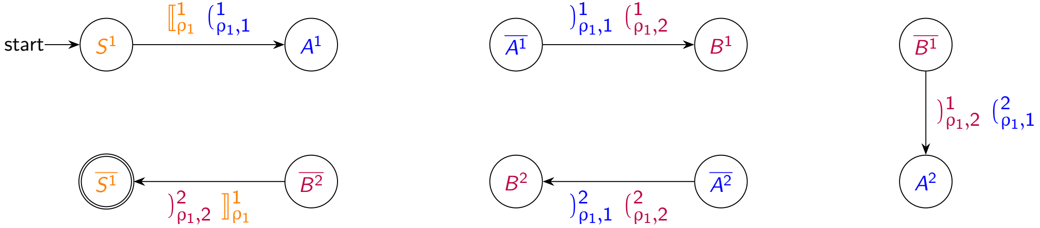
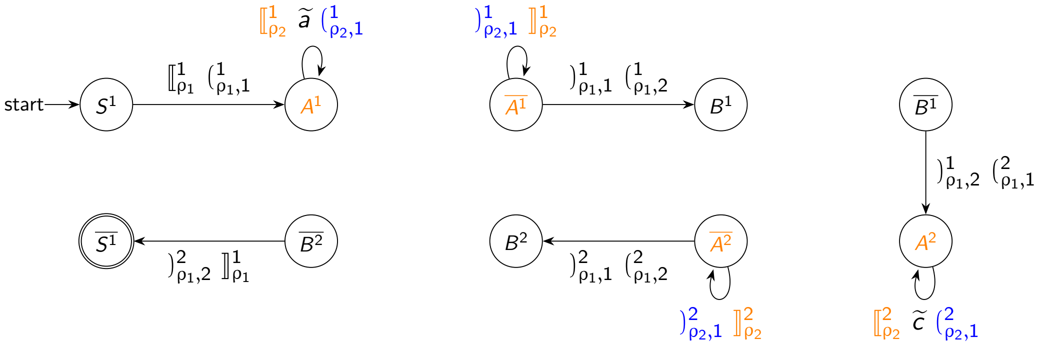


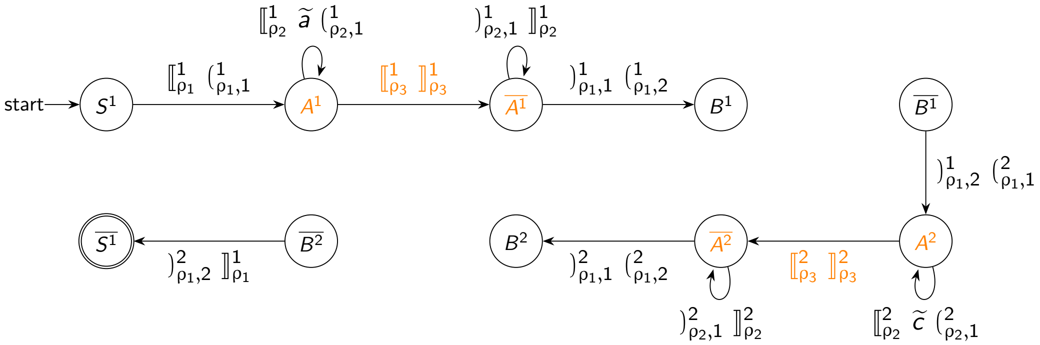
# the regular language $R$

$$\rho_2 = A \rightarrow \langle a x_1^1, c x_1^2 \rangle (A)$$



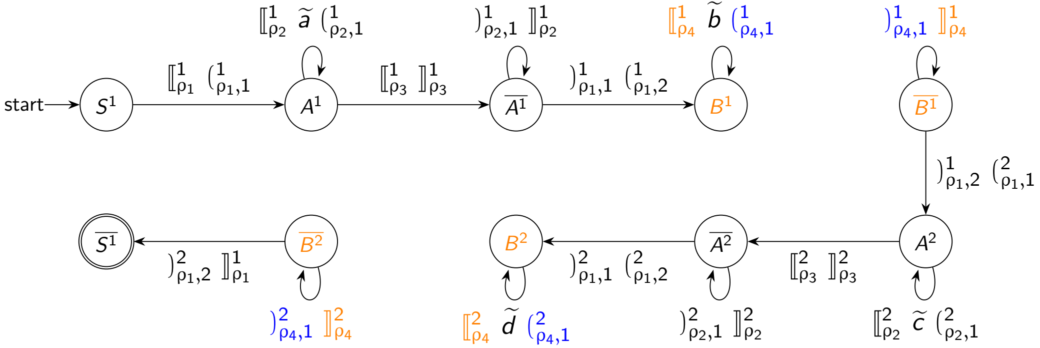
# the regular language $R$

$$\rho_3 = A \rightarrow \langle \varepsilon, \varepsilon \rangle ()$$



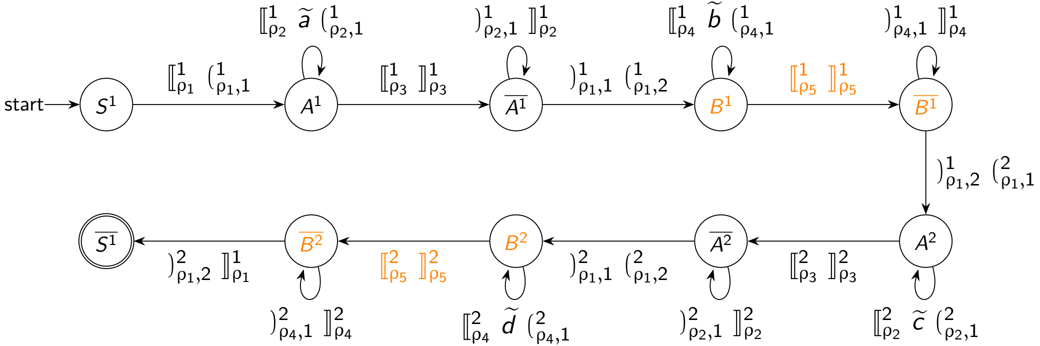
# the regular language $R$

$$\rho_4 = B \to \langle b x_1^1, d x_1^2 \rangle (B)$$



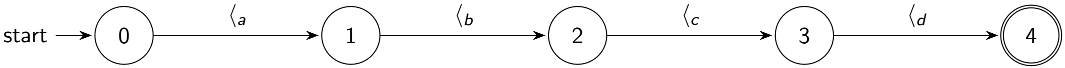
# the regular language $R$

$$\rho_5 = B \to \langle \varepsilon, \varepsilon \rangle ()$$



# the regular language $R^{abcd}$

- accept bracket words $v$ with $h(v)(abcd) \neq 0$



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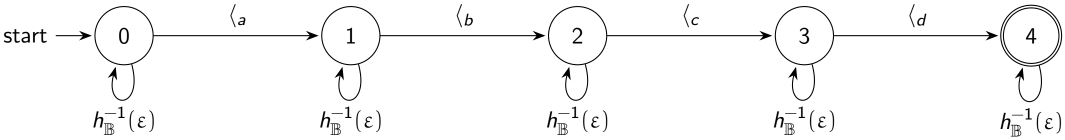

# the regular language $R^{abcd}$

- accept bracket words $v$ with $h(v)(abcd) \neq 0$
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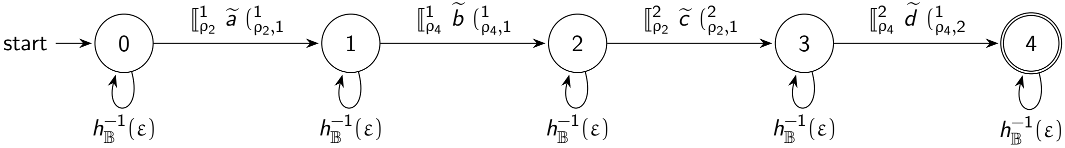
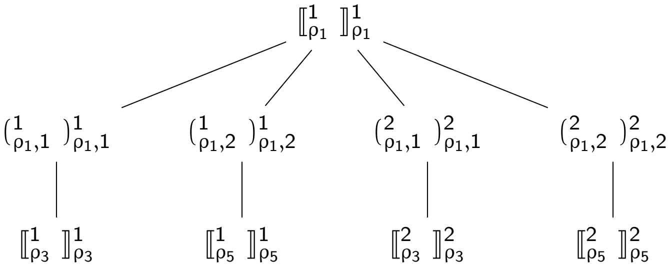
- we know context of from $R$



# the multiple Dyck language $D$: cancelation rule

- MDL over $\Delta$ is characterized by a partition $\mathfrak{P}$



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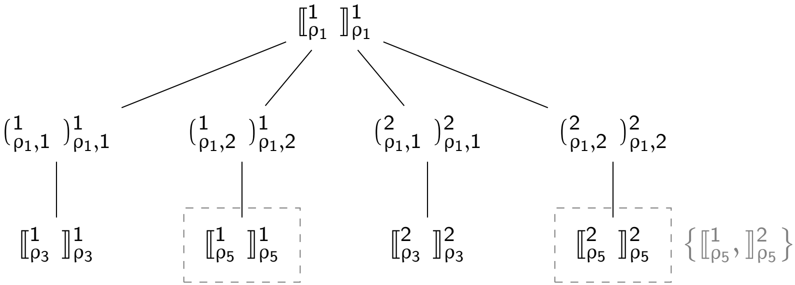

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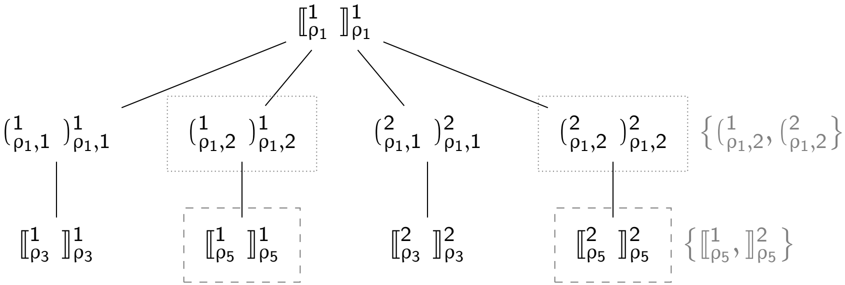

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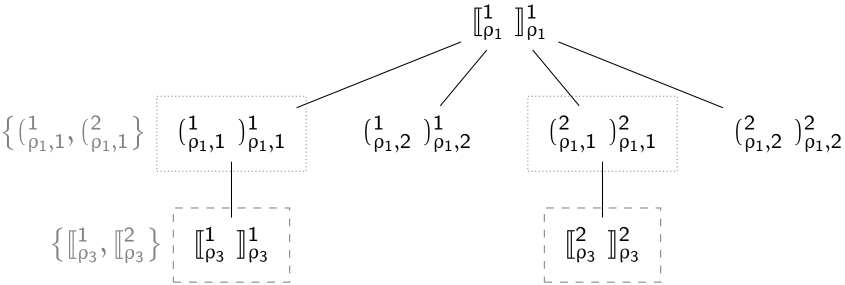

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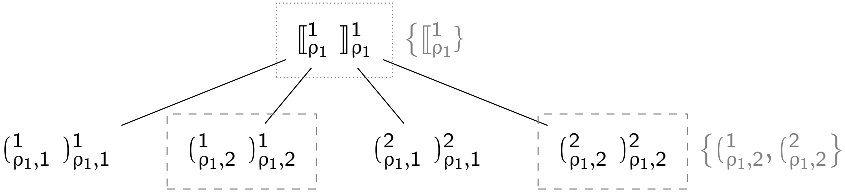

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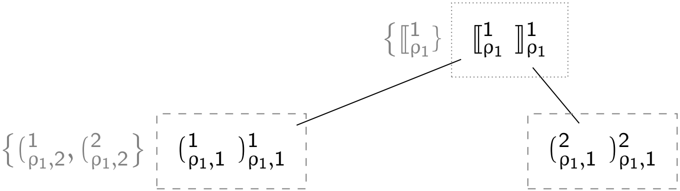

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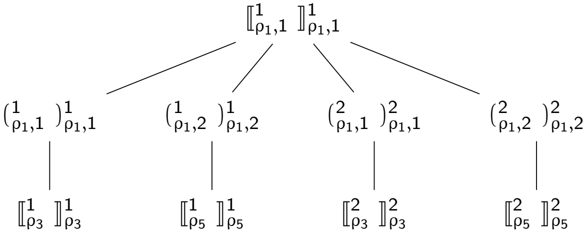
$$\llbracket^1_{\rho_1} \; \rrbracket^1_{\rho_1} \quad \{\llbracket^1_{\rho_1}\}$$

# the multiple Dyck language $D$: compact parenthesis hierarchy

- combine nodes w/ parentheses of same group



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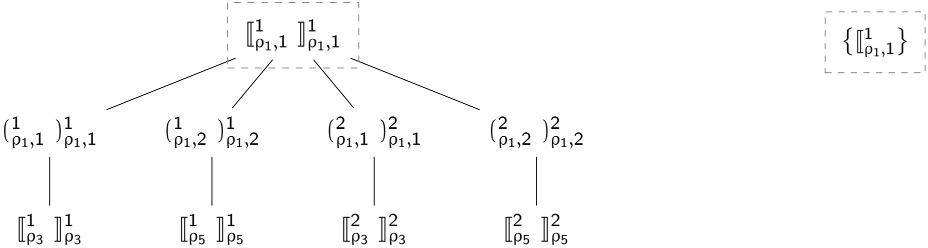

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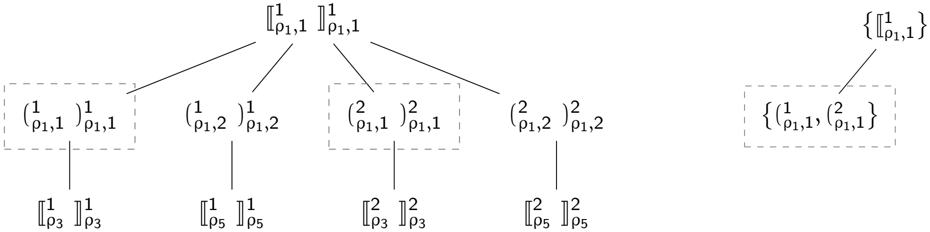

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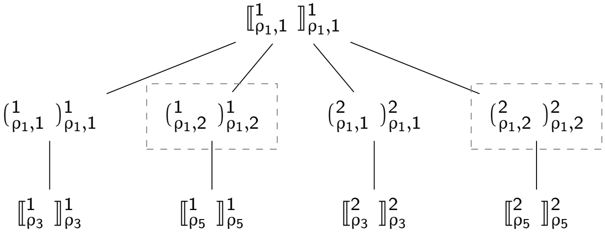


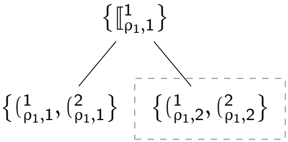

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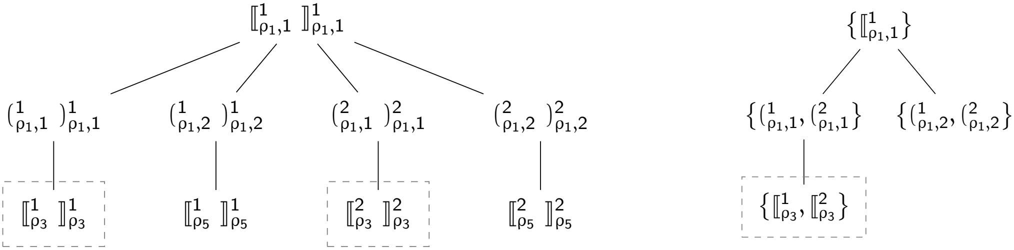

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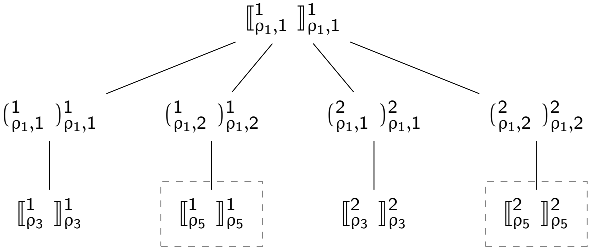


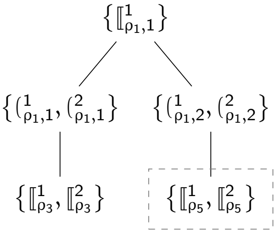

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- combine nodes w/ parentheses of same group
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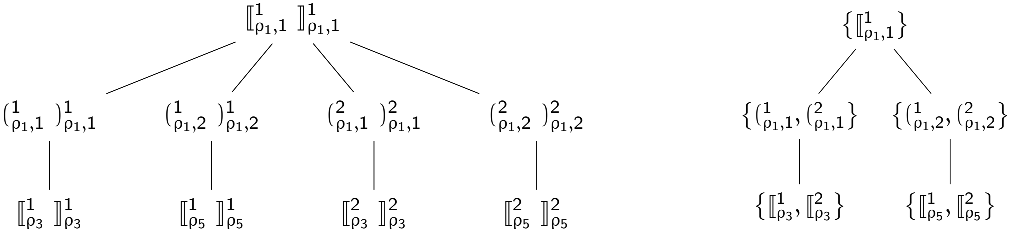

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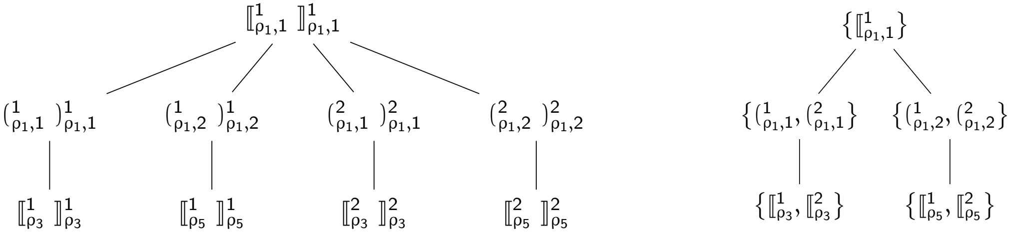

# the multiple Dyck language $D$: compact parenthesis hierarchy

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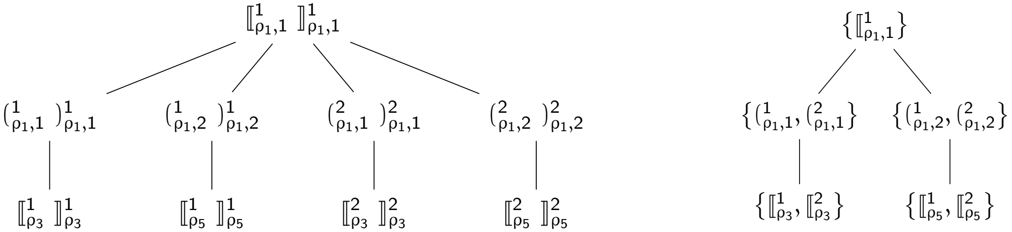
- structure relates to recognition w/ TSA
- observation: special structure of words in $R \cap D$



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  - conversion from MCFG to non-deleting MCFG

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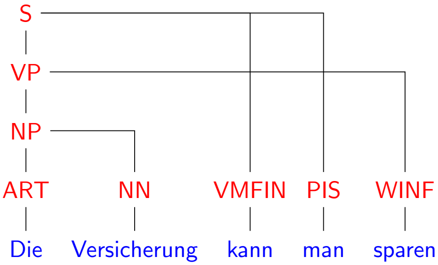


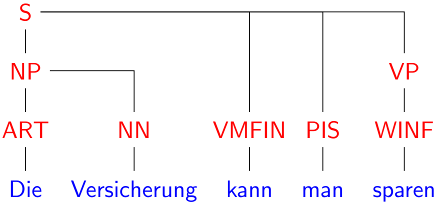

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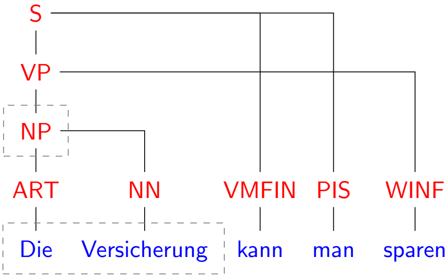


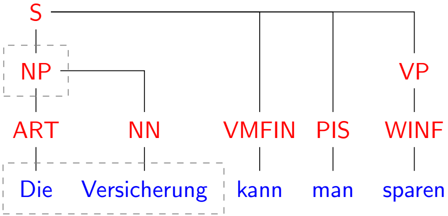


*TP*: I *FP*: *FN*:

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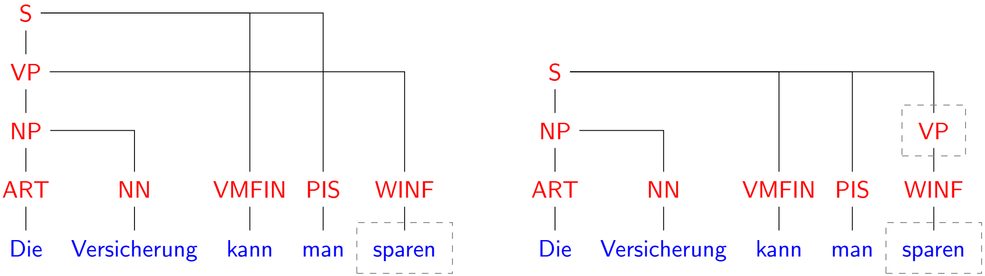


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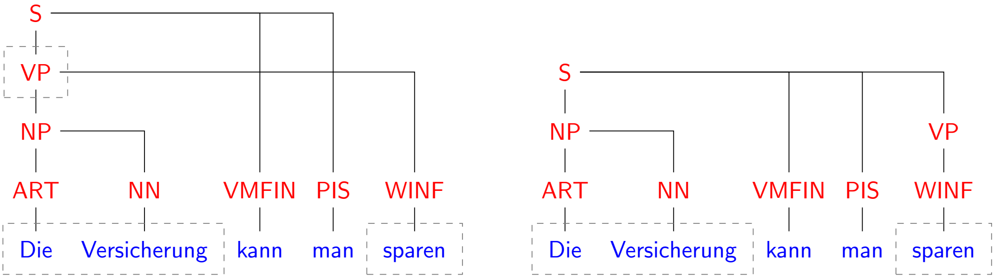


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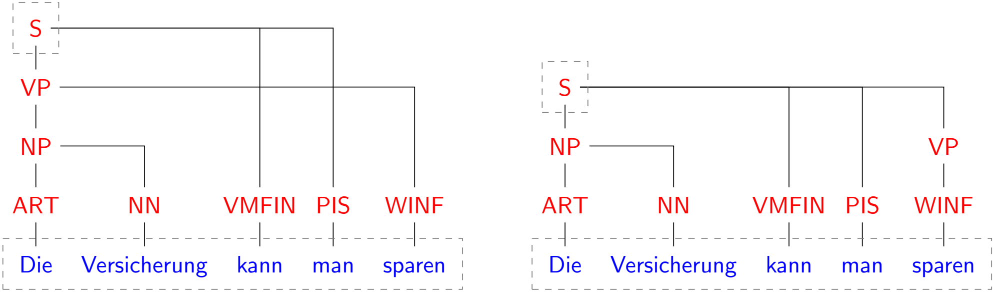


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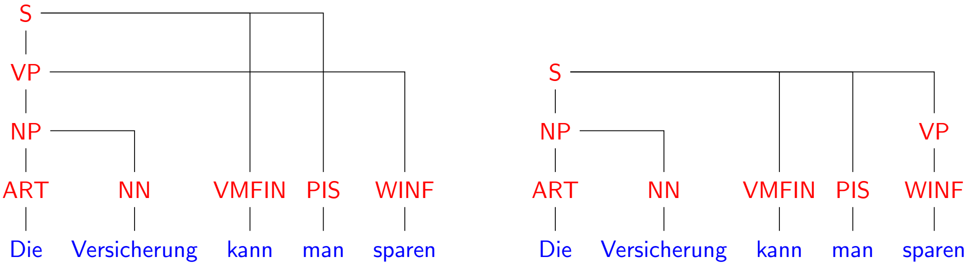
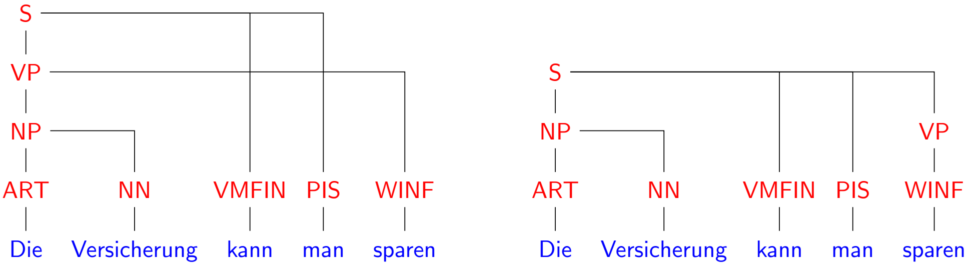



*TP*: I I *FP*: I *FN*: I

$$precision = \frac{TP}{TP+FP} = \frac{2}{3} \qquad recall = \frac{TP}{TP+FN} = \frac{2}{3}$$

# qualitative evaluation metric: parseval [Bla+91; Col97]

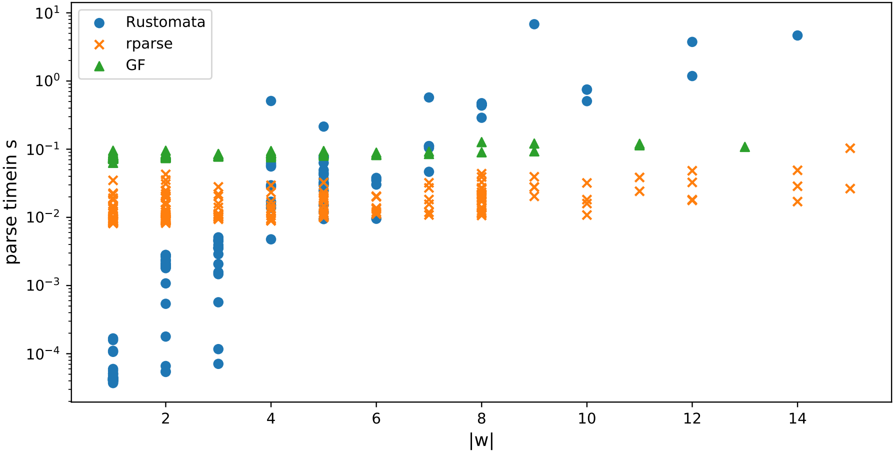
- match *constituents by span*, report precision, recall and f score
- accuracy of predicted *pos labels*



# results: quality

| parser | Rustomata | rparse | Grammatical Framework |
|---|---|---|---|
| accuracy of POS-tags | 0.64 | 0.94 | 0.44 |
| number of predicted constituents | 306 | 416 | 239 |
| labeled precision for all constituents | 0.96 | 0.86 | 0.94 |
| labeled recall for all constituents | 0.70 | 0.86 | 0.62 |
| labeled f-score for all constituents | 0.81 | 0.86 | 0.75 |

# results: parse time



# outline

weighted multiple context-free grammars

CS parsing for WMCFG

implementation

evaluation results

conclusions and future work

# conclusions and future work

- implementation works with natural language corpora

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- resolved some performance issues by more restrictive definitions
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  - filter language with minimal set of parenthesis
  - sorted multiple Dyck languages

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- compared to rparse and Grammatical Framework, our parser
  - has comparable accuracy, but
  - inferior parse time
- possible future work
  - further investigate search items in the generative approach
  - find efficient automata frameworks for FSA and PDA or improve implementation
  - $R \cap D$ by product construction of automata

# references


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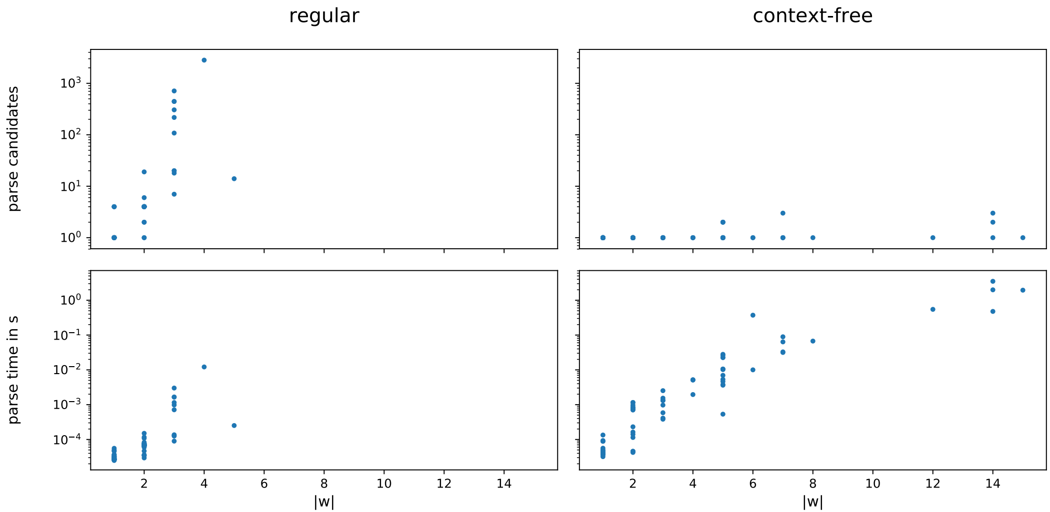
# BACKUP

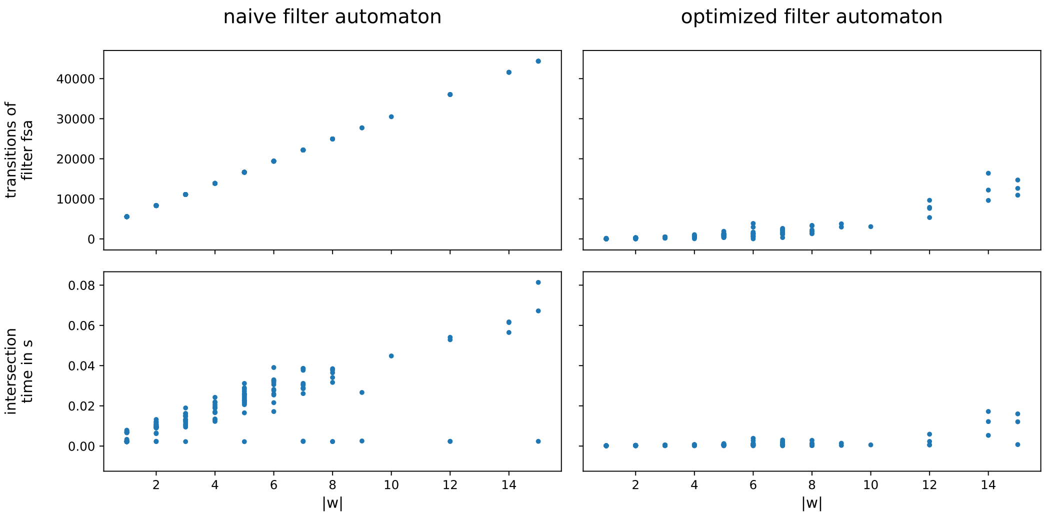
# factorizable bimonoids

- Viterbi: $([0,1], \max, \cdot, 0, 1)$, where $\sqrt[n]{w} \cdot \ldots \cdot \sqrt[n]{w}$ is a $(\geq, n)$-factorization of $w \in [0,1]$
- tropic: $(R^{\infty}_{\geq 0}, \min, +, \infty, 0)$, where $\frac{w}{n} + \ldots + \frac{w}{n}$ is a $(\leq, n)$-factorization of $w \in R^{\infty}_{\geq 0}$
- boolean: $(\{\mathrm{true}, \mathrm{false}\}, \vee, \wedge, \mathrm{false}, \mathrm{true})$, where $\mathrm{true} \cdot \ldots \cdot \mathrm{true}$ is a $(\trianglelefteq, n)$-factorization of $\mathrm{true}$ and $\mathrm{false} \cdot \ldots \cdot \mathrm{false}$ is a $(\trianglelefteq, n)$-factorization of $\mathrm{false}$

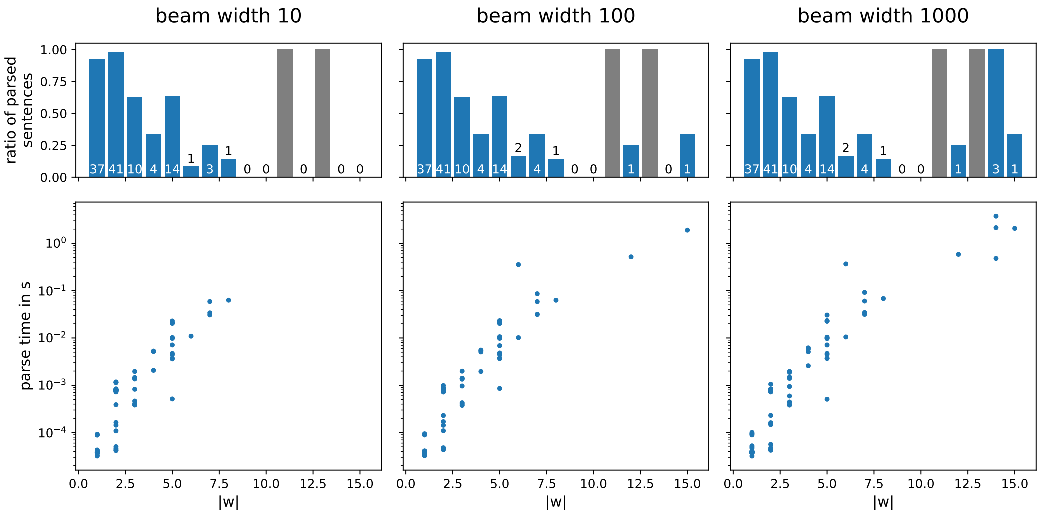
# results: optimized generator language $R$



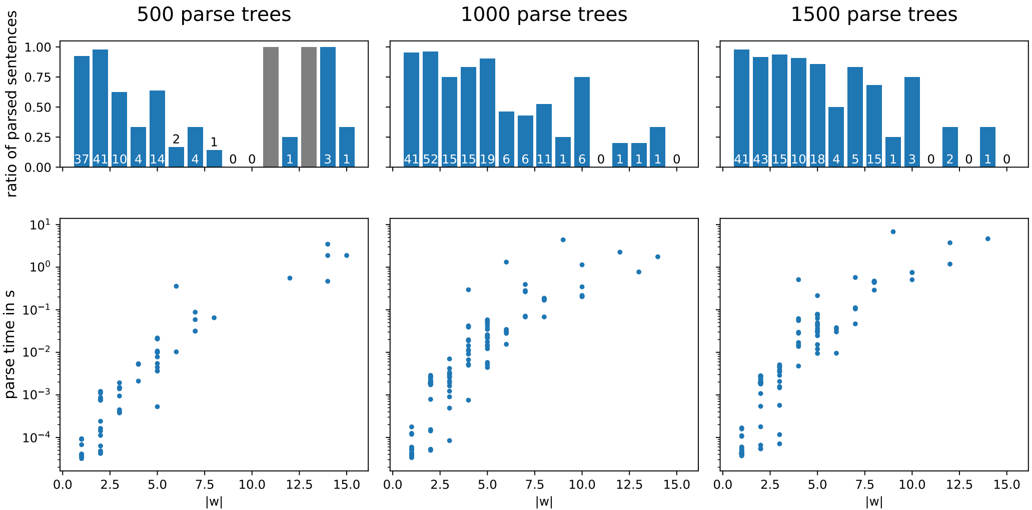
# results: optimized filter language $R^w$



# results: beam search

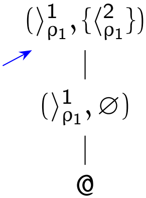


# results: grammar size



# recognizing $D$ using a TSA

- tree stack instructions
  - $\mathrm{up}_{\mathfrak{P}}(\delta)$
  - $\mathrm{down}(\delta)$



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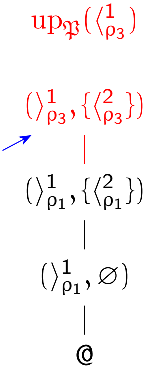

# recognizing $D$ using a TSA

- tree stack instructions
  - $\mathrm{up}_{\mathfrak{P}}(\delta)$
  - $\mathrm{down}(\delta)$

$\mathrm{down}(\rangle^{1}_{\rho_3})$

$(-, \{\langle^{2}_{\rho_3}\})$

$(\rangle^{1}_{\rho_1}, \{\langle^{2}_{\rho_1}\})$

$(\rangle^{1}_{\rho_1}, \varnothing)$

@

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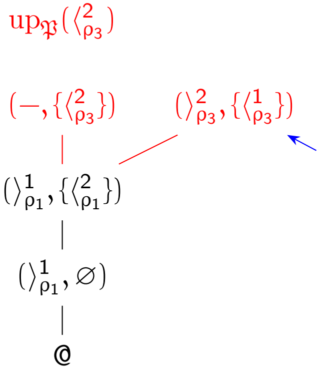

# recognizing $D$ using a TSA

- tree stack instructions
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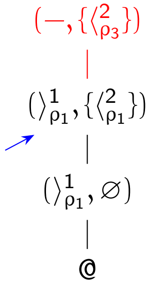
  - $\mathrm{down}(\delta)$



# recognizing $D$ using a TSA

- tree stack instructions
  - $\mathrm{up}_{\mathfrak{P}}(\delta)$ (nondeterministic)
  - $\mathrm{down}(\delta)$

or $\mathrm{up}_{\mathfrak{P}}(\langle^{2}_{\rho_3})$

$$(\rangle^{2}_{\rho_3}, \varnothing)$$

$$(\rangle^{1}_{\rho_1}, \{\langle^{2}_{\rho_1}\})$$

$$(\rangle^{1}_{\rho_1}, \varnothing)$$

@

# recognizing $D$ using a TSA

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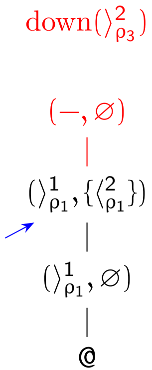

# recognizing $D$ using a TSA

- tree stack instructions
  - $\mathrm{up}_{\mathfrak{P}}(\delta)$ (nondeterministic)
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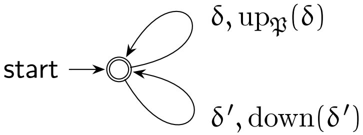
- accepting configuration only contains root symbol ($@$) and symbols o.t.f. $(-, \varnothing)$

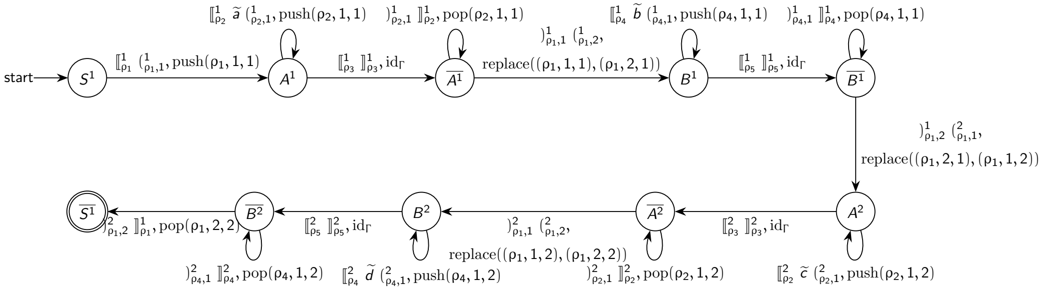
# recognizing $D$ using a TSA

- tree stack instructions
  - $\mathrm{up}_{\mathfrak{P}}(\delta)$ (nondeterministic)
  - $\mathrm{down}(\delta)$
- accepting configuration only contains root symbol (@) and symbols o.t.f. $(-, \varnothing)$
- for each $\delta \in \Delta, \delta' \in \overline{\Delta}$



# $R \cap \mathrm{D}(\Delta)$

- there are *lots* of candidates, very few are even Dyck words
- limit $R$ to Dyck words using PDA
- superset approximation of generator PDA is generator FSA



# omit unnecessary brackets

- omit all grammar rules that cannot be utilized in derivation

$$
\begin{aligned}
P:\ & S \rightarrow [x_{1,1}x_{2,1}x_{1,2}x_{2,2}](A,B),\\
& A \rightarrow [x_{1,1}a, x_{1,2}c](A),\\
& A \rightarrow [\varepsilon,\varepsilon](),\\
& B \rightarrow [x_{1,1}b, x_{1,2}d](B),\\
& B \rightarrow [\varepsilon,\varepsilon]()
\end{aligned}
$$

$$
\begin{aligned}
P_\varepsilon:\ & S \rightarrow [x_{1,1}x_{2,1}x_{1,2}x_{2,2}](A,B)\\
& A \rightarrow [\varepsilon,\varepsilon]()\\
& B \rightarrow [\varepsilon,\varepsilon]()
\end{aligned}
$$

# mutliple Dyck language

- congruence relation for each $v_1, \ldots, v_k \in D(\Sigma)$ with $v_1 \ldots v_k \equiv_{\Sigma,\mathfrak{P}} \varepsilon$

  $$\sigma_1 v_1 \overline{\sigma_1} u_1 \sigma_2 \ldots u_{k-1} \sigma_k v_k \overline{\sigma_k} \equiv_{\Sigma,\mathfrak{P}} u_1 \ldots u_{k-1}$$

  if $\{\sigma_1, \ldots, \sigma_k\} \in \mathfrak{P}$ and $u_1, \ldots, u_k$ are Dyck words over $\Sigma$
- $D = [\varepsilon]_{\equiv_{\Delta,\mathfrak{P}(G)}}$

# sorted multiple Dyck language

- $(\Sigma, \mathit{sort})$ sorted alphabet, $\mathfrak{P}$ partition of $\Sigma$ such that $\mathit{sort}(\sigma) = \mathit{sort}(\sigma') \implies \exists \mathfrak{p} \in \mathfrak{P}\colon \sigma, \sigma' \in \mathfrak{p}$
- let $\sigma_1, \ldots, \sigma_k \in \Sigma$, $v_1, \ldots, v_k \in \mathrm{D}(\Sigma)$, $\sigma_1 v_1 \overline{\sigma_1} \ldots \sigma_k v_k \overline{\sigma_k} \in \mathrm{smD}(\Sigma, \mathit{sort}, \mathfrak{P})$ iff there is a partition $\mathfrak{B}$ of $[k]$ such that for each $\{b_1, \ldots, b_\ell\} \in \mathfrak{B}$ where $b_1 < \ldots < b_\ell$:
  - $v_{b_1} \ldots v_{b_\ell} \in \mathrm{smD}(\Sigma, \mathit{sort}, \mathfrak{P})$
  - $\{\sigma_{b_1}, \ldots, \sigma_{b_\ell}\} \in \mathfrak{P}$
  - for each $\{b'_1, \ldots, b'_{\ell'}\} \in \mathfrak{B}$: $\mathit{sort}(\{\sigma_{b_1}, \ldots, \sigma_{b_\ell}\}) = \mathit{sort}(\{b'_1, \ldots, b'_{\ell'}\}) \iff \{b'_1, \ldots, b'_{\ell'}\} = \{\sigma_{b_1}, \ldots, \sigma_{b_\ell}\}$

# reachability analysis

- let $G = (N, \Sigma, P, S)$ non-deleting MCFG and $w \in \Sigma^*$
- set of $w$-productive rules $P_w$ is the smallest set $P' \subseteq P$ such that

  $$\rho = A \rightarrow [u_{1,0}x_{i(1,1)}^{j(1,1)}u_{1,1}\dots x_{i(1,m_1)}^{j(1,m_1)}u_{1,m_1},\dots,u_{s,0}x_{i(s,1)}^{j(s,1)}\dots u_{s,m_s}](A_1,\dots,A_k) \in P'$$

  iff $\rho \in P$, $u_{1,0},\dots,u_{s,m_s}$ are subsequences in $w$ and there are rules with lhs $A_1,\dots,A_k \in P'$