M-monoid parsing and reduct generation

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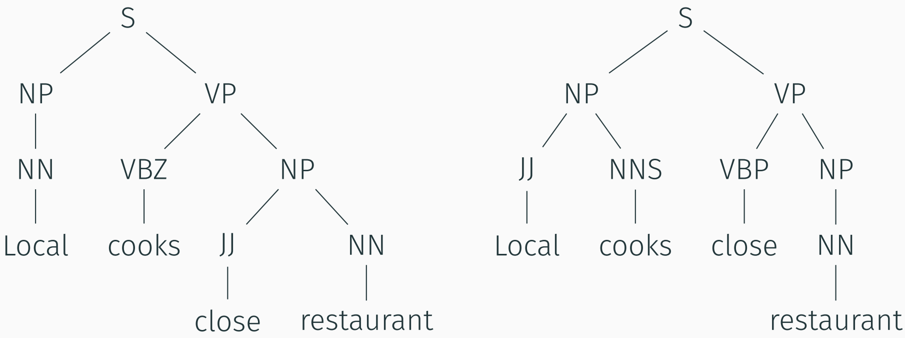
Introduction

Ambiguity in natural language

Local cooks close restaurant.

Ambiguity in natural language

Local cooks close restaurant.



Definitions

M-monoid

Definition (Multioperator monoid)

An M-monoid is an algebraic structure $(S, \oplus, 0, \Omega)$, such that

- $(S, \oplus, 0)$ is a commutative monoid,
- Ω is a set of operations on S such that $\forall \omega \in \Omega : \omega(\dots, 0, \dots) = 0$
- $0^k \in \Omega$ for all $k \in \mathbb{N}$, $0^k : S^k \to S$ such that $0^k(s_1, \dots, s_k) = 0$, and
- Ω distributes over $\oplus$

S is *complete* if the infinitary sum $\sum^{\oplus}$ exists.

Viterbi M-monoid

$(\mathbb{V}, \sum^{\max})$, where

- $\mathbb{V} = (\mathbb{R}_0^1, \max, 0, \Omega.)$
- ${\sum^{\max}}_{i \in I} s_i = \sup\{s_i \mid i \in I\}$

Operations in Ω.

$$\omega_a : (\mathbb{R}_0^1)^k \to \mathbb{R}_0^1$$
$$(s_1, \ldots, s_k) \mapsto a \cdot s_1 \cdot \ldots \cdot s_k$$

$(k \in \mathbb{N}, a \in \mathbb{R}_0^1)$

Definition (LCFRS)

Let Δ be a finite set. An LCFRS over Δ is a tuple $G = (N, \Sigma, Z, R)$, where

- N is a finite $\mathbb{N}$-sorted set (nonterminals),
- Σ is a finite $(\mathbb{N}^* \times \mathbb{N})$-sorted set (terminals) of the form $\langle \text{imagine}, x_1^{(1)} x_1^{(2)}, x_2^{(1)} \text{staff}, \varepsilon \rangle$ (sort $(2, 4)$),
- $Z \in N_1$ (initial nonterminal), and
- R is a finite ranked alphabet (rules) of the form $A \to \langle x_1^{(1)} x_1^{(3)}, x_1^{(2)} x_2^{(1)} \rangle (A_1, A_2, A_3)$ (rank 3).

Example

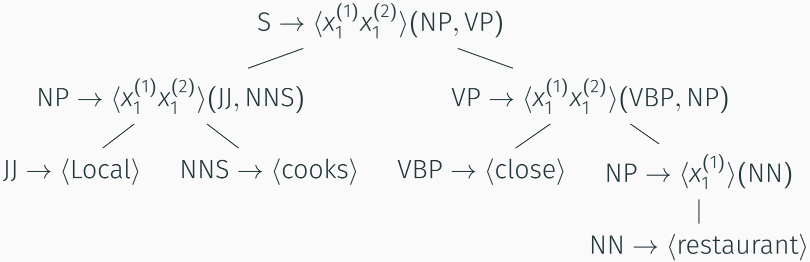
$\Delta = \{\text{Local}, \text{cooks}, \text{close}, \text{restaurant}\}$

$G = (N, \Sigma, Z, R)$, where

- $N = \{\text{S}, \text{NP}, \text{VP}, \text{NN}, \text{NNS}, \text{VBZ}, \text{VBP}, \text{JJ}\}$
- $\Sigma = \{\langle\text{Local}\rangle, \langle\text{cooks}\rangle, \langle\text{close}\rangle, \langle\text{restaurant}\rangle, \langle x_1^{(1)}\rangle, \langle x_1^{(1)} x_1^{(2)}\rangle\}$
- $Z = \text{S}$
- $R \supset \{\text{S} \rightarrow \langle x_1^{(1)} x_2^{(2)}\rangle(\text{NP VP}), \text{NP} \rightarrow \langle x_1^{(1)}\rangle(\text{NN}), \text{NN} \rightarrow \langle\text{Local}\rangle\}$

Abstract syntax trees

AST tree $d \in T_R$ such that for each $p \in \text{pos}(d)$:
if $d(p) = (A \rightarrow \sigma(A_1, \dots A_k))$,
then for each $i \in \{1, \dots, k\}$ the left-hand side of $d(pi)$ is A_i.



Definition (Range concatenation grammar)

An RCG is a tuple $G = (N, \Delta, Z, R)$, where

- N is a finite $\mathbb{N}$-sorted set (nonterminals),
- Δ is a finite set (terminals) such that $N \cap \Delta = \emptyset$,
- $Z \in N_1$ (initial nonterminal), and
- R is a ranked alphabet (rules) of the form $A(x_1^{(1)}x_1^{(3)}, x_1^{(2)}x_2^{(1)}) \to A_1(x_1^{(1)}, x_2^{(1)})A_2(x_2^{(1)})A_3(x_1^{(3)})$ (rank 3)

Equivalence

RCG $G = (N, \Delta, Z, R)$

$$A(w_1, \ldots, w_n) \to A_1(x_1^{(1)}, \ldots, x_{l_1}^{(1)}) \ldots A_k(x_k^{(1)}, \ldots, x_{l_k}^{(1)})$$

$$A \to \langle w_1, \ldots, w_n \rangle (A_1, \ldots, A_k) \text{ with } \langle w_1, \ldots, w_k \rangle \in \Sigma_{(l_1 \ldots l_k, n)}$$

LCFRS $G' = (N, \Sigma, Z, R)$ over Δ

G and G' are *related*.

Weighted LCFRS

Definition (Weighted LCFRS)

Let $(S, \oplus, 0, \Omega)$ be an M-monoid and $G = (N, \Sigma, Z, R)$ be an LCFRS. A *weighted LCFRS* is a tuple (G, wt) where $\mathsf{wt} : R \to \Omega$ is a rank-preserving mapping.

Example

$$\left(\mathrm{S} \to \langle x_1^{(1)} x_2^{(2)} \rangle(\mathrm{NP}\ \mathrm{VP})\right) \mapsto \left((s_1, s_2) \mapsto 1 \cdot s_1 \cdot s_2\right)$$
$$\left(\mathrm{NP} \to \langle x_1^{(1)} \rangle(\mathrm{NN})\right) \mapsto \left((s) \mapsto 0.7 \cdot s\right)$$
$$\left(\mathrm{NN} \to \langle \mathrm{Local} \rangle\right) \mapsto \left(() \mapsto 0.25\right)$$

M-monoid parsing problem

M-monoid parsing problem

Given

1. a complete M-monoid $(S, \sum^{\oplus})$ with $(S, \oplus, 0, \Omega)$,
2. a weighted LCFRS (G, wt) over S where
   $G = (N, \Sigma, Z, R)$ is an LCFRS over Δ and $\mathrm{wt} : R \to \Omega$, and
3. a sentence $e = e_1 \dots e_n$ with $n \geq 1$ and $e_i \in \Delta$

Compute $\mathrm{parse}(e) = \sum^{\oplus}_{d \in (T_R)_Z : [\![\pi_\Sigma(d)]\!] = e} h(d)$, where

- $h : T_R \to S$ such that $h(d) = g(\widehat{\mathrm{wt}}^1(d))$,
- g is the initial homomorphism $T_\Omega \to S$

[1] deterministic tree relabeling induced by wt

Weighted deductive parsing [Ned03]

Range vector vector $\begin{pmatrix} (l_1, r_1) \\ \dots \\ (l_k, r_k) \end{pmatrix}$ such that $0 \leq l_i < r_i \leq |e|$

Items $\mathcal{I} = \{[A, \vec{\kappa}] \mid A \in N \wedge \vec{\kappa} \in \text{ranges}(e)\}$

Inference rules

SCAN: $\frac{}{[A,(i-1,i)]}$ if $\rho = (A \to \langle e_i \rangle)$ in R

RULE: $\frac{[B_1,\vec{\kappa}_1] \quad \dots \quad [B_k,\vec{\kappa}_k]}{[A,\sigma(\vec{\kappa}_1,\dots,\vec{\kappa}_k)]}$ if $\rho = (A \to \sigma(B_1, \dots, B_k))$ in R

Goal: $[Z, (0, |e|)]$

M-monoid parsing algorithm

Input

1. an M-monoid $(S, \oplus, 0, \Omega)$,
2. an LCFRS$^-$ $G = (N, \Sigma, Z, R)$ over Δ, and $\mathrm{wt} : R \to \Omega$,
3. a function select : $2^{\mathcal{I}} \to \mathcal{I}$, and
4. a sentence $e = e_1 \ldots e_n$ with $n \geq 1$ and $e_i \in \Delta$

Variables $V : \mathcal{I} \to S$ mapping

Output $\mathrm{parse}(e)$

Algorithm 3.1 M-monoid parsing for LCFRS$^-$

$\mathcal{A}, \mathcal{C} \leftarrow \varnothing$

for each $A \in N$ and $\vec{\kappa}$ range vector over e **do**

  $V([A, \vec{\kappa}]) \leftarrow 0$

for each $\rho = (A \to \sigma)$ in R and $[A, \vec{\kappa}]$ generated by SCAN $\frac{}{[A,\vec{\kappa}]}$ **do**

  $V([A, \vec{\kappa}]) \leftarrow V([A, \vec{\kappa}]) \oplus \mathsf{wt}(\rho)()$

  $\mathcal{A} \leftarrow \mathcal{A} \cup \{[A, \vec{\kappa}]\}$

while $\mathcal{A} \neq \emptyset$ **do**

  $[A, \vec{\kappa}] \leftarrow \mathrm{select}(\mathcal{A})$

  $\mathcal{A} \leftarrow \mathcal{A} \setminus \{[A, \vec{\kappa}]\}$

  $\mathcal{C} \leftarrow \mathcal{C} \cup \{[A, \vec{\kappa}]\}$

  for each $\rho = (B \to \sigma(B_1, \ldots B_k))$ in R and $[B, \vec{\eta}]$ deduced by RULE $\frac{*}{[B,\vec{\eta}]}$ from $[A, \vec{\kappa}]$ and other items from $\mathcal{C}$ **do**

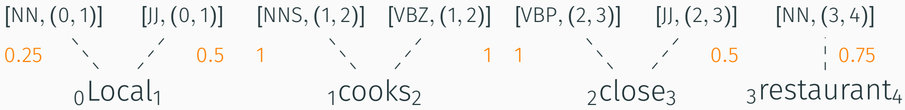
    $V([B, \vec{\eta}]) \leftarrow V([B, \vec{\eta}]) \oplus \mathsf{wt}(\rho)(V([B_1, \vec{\kappa}_1]), \ldots, V([B_k, \vec{\kappa}_k]))$

    if $[B, \vec{\eta}] \notin \mathcal{C}$ **then**

      $\mathcal{A} \leftarrow \mathcal{A} \cup \{[B, \vec{\eta}]\}$

return $V([Z, (0, n)])$

Example run of the algorithm



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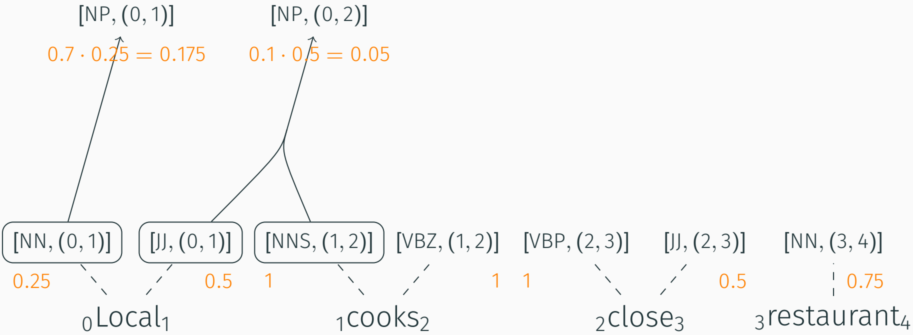

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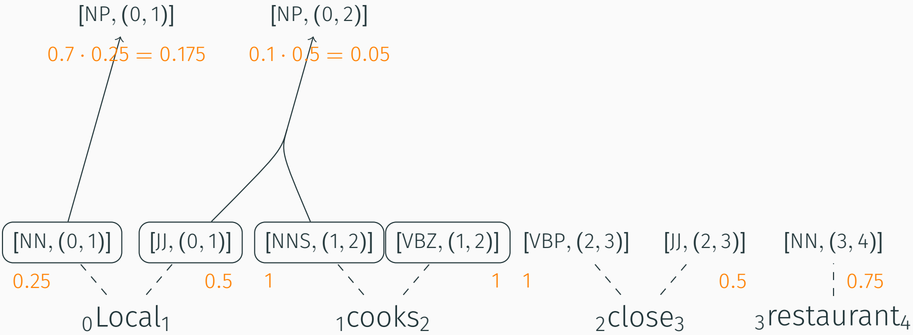

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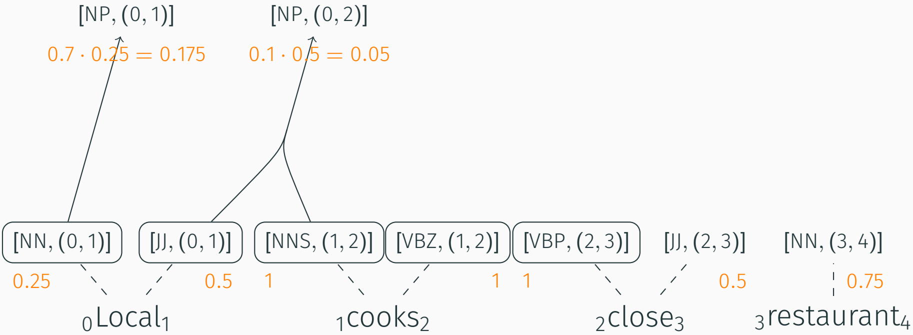

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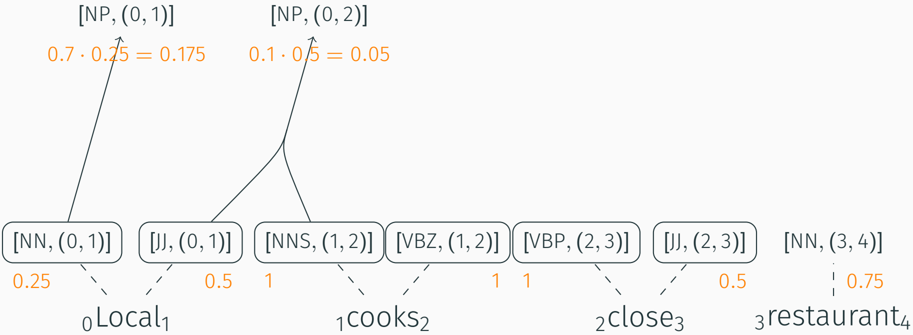

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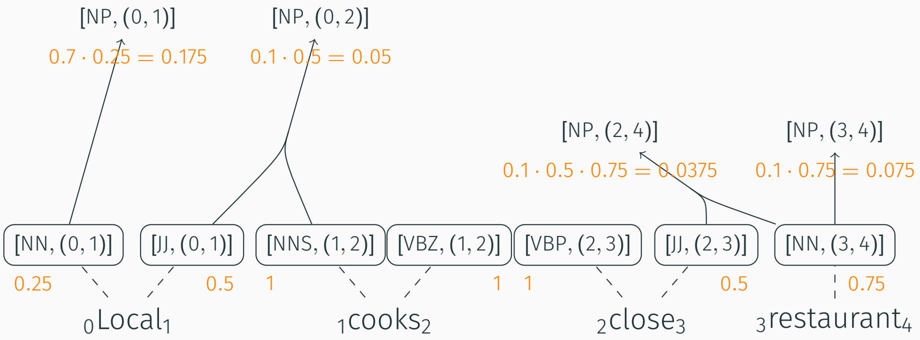

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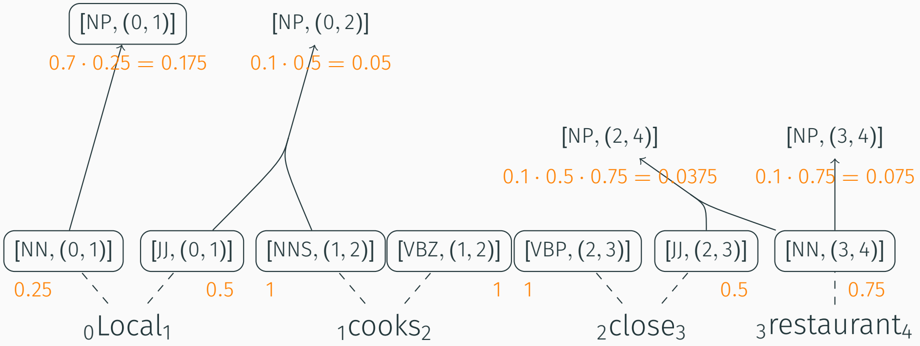

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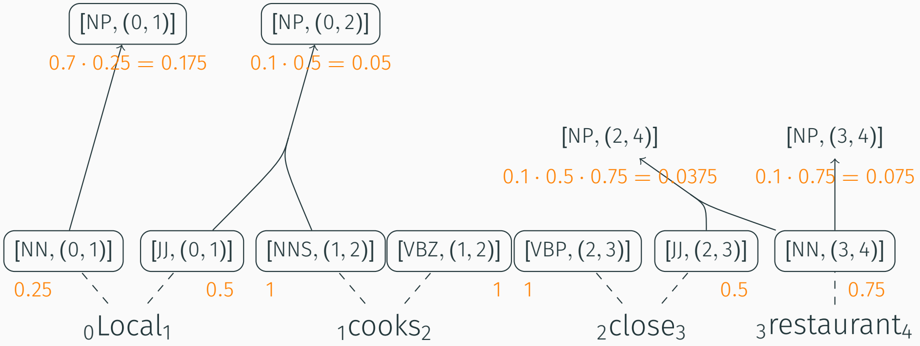

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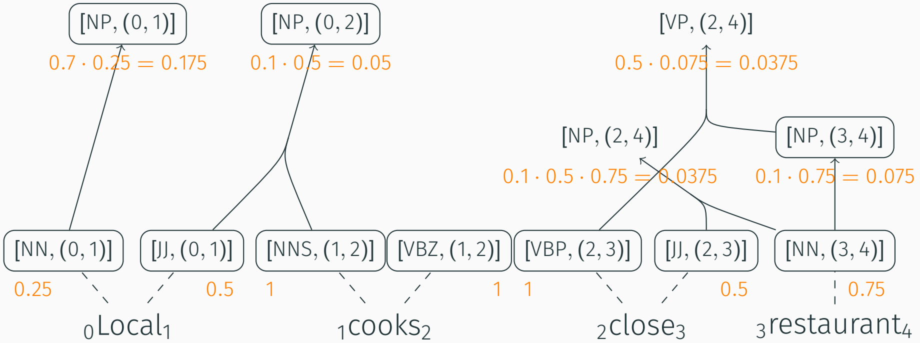

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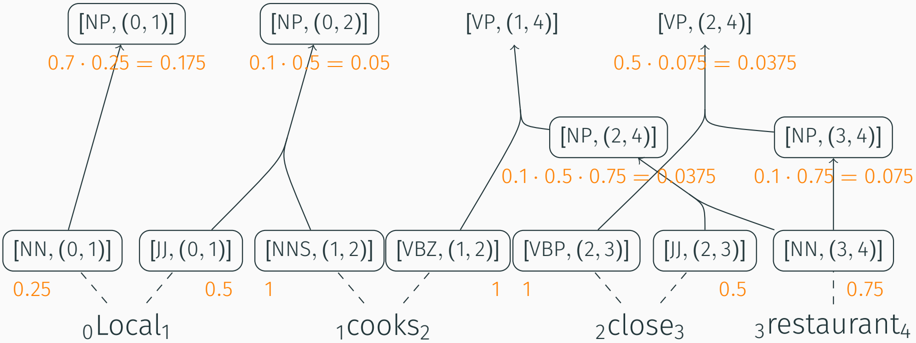

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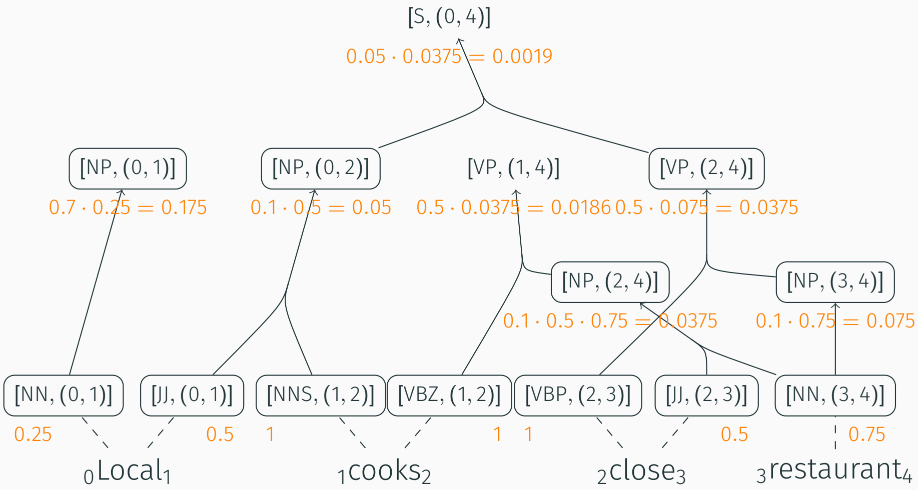

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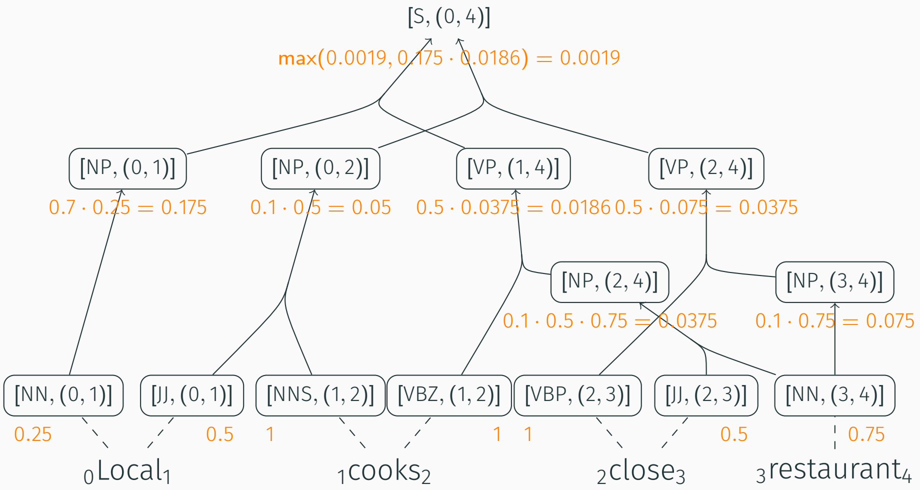

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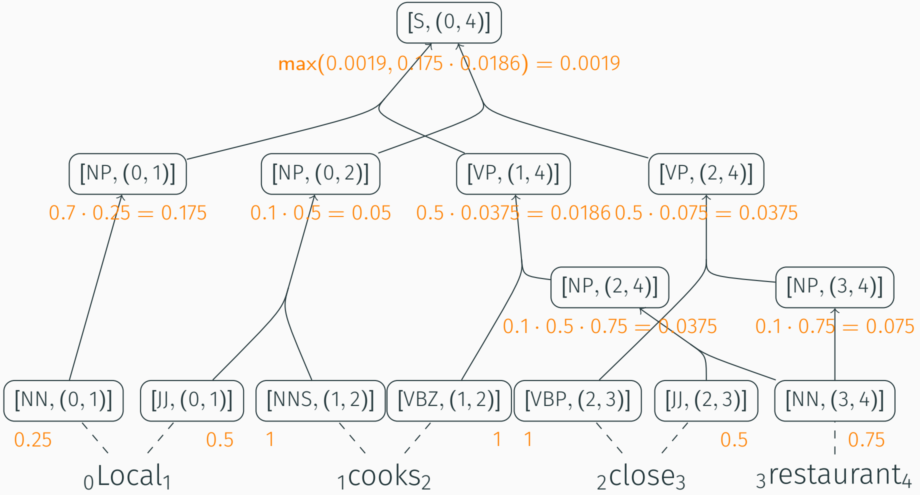


Properties of the algorithm

Termination I

Theorem

The M-monoid parsing algorithm always terminates.

Relevant variables $\mathcal{A}_i$, $\mathcal{C}_i$ (agenda and chart in ith iteration)

$$\mathcal{C}_i \xrightarrow{[A, \vec{\kappa}] \leftarrow \text{select}(\mathcal{A}),\, \mathcal{C} \leftarrow \mathcal{C} \cup \{[A, \vec{\kappa}]\}} \mathcal{C}_{i+1}$$

$\Longrightarrow \mathcal{C}_{i+1} = f(\mathcal{C}_i)$ with $f(\mathcal{C}_i) = \mathcal{C}_i \cup \{[A, \vec{\kappa}]\}$

Termination II

Interpretation $\mathsf{int} : 2^{\mathcal{I}} \rightarrow \mathbb{N}$ with $\mathsf{int}(\mathcal{C}) = n_0 - |\mathcal{C}|$
(n_0 number of all items)

$$\begin{aligned} \mathsf{int}(\mathcal{C}) &= n_0 - |\mathcal{C}| \\ &> n_0 - (|\mathcal{C}| + 1) \\ &= n_0 - |\mathcal{C} \cup \{[A, \vec{\kappa}]\}|^2 \\ &= n_0 - |f(\mathcal{C})| \\ &= \mathsf{int}(f(\mathcal{C})) \end{aligned}$$

[2] $\mathcal{A}$ and $\mathcal{C}$ are always disjoint

Cyclic and acyclic LCFRS

Let $G = (N, \Delta, Z, R)$ be an RCG and $e \in \Delta^+$.

G is cyclic for e (otherwise *acyclic*)

$$Z(e) \Rightarrow_G^* \alpha A(\vec{\kappa}(e))\beta \Rightarrow_G^+ \alpha' A(\vec{\kappa}(e))\beta' \Rightarrow_G^* \varepsilon$$

G is weakly cyclic for e (otherise *weakly acyclic*)

$$A(\vec{\kappa}(e)) \Rightarrow_G^+ \alpha A(\vec{\kappa}(e))\beta \Rightarrow_G^* \varepsilon$$

G is *(weakly) cyclic*
$\Leftrightarrow$ there is an $e \in \Delta^+$ such that G is (weakly) cyclic for e.

Correctness for acyclic LCFRS

Theorem

Let $G = (N, \Sigma, Z, R)$ *be an* LCFRS^- *over* Δ *and* $e \in \Delta^+$ *such that* G *is weakly acyclic for* e*. Then for every* M*-monoid* $(S, \oplus, 0, \Omega)$ *and* $\mathsf{wt} : R \to \Omega$ *there exists a function* $\mathsf{select} : 2^{\mathcal{I}} \to \mathcal{I}$ *such that after termination of Algorithm 3.1 for each* $[A, \vec{\kappa}] \in \mathcal{C}$ *it holds that*

$$V([A, \vec{\kappa}]) = \sum^{\bigoplus}_{d \in (T_R)_A : [\![\pi_\Sigma(d)]\!] = \vec{\kappa}(e)} h(d) \ .$$

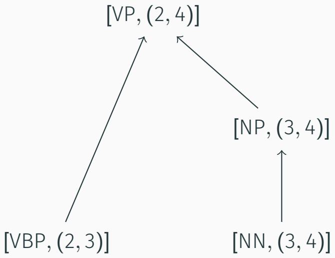
Corollary

Algorithm 3.1 is correct for the class of all acyclic LCFRS^-*.*

Item dependeny graph

$I(G, e) = (V, \leftarrow)$

- V: all items that are useful for e
- $[B, \vec{\eta}] \leftarrow [A, \vec{\kappa}]$ if $[B, \vec{\eta}]$ is derivable from $[A, \vec{\kappa}]$



Front $F_{I(G,e)}(\mathcal{C})$ candidates for select

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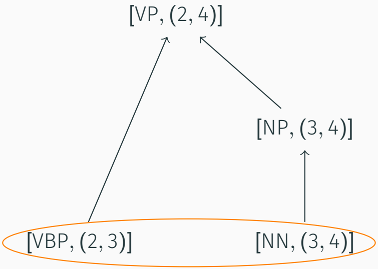

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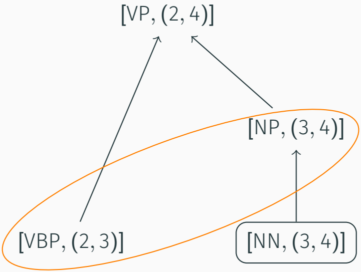

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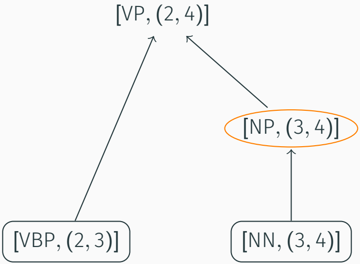

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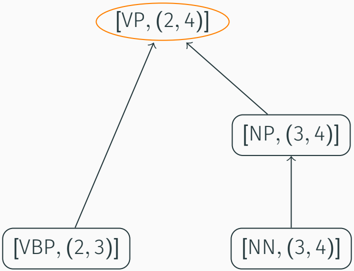

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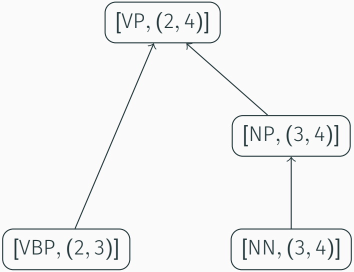

Front $F_{I(G,e)}(\mathcal{C})$ candidates for select

Select function

$$\mathsf{select}_{I(G,e)} : 2^{\mathcal{I}} \to \mathcal{I}$$
$$J \mapsto \begin{cases} \text{arb. } j \in F_{I(G,e)}(\mathcal{C}_n) \cap J & \text{if } F_{I(G,e)}(\mathcal{C}_n) \cap J \neq \emptyset \\ \text{arb. } j \in J & \text{otherwise} \end{cases}$$

Proof sketch

Lemma

If G is weakly acyclic for e then for every $n \in \mathbb{N}$ and $[A, \vec{\kappa}] \in V$ it holds that $[A, \vec{\kappa}] \in \mathcal{C}_n$ implies

$$V([A, \vec{\kappa}]) = \sum^{\bigoplus}_{d \in (T_R)_A : [\![\pi_\Sigma(d)]\!] = \vec{\kappa}(e)} h(d)$$

for the nth and all following iterations of the body of the while loop, where $I(G, e) = (V, \leftarrow)$.

Proof.

Follows from the Lemma with $\mathsf{select}_{I(G,e)}$ as the select function. □

Inferior M-monoids i

Superior functions [Knu77, Jun06]

Definition (Inferior operation)

Let $(S, \preceq)$ be a totally ordered set and $\omega : S^k \to S$ ($k \in \mathbb{N}$) be an operation. We call ω $\preceq$-*inferior* if for every $s_1, \ldots, s_k, s \in S$ and for every $i \in \{1, \ldots, k\}$ the following properties hold:

1. $s \preceq s_i \Rightarrow \omega(\ldots, s_{i-1}, s, s_{i+1}, \ldots) \preceq \omega(\ldots, s_{i-1}, s_i, s_{i+1}, \ldots)$
2. $\omega(s_1, \ldots, s_k) \preceq \min\{s_1, \ldots, s_k\}$

Definition (Inferior M-monoid)

Let $(S, \oplus, 0, \Omega)$ be an M-monoid. Moreover, let $\preceq_\oplus$ be the binary relation on S defined for every $a, b \in S$ as follows: $a \preceq_\oplus b$ if $a \oplus b = b$. If $\preceq_\oplus$ is a total order and every $\omega \in \Omega$ is $\preceq_\oplus$-inferior, then we call the M-monoid S *inferior*.

Example: *Viterbi* M-monoid

Correctness for inferior M-monoids

Theorem

Let $\mathcal{S}$ be the class of inferior M-monoids. Algorithm 3.1 is correct for $\mathcal{S}$ and the select function

$$\begin{aligned}\mathrm{select} : 2^{\mathcal{I}} &\to \mathcal{I}\\ J &\mapsto \textit{an arbitrary } j \in \operatorname*{arg\,max}_{j \in J}{}_{\preceq_\oplus} V(j)\end{aligned} \quad ,$$

where

$$\operatorname*{arg\,max}_{j \in J}{}_{\preceq_\oplus} V(j) = \{ j \in J \mid V(j') \preceq_\oplus V(j) \textit{ for every } j' \in j \} \quad .$$

$\mathcal{A}$ and $\mathcal{C}$ are always disjoint

For every $i \in \mathbb{N}$, $\mathcal{A}_i \cap \mathcal{C}_i = \emptyset$.

Induction base clear, as $\mathcal{C}_0 = \emptyset$

Induction step

$$
\begin{aligned}
\mathcal{A}_i \cap \mathcal{C}_i &= ((\mathcal{A}_i \setminus \{[A, \vec{\kappa}]\}) \cup \{i \in \mathcal{I} \mid \cdots \wedge i \notin \mathcal{C}_{i+1}\}) \cap \mathcal{C}_{i+1} \\
&= \mathcal{A}_i \setminus \{[A, \vec{\kappa}]\} \cap (\mathcal{C}_i \cup \{[A, \vec{\kappa}]\}) \\
&\quad \cup (\{i \in \mathcal{I} \mid \cdots \wedge i \notin \mathcal{C}_{i+1}\}) \cap (\mathcal{C}_i \cup \{[A, \vec{\kappa}]\})) \\
&= (\mathcal{A}_i \setminus \{[A, \vec{\kappa}]\} \cap \mathcal{C}_i) \cup (\mathcal{A}_i \setminus \{[A, \vec{\kappa}]\} \cap \{[A, \vec{\kappa}]\}) \cup \emptyset \\
&= \emptyset \cup \emptyset
\end{aligned}
$$

Proof sketch i

Lemma

If G is weakly acyclic for e, then $I(G, e)$ is acyclic.

Lemma

If G is weakly acyclic for e, then $F_{I(G,e)}(\mathcal{C}) \subseteq \mathcal{A}$ is a loop invariant in each iteration of the body of the while loop (lines 7–14).

Proof sketch ii

Corollary

If G is weakly acyclic for e then for every $n \in \mathbb{N}$ and $v, w \in V$ such that $v \leftarrow w$ it holds that $w \in F_{I(G,e)}(\mathcal{C}_n)$ implies $v = \mathsf{select}_{I(G,e)}(\mathcal{A}_i)$ for some $i < n$, where $I(G, e) = (V, \leftarrow)$.

Lemma

If G is weakly acyclic for e, then for every $n \in \mathbb{N}$ and $[A, \vec{\kappa}] \in V$ it holds that $[A, \vec{\kappa}] = \mathsf{select}_{I(G,e)}(A_n)$ implies $[A, \vec{\kappa}] \in F_{I(G,e)}(\mathcal{C}_n)$, where $I(G, e) = (V, \leftarrow)$.

Proof sketch iii

Corollary

If G is weakly acyclic for e, then for every $n \in \mathbb{N}$ *and* $[A, \vec{\kappa}] \in V$ *it holds that* $[A, \vec{\kappa}] \in \mathcal{C}_n$ *implies* $[A, \vec{\kappa}] \in F_{I(G,e)}(\mathcal{C}_i)$ *for some* $i < n$, *where* $I(G, e) = (V, \leftarrow)$.

Proof sketch i

Lemma

For every item $[A, \vec{\kappa}]$ *it holds that once* $[A, \vec{\kappa}]$ *has been added to* $\mathcal{C}$, $V([A, \vec{\kappa}])$ *will not change anymore.*

Lemma

For every item $[A, \vec{\kappa}]$ *it holds that at every point of the computation* $V([A, \vec{\kappa}]) = 0$ *or* $V([A, \vec{\kappa}]) = h(d)$ *for* $d \in (T_R)_A$ *such that* $[\![\pi_\Sigma(d)]\!] = \vec{\kappa}(e)$.

Proof sketch ii

$$W([A,\vec{\kappa}]) = \sum^{\bigoplus}_{d \in (T_R)_A : [\![\pi_\Sigma(d)]\!] = \vec{\kappa}(e)} h(d)$$

Lemma

For every item $[A,\vec{\kappa}]$ *and for every* $d \in (T_R)_A$ *such that* $[\![\pi_\Sigma(d)]\!] = \vec{\kappa}(e)$ *it holds that* $h(d) \preceq_\oplus W([A,\vec{\kappa}])$.

Lemma

For every item $[A,\vec{\kappa}]$ *it holds that at every point of the computation* $V([A,\vec{\kappa}]) \preceq_\oplus W([A,\vec{\kappa}])$.

Proof sketch iii

Proof.

$$[A, \vec{\kappa}] \in \mathcal{C} \rightarrow V([A, \vec{\kappa}]) = W([A, \vec{\kappa}]) \tag{I}$$

is a loop invariant for every $A \in N$ and $\vec{\kappa} \in \mathsf{ranges}(e)$.

Proof by contradiction similar to [Knu77, Jun06]. □

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