A Comparison of Several Models of Weighted Tree Automata*

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Abstract

In the literature several concepts of recognizability of tree series were investigated. Here we compare the classes of tree series which are recognizable by (a) multilinear mappings over some finite dimensional K-vector space, where K is a field, (b) K-Σ-tree automata, where K is a commutative semiring, (c) weighted tree automata over K, where K is a semiring, (d) finite, polynomial tree automata over K, where K is a commutative and continuous semiring, (e) polynomially-weighted tree automata, where K is a semiring, and (f) weighted tree automata over M-monoids.

1 Introduction

A tree series over some ranked alphabet Σ and some monoid $(K, \oplus, 0)$ is a mapping of the form $\varphi : T_\Sigma \to K$ where T_Σ denotes the set of trees (or: well-formed expressions) over Σ. In the literature, several concepts of recognizability of tree series appear. In this paper we will collect the following concepts and we will compare their respective power:

(a) recognizability by multilinear mappings over some finite dimensional K-vector space, where K is a field, cf. [BR82],

(b) recognizability by K-Σ-tree automata, where K is a commutative semiring, cf. [Boz99],

(c) recognizability by weighted tree automata over K, where K is a semiring, cf. [BV03],

(d) recognizability by finite, polynomial tree automata over K, where K is a commutative and continuous semiring, cf. [Kui98, ÉK03],

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(e) recognizability by polynomially-weighted tree automata, where K is a semiring, cf. [Sei92, Sei94], and

(f) recognizability by weighted tree automata over M-monoids, cf. [Mal05].

Let us now give some more details on these different concepts. The concept of recognizable tree series which has been investigated in [BR82], is based on multilinear mappings over some finite dimensional K-vector space V, where $(K, \oplus, \odot, 0, 1)$ is a field. More precisely, with every m-ary symbol $\sigma \in \Sigma$ an m-ary multilinear mapping $\mu_m(\sigma)$ over V is associated. The family $(\mu_m \mid m \geq 0)$ of such associations can be extended canonically to a Σ-homomorphism $h_\mu : T_\Sigma \to V$. The tree series $\varphi : T_\Sigma \to K$ which is recognized by (V, μ, γ), where $\gamma : V \to K$ is a linear mapping, is defined by $(\varphi, s) = \gamma(h_\mu(s))$ for every $s \in T_\Sigma$.

In [Boz99] a similar approach has been investigated which, however, is based on a larger class of algebraic structures. Roughly speaking, the field K and the K-vector space V have been replaced by a commutative semiring K and a commutative K-semimodule $(K^Q, +, 0_Q)$ of Q-vectors over K, respectively, where Q is a finite set (of states). Now, with every m-ary symbol $\sigma \in \Sigma$ a mapping $\mu_m(\sigma) : Q^m \to K^Q$ is associated. The set K^Q together with the multilinear extension $\overline{\mu_m(\sigma)}$ of the mappings $\mu_m(\sigma)$ form a Σ-algebra. (We note that multilinearity is meant with respect to the operations of the semiring K; also we note that multilinearity can only be defined reasonably if the semiring is commutative.) Then the mapping $h_\mu : T_\Sigma \to K^Q$ is the uniquely determined Σ-homomorphism to this Σ-algebra, and the tree series $\varphi : T_\Sigma \to K$ recognized by (Q, μ, f), where f maps Q to K, is defined by $(\varphi, s) = \bigoplus_{q \in Q} h_\mu(s)_q \odot f(q)$ for every $s \in T_\Sigma$. In [Boz99] a tuple (Q, μ, f) has been called K-Σ-tree automaton.

Clearly, every K-vector space V with finite basis Q is (isomorphic to) a commutative K-semimodule $(K^Q, +, 0_Q)$, and vice versa, if K is a field and Q is a finite set, then the K-semimodule $(K^Q, +, 0_Q)$ is a vector space. Hence, if K is a field, then there is no difference between the two concepts of recognizability of [BR82] and [Boz99] (cf. Theorem 4.6).

In [BV03] weighted tree automata over (arbitrary) semirings have been defined in very much the same way as the K-Σ-tree automata of [Boz99]. However, rather than defining the operations $\overline{\mu_m(\sigma)}$ of the Σ-algebra as multilinear extension of the mappings $\mu_m(\sigma)$, now they are defined explicitly. Hence, the semiring needs not be commutative. In fact, for commutative semirings, there is no difference between the concepts of recognizability of [Boz99] and [BV03] (cf. Theorem 5.4).

In [Kui98] the concept of recognizable tree series has been defined for commutative and continuous semirings. Continuity is needed, because the recognized tree series is defined as the limit of an infinite approximation sequence (or, equivalently, as least fixpoint of a continuous function over a complete poset). This concept is equivalent to the concept of [BV03] if commutative and continuous semirings are considered (cf. Theorem 6.5).

Whereas in the weighted tree automata of [BV03] with every transition on an input symbol a coefficient in the underlying semiring K is associated, the polynomially-weighted tree automata of [Sei92, Sei94] associate with every transition a polynomial over K. It is rather clear that this increases the power of the automaton model, i.e., there is a semiring K such that the class of tree series which are recognizable by polynomially-weighted tree automata over K is larger than the corresponding class which is defined by weighted tree automata over K (cf. Theorem 7.5).

Recently, weighted tree automata over multioperator monoids (for short: M-monoids) have been investigated [Mal05]. An M-monoid is an algebra $(K, \oplus, 0, \Omega)$ where $(K, \oplus, 0)$ is a commutative monoid and (K, Ω) is an Ω-algebra. In such automata, with every transition on an m-ary symbol, an m-ary operation taken from Ω is associated. The origin of this automaton concept goes back to the concepts of K-Σ-tree automata [Boz99] and of tree automata [Kui98]. However, there they required that the operations in Ω are multilinear mappings. That means, besides distributivity, they required that factors can be pulled out of their arguments. We will prove that, under this requirement, weighted tree automata over M-monoids are equivalent to weighted tree automata over commutative semirings (cf. Theorem 8.4). Also we can easily connect weighted tree automata over M-monoids with automata which use polynomials. Since polynomials over m variables are particular operations, the concept of recognizability by polynomially-weighted tree automata over semirings is captured by the concept of recognizability by weighted tree automata over M-monoids (cf. Theorem 8.6). However, these concepts are not equivalent. We will prove that there is an M-monoid K over the set of natural numbers and a tree series φ which is recognizable by a weighted tree automaton over K, but not by a polynomially-weighted tree automaton over the usual semiring of natural numbers (cf. Theorem 8.9). Finally we recall a result from [Mal05] which states that deterministic wta over a distributive M-monoid can be simulated by wta over some semiring (cf. Theorem 8.10).

The paper has the following structure. In Section 2 we recall the definitions of the used algebraic structures, tree series, and matrices. Then we devote for every of the above mentioned concepts of recognizability one chapter.

2 Definitions

For the sake of self-containedness of the paper, we will recall now the well-known concepts of partial order, semiring and field, semimodule, Ω-algebra, tree and tree series. The reader who is familiar with these concepts can skip them on first reading.

2.1 Partial orders

Let K be a set and $\leq$ a binary relation on K. If $\leq$ is reflexive, antisymmetric, and transitive, then $(K, \leq)$ is called a *partially ordered set* (for short: poset). A poset is called *(ω-)complete* if it has a least element $\perp$ and every ω-chain $k_1 \leq k_2 \leq k_3 \leq \ldots$ in K has a least upper bound in K, denoted by $\sup\{k_i \mid i \in \mathbb{N}\}$.

Let $(K, \leq)$ be a poset and $f : K \to K$ be a mapping. Then f is called *continuous* if for every ω-chain $k_1 \leq k_2 \leq k_3 \leq \ldots$ in K which has a supremum, the supremum of $\{f(k_i) \mid i \in \mathbb{N}\}$ exists and $f(\sup\{k_i \mid i \in \mathbb{N}\}) = \sup\{f(k_i) \mid i \in \mathbb{N}\}$. Let $(K, \leq)$ be a complete poset and $f : K \to K$ be continuous. Then f has a least fixpoint $\operatorname{fix} f$, and $\operatorname{fix} f = \sup\{f^n(\perp) \mid n \geq 0\}$ (cf. e.g. Section 1.5, Theorem 7 of [Wec92]).

2.2 Semirings and fields

A *semiring* $(K, \oplus, \odot, 0, 1)$ consists of a commutative monoid $(K, \oplus, 0)$ (called the underlying additive monoid) and a monoid $(K, \odot, 1)$ (called the underlying multiplicative monoid) such that the distributivity laws

$$k \cdot \bigoplus_{j=1}^{n} k_j = \bigoplus_{j=1}^{n} (k \cdot k_j)\,, \qquad \left(\bigoplus_{j=1}^{n} k_j\right) \cdot k = \bigoplus_{j=1}^{n} (k_j \cdot k) \tag{d-SR}$$

hold for every $n \geq 0$ and $k, k_1, \ldots, k_n \in K$. Note that for $n = 0$ Equation (d-SR) implies that $k \cdot 0 = 0 \cdot k = 0$. A semiring is *commutative* if the underlying multiplicative monoid is commutative. We adopt the convention that $\odot$ has a higher binding priority than $\oplus$ and drop the parentheses referring to $\odot$. Examples of (commutative) semirings are the semiring $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers and the so called arctic semiring $\underline{\mathrm{Arct}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$.

A semiring K is *naturally ordered* if the binary relation $\sqsubseteq$ on K is a partial order on K, where $\sqsubseteq$ is defined by $k \sqsubseteq k'$ iff there is an $l \in A$ such that $k \oplus l = k'$.

An *infinitary sum operation* $\bigoplus$ associates with every I-indexed family $(k_i \mid i \in I)$ of elements of K an element $\bigoplus(k_i \mid i \in I)$ in K, where I is a countable set. Rather than $\bigoplus(k_i \mid i \in I)$ we write $\bigoplus_{i \in I} k_i$. Then the tuple $(K, \bigoplus)$ (or just: K) is a $\bigoplus$-*complete semiring* (cf. e.g. [HW98, Kui98]) if the following three conditions are satisfied:

(i) $\bigoplus_{i \in \emptyset} k_i = 0$, $\bigoplus_{i \in \{j\}} k_i = k_j$, $\bigoplus_{i \in \{j,j'\}} k_i = k_j \oplus k_{j'}$ for $j \neq j'$,

(ii) $\bigoplus_{j \in J}(\bigoplus_{i \in I_j} k_i) = \bigoplus_{i \in I} k_i$, if $\bigcup_{j \in J} I_j = I$ and $I_j \cap I_{j'} = \emptyset$ for $j \neq j'$, and

(iii) $\bigoplus_{i \in I}(c \odot k_i) = c \odot (\bigoplus_{i \in I} k_i)$, $\bigoplus_{i \in I}(k_i \odot c) = (\bigoplus_{i \in I} k_i) \odot c$.

A $\bigoplus$-complete and naturally ordered semiring K is $\bigoplus$-*continuous* if, for every index set I and family $(k_i \mid i \in I)$, and $l \in K$, the following condition holds: if $\bigoplus_{i \in E} k_i \sqsubseteq l$ for every finite subset E of I, then $\bigoplus_{i \in I} k_i \sqsubseteq l$. We call a semiring *complete* (resp. *continuous*) if there is an infinitary sum operation $\bigoplus$ such that K is $\bigoplus$-complete (resp. $\bigoplus$-continuous).

We can prove that a continuous semiring K is a complete poset (with $\sqsubseteq$ as partial order) as follows. Clearly, 0 is the least element in K, because for every $k \in K$ we have $0 \oplus k = k$. Let $k_0 \sqsubseteq k_1 \sqsubseteq k_2 \sqsubseteq \ldots$ be an ω- chain of elements of K. By the definition of $\sqsubseteq$ there are $b_0, b_1, b_2, \ldots \in K$ such that $k_n = \bigoplus_{i=0}^{n} b_i$ for every $n \geq 0$. Then we can prove that $\sup\{k_n \mid n \geq 0\} = \bigoplus_{i \geq 0} b_i$. In fact, $\bigoplus_{i \geq 0} b_i$ is an upper bound of $(k_n \mid n \geq 0)$, because for every $n \geq 0$, we have that $k_n \sqsubseteq k_n \oplus \bigoplus_{i \in (\mathbb{N} \setminus \{0,\ldots,n\})} b_i = \bigoplus_{i=0}^{n} b_i \oplus \bigoplus_{i \in (\mathbb{N} \setminus \{0,\ldots,n\})} b_i$ (by the definition of k_n) $= \bigoplus_{i \geq 0} b_i$ (by property (ii) of the infinite sum). Also $\bigoplus_{i \geq 0} b_i$ is the least upper bound: assume that there is a $b \in K$ with $k_n \sqsubseteq b$ for every n; then $\bigoplus_{i=0}^{n} b_i \sqsubseteq b$ for every $n \geq 0$; then also, for every finite subset $E \subseteq \mathbb{N}$, we have that $\bigoplus_{i \in E} b_i \sqsubseteq b$ (because for every finite $E \subseteq \mathbb{N}$, there is an $n \in \mathbb{N}$, such that $\bigoplus_{i \in E} b_i \sqsubseteq \bigoplus_{i=0}^{n} b_i$, and $\sqsubseteq$ is a partial order); then, since K is continuous, $\bigoplus_{i \geq 0} b_i \sqsubseteq b$.

A *field* $(K, \oplus, \odot, 0, 1)$ is a semiring which consists of two commutative groups $(K, \oplus, 0)$ and $(K \setminus \{0\}, \odot, 1)$. An example of a field is the field $\underline{\mathbb{Q}} = (\mathbb{Q}, +, \cdot, 0, 1)$ of rational numbers.

2.3 Semimodules

Let $(K, \oplus, \odot, 0, 1)$ be a semiring, $(V, +, 0)$ a commutative monoid, and $\cdot : K \times V \to V$ a function (note that the 0 of the semiring can be different from the 0 of the monoid). Then $(V, +, 0)$ is a *(left-)K-semimodule (via $\cdot$)* if the following laws hold:

$$(k \odot k') \cdot v = k \cdot (k' \cdot v) \tag{SM1}$$

$$k \cdot (v + v') = (k \cdot v) + (k \cdot v') \tag{SM2}$$

$$(k \oplus k') \cdot v = (k \cdot v) + (k' \cdot v) \tag{SM3}$$

$$1 \cdot v = v \tag{SM4}$$

$$k \cdot 0 = 0 \cdot v = 0 \tag{SM5}$$

for every $k, k' \in K$ and $v, v' \in V$ (cf. p.149 of [Gol99]). Note that, in equation (SM5), the first and third 0 are in V and the second 0 is in K. Note also that $(K, \oplus, 0)$ is a K-semimodule via $\odot$. A K-semimodule is *commutative* if the underlying semiring K is commutative. Furthermore, we drop $\cdot$ and write, e.g., kv instead of $k \cdot v$.

Let $(V, +, 0)$ be a commutative K-semimodule. An m-ary operation $\omega : V^m \to V$ is called *multilinear* if

$$\omega(v_1, \ldots, v_{i-1}, kv + k'v', v_{i+1}, \ldots, v_m) = \tag{ml-Ω}$$
$$k\omega(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_m) + k'\omega(v_1, \ldots, v_{i-1}, v', v_{i+1}, \ldots, v_m)$$

holds for every $1 \le i \le m$, $k, k' \in K$, and $v, v', v_1, \ldots, v_m \in V$. Note that, if K is not commutative, then e.g. for $m = 2$, the equation $(k_1 \odot k_2)\omega(v_1, v_2) = \omega(k_1 v_1, k_2 v_2) = (k_2 \odot k_1)\omega(v_1, v_2)$ must hold which is in general not true. We denote by $\mathcal{L}(V^m, V)$ the set of all m-ary multilinear operations over V.

Next we define a K-semimodule which has a particular form; this will be used in Section 4. Let $(K, \oplus, \odot, 0, 1)$ be a commutative semiring and $Q = \{q_1, \ldots, q_\kappa\}$ a finite set. Now we consider the commutative monoid $(K^Q, +, 0_Q)$ where 0_Q is the κ-dimensional 0-vector and $+$ is the componentwise extension of $\oplus$ to vectors in K^Q, i.e., $(v_1+v_2)_q = (v_1)_q \oplus (v_2)_q$ for every $v_1, v_2 \in K^Q$ and $q \in Q$. Moreover, we define the scalar multiplication $\cdot$: $K \times K^Q \to K^Q$ by $(k \cdot v)_q = k \odot (v)_q$ for every $k \in K$, $v \in K^Q$, and $q \in Q$. Then it is easy to check that $(K^Q, +, 0_Q)$ is a commutative K-semimodule. For every $q \in Q$, we denote by 1_q the q-unit vector in K^Q, i.e., the vector in K^Q of which the qth component is 1 and the other components are 0.

Let $(K, \oplus, \odot, 0, 1)$ be a field. A *K-vector space* is a K-semimodule $(V, +, 0)$ where $(V, +, 0)$ is a commutative group. The vectors $v_1, \ldots, v_m \in V$ are *linearly independent* if, for every $k_1, \ldots, k_m \in K$, the equality $k_1 v_1 + \ldots + k_m v_m = 0$ implies that $k_1 = \ldots = k_m = 0$. A subset V' of V is linearly independent if every $v_1, \ldots, v_m \in V'$ are linearly independent. Moreover, V' *generates* V if, for every $v \in V$, there are $m \ge 1$, $v_i \in V$ and $k_i \in K$ for $1 \le i \le m$ such that $v = k_1 v_1 + \ldots + k_m v_m$. Finally, V' is a *basis of* V if it is linearly independent and generates V. A vector space V that admits a finite basis is called *finite-dimensional*. In a finite-dimensional vector space each basis has the same number of elements. V is κ-dimensional, where $\kappa < \infty$ is a nonnegative integer, if it has a basis consisting of κ elements.

Let $(V, +, 0)$ be a K-vector space with the finite basis $Q = \{q_1, \ldots, q_\kappa\}$. Moreover, we consider the K-semimodule $(K^Q, +', 0_Q)$ which clearly is also a K-vector space, because

K is a field. These two vector spaces are isomorphic via the bijective mapping $f : V \to K^Q$ defined by

$$f(k_1 q_1 + \ldots + k_\kappa q_\kappa) = k_1 1_{q_1} +' \ldots +' k_\kappa 1_{q_\kappa}.$$

Hence, we identify V and K^Q. Since the two addition operations $+$ and $+'$ are defined in the same way (viz. taking the $\oplus$-addition of every q-component), we will drop the prime from $+'$ and simply write $(K^Q, +, 0_Q)$ in the sequel.

Let V and V' be K-vector spaces. A mapping $\gamma : V \to V'$ is *linear* if, for every $k, l \in K$ and $u, v \in V$, the equation $\gamma(ku+lv) = k\gamma(u)+l\gamma(v)$ holds. A linear mapping $\lambda : V \to K$ is called a *linear form*.

2.4 Ω-algebras

A *ranked set* is a pair (Ω, rk) where Ω is a set and $\mathrm{rk}\colon \Omega \to \mathbb{N}$ (called rank function). Usually we denote the ranked set simply by Ω. For every $m \geq 0$ we denote the set $\{\omega \in \Omega \mid \mathrm{rk}(\omega) = m\}$ by $\Omega^{(m)}$. In the rest of the paper we assume that $\Omega^{(0)} \neq \emptyset$ for every ranked set Ω that we consider.

Let Ω be a ranked set. An Ω*-algebra* (K, Ω_K) consists of a nonempty set K and a family

$$\Omega_K = (\omega_K : K^m \to K \mid m \geq 0,\, \omega \in \Omega^{(m)})$$

of operations on K. If the meaning is clear from the context, we do not make a distinction between ω and ω_K and simply drop the K from ω_K. Also we identify the ranked set Ω with the family Ω_K. We denote the m-ary operation on K which maps every argument tuple to 0 by 0^m.

A *multioperator monoid* (for short: M-monoid) $(K, \oplus, 0, \Omega)$ consists of a commutative monoid $(K, \oplus, 0)$ (called the underlying monoid) and an Ω-algebra (K, Ω). A multioperator monoid is *distributive* if the law

$$\omega_K(k_1, \ldots, k_{i-1}, \bigoplus_{j=1}^{n} a_j, k_{i+1}, \ldots, k_m) = \bigoplus_{j=1}^{n} \omega_K(k_1, \ldots, k_{i-1}, a_j, k_{i+1}, \ldots, k_m) \qquad \text{(d-}\Omega\text{)}$$

holds for every $m \geq 0$, $\omega \in \Omega^{(m)}$, $k_1, \ldots, k_m \in K$, $1 \leq i \leq m$, and $a_1, \ldots, a_n \in K$ (cf. p.727 of [Mal05], p.222 of [ÉK03]). Note that, in particular, (d-Ω) implies that $\omega_K(\ldots, 0, \ldots,) = 0$. We also note that every semiring $(K, \oplus, \odot, 0, 1)$ can be viewed as a distributive M-monoid $(K, \oplus, 0, \Omega)$ where $\Omega = \{\odot^{(2)}, 1^{(0)}\}$.

A *semiring M-monoid* is a tuple $(K, \oplus, \odot, 0, 1, \Omega)$ such that

- $(K, \oplus, \odot, 0, 1)$ is a commutative semiring (hence $(K, \oplus, 0)$ is a commutative K-semimodule via $\odot$) and
- Ω is a set of multilinear operations.

2.5 Trees and tree series

Let Σ be a ranked alphabet, i.e., a finite ranked set, and H an arbitrary set. The set of ΣH*-trees*, denoted by $T_\Sigma(H)$, is defined as usual inductively as the set of well-formed expressions which may have elements of H as leaf labels. If $H = \emptyset$, then we write T_Σ rather than $T_\Sigma(H)$.

Let $(K, \oplus, 0)$ be a commutative monoid and Σ a ranked alphabet. A *tree series* (*over* Σ *and* K) is a mapping $\varphi : T_\Sigma \to K$. For every $s \in T_\Sigma$, the element $\varphi(s) \in K$ is called the *coefficient* of s and it is also denoted by (φ, s). If there is a $k \in K$ such that for every $s \in T_\Sigma$, we have $(\varphi, s) = k$, then φ is a *constant* and it is denoted by $\tilde{k}$. The *set of all tree series over* Σ *and* K is denoted by $K\langle\langle T_\Sigma\rangle\rangle$.

The *support of a tree series* $\varphi \in K\langle\langle T_\Sigma\rangle\rangle$ is defined as the set $supp(\varphi) = \{s \in T_\Sigma \mid (\varphi, s) \neq 0\}$. Moreover, φ is *polynomial* if $supp(\varphi)$ is finite. The *set of all polynomial tree series over* Σ *and* K is denoted by $K\langle T_\Sigma\rangle$. A polynomial trees series $\varphi \in K\langle T_\Sigma\rangle$ is written as $k_1 s_1 \oplus \ldots \oplus k_n s_n$, where $supp(\varphi) = \{s_1, \ldots, s_n\}$ and $k_i = (\varphi, s_i)$ for every $1 \leq i \leq n$.

Now we define the addition over $K\langle\langle T_\Sigma\rangle\rangle$ in the following way. Let I be an index set and $(\varphi_i \in K\langle\langle T_\Sigma\rangle\rangle \mid i \in I)$ a family of tree series. This family is called *locally finite* if for every $s \in T_\Sigma$, the set $\{i \in I \mid (\varphi_i, s) \neq 0\}$ is finite. Now, if the family of tree series is locally finite, or the semiring K is complete, then we can define the sum $\bigoplus_{i\in I} \varphi_i \in K\langle\langle T_\Sigma\rangle\rangle$ by $((\bigoplus_{i\in I} \varphi_i), s) = \bigoplus_{i\in I}(\varphi_i, s)$ for every $s \in T_\Sigma$. It is easy to see that, for every tree series $\varphi : T_\Sigma \to K$, the equation $\varphi = \bigoplus_{s\in T_\Sigma}(\varphi, s)s$ holds, because the family $((\varphi, s)s \mid s \in T_\Sigma)$ of monomials is locally finite. In fact, $(K\langle\langle T_\Sigma\rangle\rangle, \oplus, \tilde{0})$ is a commutative monoid (cf. page 615 of [Kui97]).

Now let $(K, \oplus, \odot, 0, 1)$ be a semiring. If we define the scalar multiplication $\cdot : K \times K\langle\langle T_\Sigma\rangle\rangle \to K\langle\langle T_\Sigma\rangle\rangle$ by $(k \cdot \varphi, s) = k \odot (\varphi, s)$ for every $k \in K$, $\varphi \in K\langle\langle T_\Sigma\rangle\rangle$, and $s \in T_\Sigma$, then it is obvious that $(K\langle\langle T_\Sigma\rangle\rangle, \oplus, \tilde{0})$ is a K-semimodule via $\cdot$.

For every $m \geq 0$ and $\sigma \in \Sigma^{(m)}$, we define the mapping $\mathrm{top}_\sigma : (K\langle\langle T_\Sigma\rangle\rangle)^m \to K\langle\langle T_\Sigma\rangle\rangle$ by

$$\mathrm{top}_\sigma(\varphi_1, \ldots, \varphi_m) = \bigoplus_{s_1,\ldots,s_m\in T_\Sigma} \Big((\varphi_1, s_1) \odot \ldots \odot (\varphi_m, s_m)\Big)\sigma(s_1, \ldots, s_m).$$

Now let $l \geq 0$ and $Y_l = \{y_1, \ldots, y_l\}$ be a set of variables. Let $\varphi \in K\langle\langle T_\Sigma(Y_l)\rangle\rangle$ and $\psi_i \in K\langle\langle T_\Sigma\rangle\rangle$ for every $1 \leq i \leq l$. We define the *OI-substitution* of trees series $\varphi \leftarrow_{OI} (\psi_1, \ldots, \psi_l)$, cf. [Kui98], [ÉK03], and [Boz99]. First, for every tree $s \in T_\Sigma(Y_l)$ we define the tree series $s \leftarrow_{OI} (\psi_1, \ldots, \psi_l)$ in $K\langle\langle T_\Sigma\rangle\rangle$ in the following way:

- if $s = \alpha \in \Sigma^{(0)}$, then $s \leftarrow_{OI} (\psi_1, \ldots, \psi_l) = \alpha$,
- if $s = y_i$ for some $1 \leq i \leq l$, then $s \leftarrow_{OI} (\psi_1, \ldots, \psi_l) = \psi_i$,
- if $s = \sigma(s_1, \ldots, s_m)$ for some $m > 0$, $\sigma \in \Sigma^{(m)}$, and $s_1, \ldots, s_m \in T_\Sigma(Y_l)$, then $s \leftarrow_{OI} (\psi_1, \ldots, \psi_l) = \mathrm{top}_\sigma(s_1 \leftarrow_{OI} (\psi_1, \ldots, \psi_l), \ldots, s_m \leftarrow_{OI} (\psi_1, \ldots, \psi_l))$.

Finally, we define $\varphi \leftarrow_{OI} (\psi_1, \ldots, \psi_l)$ by

$$\varphi \leftarrow_{OI} (\psi_1, \ldots, \psi_l) = \bigoplus_{s\in T_\Sigma(Y_l)} (\varphi, s) \cdot \Big(s \leftarrow_{OI} (\psi_1, \ldots, \psi_l)\Big).$$

Note that the family $(s \leftarrow_{OI} (\psi_1, \ldots, \psi_l) \mid s \in T_\Sigma(Y_l))$ is locally finite.

2.6 Matrices

We recall some notions concerning matrices (cf. [Kui97]). Let I, J and S be arbitrary sets. An $I \times J$ *matrix over* S is a mapping $\mathcal{M} : I \times J \to S$. The set of all $I \times J$ matrices over S is denoted by $S^{I \times J}$. An element $\mathcal{M}(i,j) \in S$ is also denoted by $\mathcal{M}_{i,j}$. If I (or J) is a singleton, then we identify it by $\{1\}$ and we write just 1 for I (or J). Moreover in that case an element of the form $\mathcal{M}_{i,1}$ (or $\mathcal{M}_{1,j}$) is written just $\mathcal{M}_i$ (or $\mathcal{M}_j$).

3 Multilinear mappings over finite-dimensional vector spaces

In [BR82] the recognizability of tree series was defined in terms of multilinear mappings over finite-dimensional vector spaces in the following way.

Let Σ be a ranked alphabet. A *multilinear representation* of T_Σ is a tuple (V, μ, γ) where $(V, +, 0)$ is a finite-dimensional vector space over a field K, $\mu = (\mu_m \mid m \geq 0)$ is a family with $\mu_m : \Sigma^{(m)} \to \mathcal{L}(V^m, V)$, and $\gamma : V \to K$ is a linear form. (Note that $\mu_m(\sigma)$ is an m-ary function and we do not consider V^m as the vector space of m-tuples.)

The Σ-algebra associated with (V, μ, γ) is $V_{alg} = (V, \overline{\Sigma}_\mu)$, where $\overline{\Sigma}_\mu = (\mu_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$. Note that V_{alg} does not depend on γ. The unique Σ-homomorphism from T_Σ to V_{alg} is denoted by h_μ.

Definition 3.1. Let K be a field and Σ a ranked alphabet. A tree series $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ is *recognizable by multilinear mappings over a K-vector space* if there is a K-vector space V and a multilinear representation (V, μ, γ) of T_Σ such that $(\varphi, s) = \gamma(h_\mu(s))$ for every $s \in T_\Sigma$.

First we give an example of a tree series which is recognizable by multilinear mappings over a $\underline{\mathbb{Q}}$-vector space. For this, let Σ be a ranked alphabet and let us define the tree series size : $T_\Sigma \to \mathbb{N}$ in the usual way, i.e, let $\text{size}(\sigma) = 1$ for $\sigma \in \Sigma^{(0)}$, and $\text{size}(\sigma(s_1, \ldots, s_k)) = 1 + \text{size}(s_1) + \ldots + \text{size}(s_k)$.

Proposition 3.2. (Cf. Example 4.1 of [BR82]). The tree series size is recognizable by multilinear mappings over the $\underline{\mathbb{Q}}$-vector space $(\mathbb{Q}^2, +, 0_2)$.

Proof. First, for every $\delta \in \Sigma$, we define the δ-size of a tree $s = \sigma(s_1, \ldots, s_k)$ as follows.

$$\text{size}_\delta(s) = \begin{cases} 1 + \text{size}_\delta(s_1) + \ldots + \text{size}_\delta(s_k) & \text{if } \sigma = \delta \\ \text{size}_\delta(s_1) + \ldots + \text{size}_\delta(s_k) & \text{otherwise.} \end{cases}$$

Certainly, $\text{size}(s) = \sum_{\delta \in \Sigma} \text{size}_\delta(s)$. Since, for every field K, tree series recognizable by multilinear mappings over a K-vector space are closed under finite sum, see Proposition 3.1 of [BR82], it is sufficient to show that the tree series size_δ is recognizable by multilinear mappings over the $\underline{\mathbb{Q}}$-vector space $\mathbb{Q}^2$.

For this, let $\delta \in \Sigma^{(l)}$ and $l \geq 0$ be arbitrary, but now fixed. We define the multilinear tree representation (V, μ, γ) with $V = \mathbb{Q}^2$ in the following way. Certainly it is sufficient to define the multilinear mappings $\mu_m(\sigma)$ (for $m \geq 0$ and $\sigma \in \Sigma^{(m)}$) and γ on the basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For every $e_{i_1}, \ldots, e_{i_l} \in \{e_1, e_2\}$, we define

$$\mu_l(\delta)(e_{i_1},\dots,e_{i_l}) = \begin{cases} e_1+e_2 & \text{if } i_1=\dots=i_l=1 \\ e_2 & \text{if there is exactly one } 1\le j\le l \text{ with } i_j=2 \\ 0_2 & \text{otherwise,} \end{cases}$$

and, for every $\sigma \in \Sigma^{(m)}$ with $\sigma \neq \delta$ and $e_{i_1},\dots,e_{i_m} \in \{e_1,e_2\}$, we define

$$\mu_m(\sigma)(e_{i_1},\dots,e_{i_m}) = \begin{cases} e_1 & \text{if } i_1=\dots=i_m=1 \\ e_2 & \text{if there is exactly one } 1\le j\le m \text{ with } i_j=2 \\ 0_2 & \text{otherwise.} \end{cases}$$

Thus, in particular, for $\alpha \in \Sigma^{(0)}$, this gives

$$\mu_0(\alpha) = \begin{cases} e_1+e_2 & \text{if } \alpha=\delta \\ e_1 & \text{otherwise.} \end{cases}$$

Finally, we define $\gamma(e_1)=0$ and $\gamma(e_2)=1$.

One can show easily that, for every $s \in T_\Sigma$, we have $\gamma(h_\mu(s)) = (\mathrm{size}_\delta, s)$. For this, let us first show that $h_\mu(s) = e_1 + (\mathrm{size}_\delta, s)e_2$. This is clear if $s=\alpha\in\Sigma^{(0)}$, thus let $s=\sigma(s_1,\dots,s_m)$ for some $m\ge 1$. Then, using the multilinearity at equations marked by $*$, we can calculate as follows:

$$\begin{aligned}
h_\mu(s) &= \mu_m(\sigma)(h_\mu(s_1),\dots,h_\mu(s_m)) \\
&= \mu_m(\sigma)(e_1+(\mathrm{size}_\delta,s_1)e_2,\dots,e_1+(\mathrm{size}_\delta,s_m)e_2) \\
&=^* \mu_m(\sigma)(e_1,\dots,e_1) + \textstyle\sum_{i=1}^m \mu_m(\sigma)(e_1,\dots,e_1,(\mathrm{size}_\delta,s_i)e_2,e_1,\dots,e_1) \\
&\quad + \textstyle\sum_{1\le i<j\le m} \mu_m(\sigma)(\dots,(\mathrm{size}_\delta,s_i)e_2,\dots,(\mathrm{size}_\delta,s_j)e_2,\dots) \\
&=^* \mu_m(\sigma)(e_1,\dots,e_1) + \textstyle\sum_{i=1}^m (\mathrm{size}_\delta,s_i)\mu_m(\sigma)(e_1,\dots,e_1,e_2,e_1,\dots,e_1) \\
&\quad + \textstyle\sum \dots(\mathrm{size}_\delta,s_i)\dots(\mathrm{size}_\delta,s_j)\dots\mu_m(\sigma)(\dots,e_2,\dots,e_2,\dots) \\
&= \mu_m(\sigma)(e_1,\dots,e_1) + \textstyle\sum_{i=1}^m (\mathrm{size}_\delta,s_i)e_2 + 0 \\
&= \begin{cases} e_1+e_2+\sum_{i=1}^m(\mathrm{size}_\delta,s_i)e_2 & \text{if } \sigma=\delta \\ e_1+\sum_{i=1}^m(\mathrm{size}_\delta,s_i)e_2 & \text{otherwise.} \end{cases}
\end{aligned}$$

Then it should be clear that $\gamma(h_\mu(s)) = (\mathrm{size}_\delta, s)$. This means that size_δ and thus size is recognizable by multilinear mappings over the $\underline{\mathbb{Q}}$-vector space $\mathbb{Q}^2$. □

Now we give an example of a tree series which is not recognizable by multilinear mappings over any $\underline{\mathbb{Q}}$-vector space V. For this, let us define the tree series height : $T_\Sigma \to \mathbb{N}$ in the usual way, i.e, let $\mathrm{height}(\sigma)=0$ for $\sigma\in\Sigma^{(0)}$, and $\mathrm{height}(\sigma(s_1,\dots,s_k)) = 1 + \max\{\mathrm{height}(s_1),\dots,\mathrm{height}(s_k)\}$.

Proposition 3.3. (See Example 9.2 in [BR82].) The tree series height is not recognizable by multilinear mappings over any $\underline{\mathbb{Q}}$-vector space.

4 K-Σ-tree automata

In [Boz99] the recognizability of tree series by K-Σ-tree automata (K-Σ-ta) was defined.

Let $(K,\oplus,\odot,0,1)$ be a commutative semiring and $Q=\{q_1,\dots,q_\kappa\}$ a finite set. Now we consider the commutative K-semimodule $(K^Q,+,0_Q)$ as defined in Section 2.3. In particular, we have that $v = \sum_{1 \leq i \leq \kappa} (v)_{q_i} \cdot 1_{q_i}$ for every $v \in K^Q$ (recall that 1_{q_i} is the q_i-unit vector in K^Q).

Now let $m \geq 0$ and consider a mapping $\nu : Q^m \to K^Q$. A *multilinear extension of* ν is a mapping $\overline{\nu} : \underbrace{K^Q \times \ldots \times K^Q}_{m} \to K^Q$ such that

- $\overline{\nu}$ is multilinear (i.e., Equation (ml-Ω) holds) and
- for every $p_1, \ldots, p_m \in Q$ we have $\overline{\nu}(1_{p_1}, \ldots, 1_{p_m}) = \nu(p_1, \ldots, p_m)$.

It can be easily seen that such a multilinear extension of ν exists and is unique which can be seen as follows. We consider the mapping $\overline{\nu} : \underbrace{K^Q \times \ldots \times K^Q}_{m} \to K^Q$ defined by

$$\overline{\nu}(v_1, \ldots, v_m)_q = \bigoplus_{p_1, \ldots, p_m \in Q} \left(\bigodot_{1 \leq i \leq m} (v_i)_{p_i} \right) \odot \nu(p_1, \ldots, p_m)_q,$$

note that K is commutative. It can be shown easily that $\overline{\nu}$ is multilinear and an extension of ν. Moreover it can be seen that any two multilinear extensions of ν are equal. Henceforth, we speak about *the multilinear extension of* ν.

Let Σ be a ranked alphabet and K a commutative semiring. A *K-Σ-tree automaton* (for short: K-Σ-ta) is a system $M = (Q, \mu, f)$, where Q is a finite set, $\mu = (\mu_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$ is a family of transition functions $\mu_m(\sigma) : Q^m \to K^Q$, and $f : Q \to K$ is a final weight function.

Now we define the tree series recognized by M. For every $m \geq 0$ and $\sigma \in \Sigma^{(m)}$, let $\overline{\mu_m(\sigma)} : (K^Q)^m \to K^Q$ be the multilinear extension of $\mu_m(\sigma)$. Based on this definition of the mappings $\overline{\mu_m(\sigma)}$, the *Σ-algebra associated with* M is $M_{alg} = (K^Q, \overline{\Sigma}_\mu)$ where $\overline{\Sigma}_\mu = (\overline{\mu_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$. Let us denote the unique Σ-homomorphism from T_Σ to M_{alg} by $h_\mu : T_\Sigma \to K^Q$. *The tree series recognized by* M is the tree series $\varphi_M \in K\langle\langle T_\Sigma \rangle\rangle$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} h_\mu(s)_q \odot f(q).$$

Definition 4.1. Let K be a commutative semiring and Σ a ranked alphabet. A tree series $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ is *recognizable by a K-Σ-ta* if there is a K-Σ-ta $M = (Q, \mu, f)$ such that $\varphi = \varphi_M$.

The following observation is trivial.

Observation 4.2. If a tree series is not recognizable by any K-Σ-ta, then it is not recognizable by any K'-Σ-ta, where K' is a subsemiring of K.

Now we give an example of a tree series recognizable by an $\underline{\text{Arct}}$-Σ-ta.

Example 4.3. For the sake of simplicity, let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We show that the tree series height is recognizable by an $\underline{\text{Arct}}$-Σ-ta. For this, let us consider the $\underline{\text{Arct}}$-Σ-ta $M = (Q, \mu, f)$ where

- $Q = \{p_1, p_2\}$,
- $f(p_1) = 0$ and $f(p_2) = -\infty$,
- μ is defined in the following way:

  - $\mu_0(\alpha)()_{p_1} = 0$,
  - $\mu_0(\alpha)()_{p_2} = 0$,
  - $\mu_2(\sigma)(p_1, p_2)_{p_1} = 1$,
  - $\mu_2(\sigma)(p_2, p_1)_{p_1} = 1$,
  - $\mu_2(\sigma)(p_2, p_2)_{p_2} = 0$,
  - $\mu_2(\sigma)(p, q)_r = -\infty$ for every other $p, q, r \in Q$.

We claim that $(\varphi_M, s) = \text{height}(s)$ for every $s \in T_\Sigma$. Indeed, it can be proved by a straightforward induction that for every $s \in T_\Sigma$: $h_\mu(s)_{p_1} = \text{height}(s)$ and $h_\mu(s)_{p_2} = 0$.

$\underline{s = \alpha:}$ $h_\mu(\alpha)_{p_1} = \overline{\mu_0(\alpha)}(\,)_{p_1} = \mu_0(\alpha)(\,)_{p_1} = 0 = \text{height}(\alpha)$ and similarly $h_\mu(\alpha)_{p_2} = 0$.

$\underline{s = \sigma(s_1, s_2):}$

$$\begin{aligned}
h_\mu(\sigma(s_1, s_2))_{p_1} &= \overline{\mu_2(\sigma)}(h_\mu(s_1), h_\mu(s_2))_{p_1} \\
&= \max_{q_1, q_2 \in Q}\{h_\mu(s_1)_{q_1} + h_\mu(s_2)_{q_2} + \mu_2(\sigma)(q_1, q_2)_{p_1}\} \\
&= \max\{h_\mu(s_1)_{p_1} + h_\mu(s_2)_{p_1} + (-\infty),\ \ h_\mu(s_1)_{p_1} + h_\mu(s_2)_{p_2} + 1, \\
&\qquad\quad h_\mu(s_1)_{p_2} + h_\mu(s_2)_{p_1} + 1,\ \ h_\mu(s_1)_{p_2} + h_\mu(s_2)_{p_2} + (-\infty)\} \\
&= \max\{h_\mu(s_1)_{p_1} + h_\mu(s_2)_{p_2} + 1,\ \ h_\mu(s_1)_{p_2} + h_\mu(s_2)_{p_1} + 1\} \\
&= \max\{\text{height}(s_1) + 0 + 1,\ \ 0 + \text{height}(s_2) + 1\} \\
&\quad \text{(by the I. H.)} \\
&= 1 + \max\{\text{height}(s_1), \text{height}(s_2)\} \\
&= \text{height}(s),
\end{aligned}$$

$$\begin{aligned}
h_\mu(\sigma(s_1, s_2))_{p_2} &= \overline{\mu_2(\sigma)}(h_\mu(s_1), h_\mu(s_2))_{p_2} \\
&= \max_{q_1, q_2 \in Q}\{h_\mu(s_1)_{q_1} + h_\mu(s_2)_{q_2} + \mu_2(\sigma)(q_1, q_2)_{p_2}\} \\
&= \max\{h_\mu(s_1)_{p_1} + h_\mu(s_2)_{p_1} + (-\infty),\ \ h_\mu(s_1)_{p_1} + h_\mu(s_2)_{p_2} + (-\infty), \\
&\qquad\quad h_\mu(s_1)_{p_2} + h_\mu(s_2)_{p_1} + (-\infty),\ \ h_\mu(s_1)_{p_2} + h_\mu(s_2)_{p_2} + 0\} \\
&= h_\mu(s_1)_{p_2} + h_\mu(s_2)_{p_2} + 0 \\
&= 0 + 0 + 0 \\
&\quad \text{(by the I. H.)} \\
&= 0.
\end{aligned}$$

Then, $(\varphi_M, s) = \max_{q \in Q}\{h_\mu(s)_q + f(q)\} = \max\{h_\mu(s)_{p_1} + 0,\ h_\mu(s)_{p_2} + (-\infty)\} = \max\{\text{height}(s), -\infty\} = \text{height}(s)$. □

Let now K be a field. In the rest of this section we relate recognizability by multilinear mappings over a K-vector space and recognizability by K-Σ-ta over K (viewed as a commutative semiring).

Definition 4.4. Let $(V, +, 0)$ be a vector space over the field $(K, \oplus, \odot, 0, 1)$ of dimension $\kappa < \infty$; also let (V, μ, γ) be a multilinear representation of T_Σ. Moreover, let $M = (Q, \nu, f)$ be a K-Σ-ta over K. We say that (V, μ, γ) and M are *related* if

- Q is a basis of V,
- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $p, p_1, \ldots, p_m \in Q$, the equation $\nu_m(\sigma)(p_1, \ldots, p_m)_p = \mu_m(\sigma)(p_1, \ldots, p_m)_p$ holds, and
- for every $p \in Q$, the equation $f(p) = \gamma(p)$ holds.

Lemma 4.5. Let (V, μ, γ) and M be specified as in Definition 4.4. If (V, μ, γ) and M are related, then

- $\mu_m(\sigma)(v_1, \ldots, v_m)_p = \overline{\nu_m(\sigma)}(v_1, \ldots, v_m)_p$ for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, $v_1, \ldots, v_m \in V (= K^Q)$ and $p \in Q$,
- for every $s \in T_\Sigma$ and $p \in Q$ we have that $h_\mu(s)_p = h_\nu(s)_p$, and
- for every $s \in T_\Sigma$ we obtain: $\gamma(h_\mu(s)) = (\varphi_M, s)$.

Proof. Let (V, μ, γ) be a vector space of dimension κ, and let $M = (Q, \nu, f)$ be a K-Σ-ta where, say, $Q = \{q_1, \ldots, q_\kappa\}$. Moreover, let (V, μ, γ) and M be related. Hence we consider the two vector spaces V and K^Q as being equal. Let $\{1_{q_1}, \ldots, 1_{q_\kappa}\}$ be a basis of V where 1_{q_i} is the q_i-unit vector.

Now let $v_1, \ldots, v_m \in V$ and $p \in Q$. Thus, for every $1 \leq i \leq m$ there are $k_{i1}, \ldots, k_{i\kappa} \in K$ such that $v_i = k_{i1} 1_{q_1} + \ldots + k_{i\kappa} 1_{q_\kappa}$. Then:

$$
\begin{aligned}
& \mu_m(\sigma)(v_1, \ldots, v_m)_p \\
= \; & \Big(\textstyle\sum_{1 \leq j_1, \ldots, j_m \leq \kappa} \mu_m(\sigma)(k_{1j_1} 1_{q_{j_1}}, \ldots, k_{mj_m} 1_{q_{j_m}}) \Big)_p \\
= \; & \textstyle\bigoplus_{1 \leq j_1, \ldots, j_m \leq \kappa} \mu_m(\sigma)(k_{1j_1} 1_{q_{j_1}}, \ldots, k_{mj_m} 1_{q_{j_m}})_p \\
= \; & \textstyle\bigoplus_{1 \leq j_1, \ldots, j_m \leq \kappa} \Big(\big(\bigodot_{1 \leq i \leq m} k_{ij_i} \big) \cdot \mu_m(\sigma)(1_{q_{j_1}}, \ldots, 1_{q_{j_m}}) \Big)_p \\
& \text{(recall that } \cdot : K \times V \to V \text{ is the scalar multiplication and that } \odot \text{ is commutative)} \\
= \; & \textstyle\bigoplus_{1 \leq j_1, \ldots, j_m \leq \kappa} \big(\bigodot_{1 \leq i \leq m} k_{ij_i} \big) \odot \mu_m(\sigma)(1_{q_{j_1}}, \ldots, 1_{q_{j_m}})_p \\
= \; & \textstyle\bigoplus_{1 \leq j_1, \ldots, j_m \leq \kappa} \big(\bigodot_{1 \leq i \leq m} k_{ij_i} \big) \odot \overline{\nu_m(\sigma)}(1_{q_{j_1}}, \ldots, 1_{q_{j_m}})_p \\
= \; & \textstyle\bigoplus_{1 \leq j_1, \ldots, j_m \leq \kappa} \big(\bigodot_{1 \leq i \leq m} k_{ij_i} \big) \odot \nu_m(\sigma)(q_{j_1}, \ldots, q_{j_m})_p \\
= \; & \textstyle\bigoplus_{1 \leq j_1, \ldots, j_m \leq \kappa} \big(\bigodot_{1 \leq i \leq m} (v_i)_{q_{j_i}} \big) \odot \nu_m(\sigma)(q_{j_1}, \ldots, q_{j_m})_p \\
= \; & \textstyle\bigoplus_{p_1, \ldots, p_m \in Q} \big(\bigodot_{1 \leq i \leq m} (v_i)_{p_i} \big) \odot \nu_m(\sigma)(p_1, \ldots, p_m)_p \\
= \; & \overline{\nu_m(\sigma)}(v_1, \ldots, v_m)_p.
\end{aligned}
$$

This proves the first statement. The second statement follows trivially from the first one. Now let $s \in T_\Sigma$. Then

$(\varphi_M, s) = \bigoplus_{q \in Q} h_\nu(s)_q \odot f(q) = \bigoplus_{1 \le j \le \kappa} h_\nu(s)_{q_j} \odot f(q_j) = \bigoplus_{1 \le j \le \kappa} h_\mu(s)_{q_j} \odot \gamma(q_j) = \gamma(h_\mu(s))$. ☐

Theorem 4.6. *Let $(K, \oplus, \odot, 0, 1)$ be a field, and $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$. Then φ is recognizable by multilinear mappings over a finite-dimensional K-vector space if, and only if φ is recognizable by a K-Σ-ta over the commutative semiring K.*

Proof. Let $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ be recognizable by multilinear mappings over a K-vector space $(V, +, 0)$ of dimension $\kappa < \infty$. Thus there is a multilinear representation (V, μ, γ) of T_Σ such that $(\varphi, s) = \gamma(h_\mu(s))$ for every $s \in T_\Sigma$. It is easy to see that there is a K-Σ-ta $M = (Q, \nu, f)$ such that (V, μ, γ) and M are related. Hence, by Lemma 4.5, for every $s \in T_\Sigma$ we obtain: $\gamma(h_\mu(s)) = (\varphi_M, s)$.

Let $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ be recognizable by the K-Σ-ta $M = (Q, \nu, f)$ over (the commutative semiring) K with $Q = \{q_1, \ldots, q_\kappa\}$. Then define the κ-dimensional vector space K^Q with the componentwise addition (as usual). Consider the multilinear representation (K^Q, μ, λ) with $\mu = (\overline{\nu_m(\sigma)} \mid m \ge 0, \sigma \in \Sigma^{(m)})$ and with linear form $\gamma : K^Q \to K$ defined by $\gamma(v) = \bigoplus_{j=1}^{\kappa} k_j \odot f(q_j)$ for every $v = k_1 q_1 + \ldots + k_\kappa q_\kappa \in K^Q$. Then (V, μ, γ) and M are related and by Lemma 4.5, we have that for every $s \in T_\Sigma$ we obtain: $\gamma(h_\mu(s)) = (\varphi_M, s)$. ☐

On the contrary to Example 4.3, we can prove the following result.

Corollary 4.7. The tree series height is not recognizable by any $\underline{\mathbb{N}}$-Σ-ta.

Proof. By Proposition 3.3, height is not recognizable by multilinear mappings over any $\underline{\mathbb{Q}}$-vector space. Hence, by Theorem 4.6, it is not recognizable by any $\underline{\mathbb{Q}}$-Σ-ta, where now $\underline{\mathbb{Q}}$ is viewed as a commutative semiring. Thus, by Observation 4.2, it is not recognizable by any $\underline{\mathbb{N}}$-Σ-ta. ☐

5 Weighted tree automata over semirings

In [BV03] recognizability of tree series by a weighted tree automaton over arbitrary semirings has been defined.

A *weighted tree automaton (for short: wta) over a semiring* is a tuple $M = (Q, \Sigma, K, F, \delta)$, where Q is a finite set, Σ is a ranked alphabet, $(K, \oplus, \odot, 0, 1)$ is a semiring, $F = (F_q \mid q \in Q)$ is a Q-family with $F_q \in K$ and δ is a family $(\delta_m : \Sigma^{(m)} \to K^{Q^m \times Q} \mid m \ge 0)$ of mappings.

For every $m \ge 0$ and $\sigma \in \Sigma^{(m)}$, we define the mapping $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$ in the following way. For every $v_1, \ldots, v_m \in K^Q$ and $q \in Q$,

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} (v_1)_{q_1} \odot \ldots \odot (v_m)_{q_m} \odot \delta_m(\sigma)_{q_1 \ldots q_m, q}.$$

Based on this definition of the mappings $\overline{\delta_m(\sigma)}$, the Σ-*algebra associated with* M is the Σ-algebra $M_{alg} = (K^Q, \overline{\Sigma}_\delta)$ where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \ge 0, \sigma \in \Sigma^{(m)})$, and the *unique* Σ-*homomorphism from* T_Σ *to* M_{alg} is denoted by $h_\delta : T_\Sigma \to K^Q$.

The tree series recognized by M is the tree series $\varphi_M \in K\langle\langle T_\Sigma \rangle\rangle$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} h_\delta(s)_q \odot F_q.$$

Definition 5.1. Let K be a semiring. A tree series $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ is *recognizable by a wta over the semiring* K if there is a wta $M = (Q, \Sigma, K, F, \delta)$ such that $\varphi = \varphi_M$.

In the rest of this section we show that, for every commutative semiring K, K-Σ-ta and wta over K are equally powerful.

Definition 5.2. Let K be a commutative semiring and Σ a ranked alphabet. A K-Σ-ta $M = (Q, \mu, f)$ and a wta $N = (Q, \Sigma, K, F, \delta)$ are *related* if

- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q, q_1, \ldots, q_m \in Q$, $\mu_m(\sigma)(q_1, \ldots, q_m)_q = \delta_m(\sigma)_{q_1 \ldots q_m, q}$,
- for every $q \in Q$, we have $f(q) = F_q$.

Lemma 5.3. Let K be a commutative semiring and Σ a ranked alphabet. If a K-Σ-ta $M = (Q, \mu, f)$ and a wta $N = (Q, \Sigma, K, F, \delta)$ are related, then $\varphi_M = \varphi_N$.

Proof. If M and N are related, then it is obvious that, for every $m \geq 0$ and $\sigma \in \Sigma^{(m)}$, we have $\overline{\mu_m(\sigma)} = \overline{\delta_m(\sigma)}$; the proof of this equality needs the commutativity of $\odot$. Consequently, the algebras $(K^Q, \overline{\Sigma}_\mu)$ and $(K^Q, \overline{\Sigma}_\delta)$ are the same and thus $h_\mu = h_\delta$. Moreover, since $f(q) = F_q$ for every $q \in Q$, we obtain that, for every $s \in T_\Sigma$, the equality $(\varphi_M, s) = (\varphi_N, s)$ holds. □

Theorem 5.4. *Let* $(K, \oplus, \odot, 0, 1)$ *be a commutative semiring, and* $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$. *Then* φ *is recognizable by a* K-Σ-*ta if, and only if* φ *is recognizable by a wta over the semiring* K.

Proof. Assume that $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ is recognizable by a K-Σ-ta M. Then it is an easy task to construct a wta N such that M and N are related, cf. Definition 5.2. By Lemma 5.3, we have $\varphi = \varphi_N$, thus φ is recognizable by a wta over N. The converse of the statement can be proved similarly. □

Corollary 5.5. The tree series height is not recognizable by any wta over the semiring $\underline{\mathbb{N}}$ of natural numbers.

Proof. Since $\underline{\mathbb{N}}$ is commutative, the statement obviously follows from Corollary 4.7 and Theorem 5.4. □

6 Finite tree automata over semirings with fixpoint semantics

In [Kui98] and [ÉK03] a tree automaton model was defined over commutative and continuous semirings.

Let $\underline{K} = (K, \oplus, \odot, 0, 1)$ be a commutative and $\bigoplus$-continuous semiring. Moreover, let Σ be a ranked alphabet. A *finite tree automaton (over K and Σ)* is a tuple $M = (Q, \mathcal{M}, S, P)$ where

- Q is a finite set (of *states*),
- $\mathcal{M} = (\mathcal{M}_m \mid m \geq 0)$ is a family of *transition matrices* $\mathcal{M}_m$ such that $\mathcal{M}_m \in (K\langle\langle T_\Sigma(Y_m)\rangle\rangle)^{Q \times Q^m}$ and for almost every m it holds that every entry of $\mathcal{M}_m$ is $\tilde{0} \in K\langle\langle T_\Sigma(Y_m)\rangle\rangle$,
- $S \in (K\langle\langle T_\Sigma(Y_1)\rangle\rangle)^{1 \times Q}$ is the *initial state vector*, and
- $P \in (K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}$ is the *final state vector.*

In order to prepare the definition of the semantics we define the substitution of matrices over tree series. Let $\mathcal{M}_m \in (K\langle\langle T_\Sigma(Y_m)\rangle\rangle)^{Q \times Q^m}$ be a transition matrix and $\theta_1, \ldots, \theta_m \in (K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}$. Then we define for every $q \in Q$ the matrix $\mathcal{M}_m(\theta_1, \ldots, \theta_m) \in (K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}$ of tree series for every $q \in Q$ by

$$\mathcal{M}_m(\theta_1, \ldots, \theta_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} (\mathcal{M}_m)_{q, q_1 \ldots q_m} \leftarrow_{OI} ((\theta_1)_{q_1}, \ldots, (\theta_m)_{q_m}).$$

Since K is a $\bigoplus$-continuous semiring with $\sqsubseteq$ as partial order, $(K, \sqsubseteq)$ is a complete poset. By extending $(K, \sqsubseteq)$ to $(K\langle\langle T_\Sigma\rangle\rangle, \sqsubseteq)$ by defining $\varphi \sqsubseteq \psi$ if $(\varphi, s) \sqsubseteq (\psi, s)$ for every $s \in T_\Sigma$, and in its turn, extending $(K\langle\langle T_\Sigma\rangle\rangle, \sqsubseteq)$ to $((K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}, \sqsubseteq)$ componentwise, also $((K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}, \sqsubseteq)$ is a complete poset. Moreover, the mapping

$$\Phi : (K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1} \to (K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}$$

defined for every $\theta \in (K\langle\langle T_\Sigma\rangle\rangle)^{Q \times 1}$ by

$$\Phi(\theta) = \bigoplus_{m \geq 0} \mathcal{M}_m(\underbrace{\theta, \ldots, \theta}_{m}) \oplus P$$

is continuous (cf. page 228 of [ÉK03]; we note that, due to the restriction on $\mathcal{M}$, the family $(\mathcal{M}_m(\theta, \ldots, \theta) \oplus P \mid m \geq 0)$ is locally finite; hence the sum exists). Thus, by the fixpoint theorem (cf. Section 2.1), Φ has a least fixpoint $\operatorname{fix} \Phi$ and this is the least upper bound of the *approximation sequence* $(\Phi^n(\bot) \mid n \geq 0)$ *of M* with $\bot_q = \tilde{0}$ for every $q \in Q$, i.e.,

$$\operatorname{fix} \Phi = \sup\{\Phi^n(\bot) \mid n \geq 0\}.$$

Then, the *tree series recognized by M* is the tree series

$$\varphi_M = \bigoplus_{q \in Q} \big(S_q \leftarrow_{OI} ((\operatorname{fix} \Phi)_q)\big).$$

In order to define the notion of recognizability for this automaton model, we have to restrict it. Let $M = (Q, \mathcal{M}, S, P)$ be a finite tree automaton. We call M *polynomial* if for every $m \geq 0$ all the entries of $\mathcal{M}_m$ are in $K\langle T_\Sigma(Y_m)\rangle$; moreover, for every $q \in Q$ there is a $k \in K$ such that $S_q = k y_1$; and finally, for every $q \in Q$, the entry P_q is in $K\langle T_\Sigma\rangle$.

Definition 6.1. Let K be a commutative and continuous semiring. A tree series $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ is *recognizable by a finite, polynomial tree automaton over* K if there is a finite and polynomial tree automaton $M = (Q, \mathcal{M}, S, P)$ over K such that $\varphi = \varphi_M$.

In the following we will compare the concept of recognizability by finite, polynomial tree automata with that of weighted tree automata. For this we use a restricted version of finite, polynomial tree automata which, however, is semantically equivalent with the general model.

Let $M = (Q, \mathcal{M}, S, P)$ be a finite, polynomial tree automaton. We call M *simple and proper* if

- for every $m \geq 0$, every entry of $\mathcal{M}_m$ is of the form $\bigoplus_{\sigma \in \Sigma^{(m)}} k_\sigma \sigma(y_1, \ldots, y_m)$,
- every entry of P is of the form $\bigoplus_{\alpha \in \Sigma^{(0)}} k_\alpha \alpha$, and
- every entry of S is of the form $k y_1$

for some $k_\sigma, k_\alpha, k \in K$.

Proposition 6.2. (cf. Corollary 3.6 of [ÉK03], also cf. Corollary 4.5 of [Kui98]) Let $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ be a tree series. Then, φ is recognizable by a finite and polynomial tree automaton if, and only if φ is recognizable by a finite, simple, and proper tree automaton.

Definition 6.3. Let Σ be a ranked alphabet and $M = (Q, \mathcal{M}, S, P)$ be a finite, simple, and proper tree automaton over Σ and K. Furthermore, let $N = (Q, \Sigma, K, \delta, F)$ be a wta over K. Then M and N are called *related* if

- for every $m \geq 1$, $q_1, \ldots, q_m, q \in Q$ and $\sigma \in \Sigma^{(m)}$,

  $$((\mathcal{M}_m)_{q, q_1 \ldots q_m}, \sigma(y_1, \ldots, y_m)) = \delta_m(\sigma)_{q_1 \ldots q_m, q}$$

  and for every $q \in Q$ and $\alpha \in \Sigma^{(0)}$,

  $$((\mathcal{M}_0)_{q, \varepsilon}, \alpha) \oplus (P_q, \alpha) = \delta_0(\alpha)_{\varepsilon, q}$$

- and for every $q \in Q$, $F_q = (S_q, y_1)$.

Lemma 6.4. Let Σ, K, M, and N be as in Definition 6.3. If M and N are related, then $\varphi_M = \varphi_N$.

Proof. First let us prove the following statement by induction on n: (†) for every $s \in T_\Sigma$, $q \in Q$, $n \geq 0$, if $\mathrm{height}(s) \leq n$, then $((\Phi^{n+1}(\perp))_q, s) = h_\delta(s)_q$.

<u>$n = 0$:</u> Then $s = \alpha$ for some $\alpha \in \Sigma^{(0)}$. Then:

$$
\begin{aligned}
& h_\delta(\alpha)_q \\
= \;& \delta_0(\alpha)_{\varepsilon, q} \\
= \;& ((\mathcal{M}_0)_{q, \varepsilon}, \alpha) \oplus (P_q, \alpha) \\
& \text{(by the fact that } M \text{ and } N \text{ are related)} \\
= \;& (\mathcal{M}_0()_q, \alpha) \oplus (P_q, \alpha) \\
= \;& (\mathcal{M}_0()_q \oplus P_q, \alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= (\textstyle\bigoplus_{l\geq 0}(\mathcal{M}_l)(\perp,\dots,\perp)_q \oplus P_q, \alpha)\\
&\quad \text{(because for every } l \neq 0 \text{ we have that } ((\mathcal{M}_l)(\perp,\dots,\perp)_q, \alpha) = 0)\\
&= ((\Phi^1(\perp))_q, \alpha).
\end{aligned}
$$

$\underline{n \Rightarrow n+1}$: Let $s \in T_\Sigma$ such that $\mathrm{height}(s) \leq n+1$. Then either $s = \alpha$ or $s = \sigma(s_1,\dots,s_m)$ for some $m \geq 1$. The first case is already proved in the induction base. Now let $m \geq 1$. Then we can calculate as follows:

$$
\begin{aligned}
&h_\delta(\sigma(s_1,\dots,s_m))_q\\
&= \textstyle\bigoplus_{q_1,\dots,q_m\in Q} \bigodot_{1\leq i\leq m} h_\delta(s_i)_{q_i} \odot \delta_m(\sigma)_{q_1\dots q_m,q}\\
&= \textstyle\bigoplus_{q_1,\dots,q_m\in Q} \bigodot_{1\leq i\leq m} ((\Phi^{n+1}(\perp))_{q_i}, s_i) \odot ((\mathcal{M}_m)_{q,q_1\dots q_m}, \sigma(y_1,\dots,y_m))\\
&\quad \text{(by I.H. and the fact that } M \text{ and } N \text{ are related)}\\
&= ((\mathcal{M}_m)(\Phi^{n+1}(\perp),\dots,\Phi^{n+1}(\perp))_q, \sigma(s_1,\dots,s_m))\\
&\quad \text{(by definition of } \leftarrow_{OI})\\
&= ((\mathcal{M}_m)(\Phi^{n+1}(\perp),\dots,\Phi^{n+1}(\perp))_q \oplus P_q, \sigma(s_1,\dots,s_m))\\
&\quad \text{(because } m \geq 1 \text{ and thus } (P_q, \sigma(s_1,\dots,s_m)) = 0)\\
&= (\textstyle\bigoplus_{l\geq 0}(\mathcal{M}_l)(\Phi^{n+1}(\perp),\dots,\Phi^{n+1}(\perp))_q \oplus P_q, \sigma(s_1,\dots,s_m))\\
&\quad \text{(because } m \geq 1 \text{ and thus } ((\mathcal{M}_l)(\Phi^{n+1}(\perp),\dots,\Phi^{n+1}(\perp))_q, \sigma(s_1,\dots,s_m)) = 0\\
&\quad \text{for every } l \neq m)\\
&= ((\Phi^{n+2}(\perp))_q, \sigma(s_1,\dots,s_m)).
\end{aligned}
$$

We note that for every $s \in T_\Sigma$, $n \geq 0$, and $q \in Q$, if $\mathrm{height}(s) = n$, then

1. for every $0 \leq j \leq n$, we have that $((\Phi^j(\perp))_q, s) = 0$ and
2. for every $j > n$, we have that $((\Phi^j(\perp))_q, s) = h_\delta(s)_q$.

Statement (1) is obvious, because for $j < n$, $(\Phi^j(\perp))_q$ did not yet build up trees of height n. Statement (2) follows from Statement (†). Then for every $s \in T_\Sigma$, we calculate as follows:

$$
\begin{aligned}
&(\varphi_M, s)\\
&= \big(\textstyle\bigoplus_{q\in Q} S_q \leftarrow_{OI} ((\mathrm{fix}\,\Phi)_q), s\big)\\
&= \textstyle\bigoplus_{q\in Q}(S_q \leftarrow_{OI} ((\mathrm{fix}\,\Phi)_q), s)\\
&= \textstyle\bigoplus_{q\in Q}(S_q, y_1) \odot ((\mathrm{fix}\,\Phi)_q, s)\\
&= \textstyle\bigoplus_{q\in Q} F_q \odot \big((\sup_{(K\langle\langle T_\Sigma\rangle\rangle)^{Q\times 1}}\{\Phi^j(\perp) \mid j \geq 0\})_q, s\big)\\
&\quad \text{(by the fact that } M \text{ and } N \text{ are related and by definition of } \mathrm{fix}\,\Phi)\\
&= \textstyle\bigoplus_{q\in Q} F_q \odot \big(\sup_{K\langle\langle T_\Sigma\rangle\rangle}\{(\Phi^j(\perp))_q \mid j \geq 0\}, s\big)\\
&= \textstyle\bigoplus_{q\in Q} F_q \odot \sup_K\{((\Phi^j(\perp))_q, s) \mid j \geq 0\}\\
&= \textstyle\bigoplus_{q\in Q} F_q \odot ((\Phi^{\mathrm{height}(s)+1}(\perp))_q, s)\\
&\quad \text{(by Statements (1) and (2))}\\
&= \textstyle\bigoplus_{q\in Q} F_q \odot h_\delta(s)_q
\end{aligned}
$$

(by Statement (†))

$$= \bigoplus_{q\in Q} h_\delta(s)_q \odot F_q$$

$$= (\varphi_N, s).$$

□

Theorem 6.5. *Let K be a commutative and continuous semiring and $\varphi \in K\langle\langle T_\Sigma\rangle\rangle$. Then φ is recognizable by a finite, polynomial tree automaton over K if, and only if φ is recognizable by a wta over K.*

Proof. Let $M = (Q, \mathcal{M}, S, P)$ be a finite and polynomial tree automaton over Σ and K such that $\varphi = \varphi_M$. By Proposition 6.2 we can assume that M is finite, simple, and proper. Then construct the wta $N = (Q, \Sigma, K, \delta, F)$ by using the equations of Definition 6.3 as definition of δ and F. Clearly, M and N are related and hence $\varphi = \varphi_N$.

Now let $N = (Q, \Sigma, K, \delta, F)$ be wta such that $\varphi = \varphi_N$. We construct the finite, polynomial tree automaton $M = (Q, \mathcal{M}, S, P)$ over Σ and K by letting $(P_q, \alpha) = 0$ for every $\alpha \in \Sigma^{(0)}$ and $S_q = F_q y_1$ for every $q \in Q$. We use the equations of Definition 6.3 as definition of $\mathcal{M}$. Then we see that M is a finite, simple, and proper. Again, M and N are related and hence $\varphi = \varphi_M$. □

7 Polynomially-weighted tree automata over semirings

In [Sei92, Sei94] the concept of weighted tree automata over semirings was extended by associating with every transition a polynomial over some semiring.

Let $(K, \oplus, \odot, 0, 1)$ be a semiring. The set of *m-ary polynomial symbols over K* is the set $T_{\{\cdot^{(2)},+^{(2)}\}}(K \cup X_m)$, which we abbreviate to $P_m(K)$. Every polynomial symbol $f \in P_m(K)$ induces an m-ary mapping, called polynomial function (or just polynomial), $\overline{f} : K^m \to K$ which is defined as follows for every $a_1, \ldots, a_m \in K$:

(i) if $f = a \in K$, then $\overline{f}(a_1, \ldots, a_m) = a$,

(ii) if $f = x_j \in X_m$, then $\overline{f}(a_1, \ldots, a_m) = a_j$, for every index $1 \le j \le m$,

(iii) if $f = f_1 + f_2$, then $\overline{f}(a_1, \ldots, a_m) = \overline{f_1}(a_1, \ldots, a_m) \oplus \overline{f_2}(a_1, \ldots, a_m)$, and

(iv) if $f = f_1 \cdot f_2$, then $\overline{f}(a_1, \ldots, a_m) = \overline{f_1}(a_1, \ldots, a_m) \odot \overline{f_2}(a_1, \ldots, a_m)$.

We denote the class of m-ary polynomial functions also by $P_m(K)$.

A *polynomially-weighted tree automaton (for short: pwta) over a semiring* is a tuple $M = (Q, \Sigma, K, F, \delta)$, where Q is a finite set, Σ is a ranked alphabet, $(K, \oplus, \odot, 0, 1)$ is a semiring, $F = (F_q \mid q \in Q)$ is a Q-family of unary polynomials $F_q \in P_1(K)$, and δ is a family $(\delta_m : \Sigma^{(m)} \to P_m(K)^{Q^m \times Q} \mid m \ge 0)$ of mappings.

For every $m \ge 0$, $\sigma \in \Sigma^{(m)}$, we define the mapping $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$ in the following way. For every $v_1, \ldots, v_m \in K^Q$ and $q \in Q$,

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1,\ldots,q_m \in Q} \delta_m(\sigma)_{q_1\cdots q_m,q}((v_1)_{q_1}, \ldots, (v_m)_{q_m}).$$

Based on this definition of the mappings $\overline{\delta_m(\sigma)}$, the Σ-*algebra associated with* M is the Σ-algebra $M_{alg} = (K^Q, \overline{\Sigma}_\delta)$ where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$, and the *unique* Σ-*homomorphism from* T_Σ *to* M_{alg} is denoted by $h_\delta : T_\Sigma \to K^Q$.

The tree series recognized by M is the tree series $\varphi_M \in K\langle\langle T_\Sigma\rangle\rangle$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} F_q(h_\delta(s)_q).$$

Definition 7.1. Let K be a semiring. A tree series $\varphi \in K\langle\langle T_\Sigma\rangle\rangle$ is *recognizable by a pwta over the semiring* K if there is a pwta $M = (Q, \Sigma, K, F, \mu)$ such that $\varphi = \varphi_M$.

Theorem 7.2. *Let* $(K, \oplus, \odot, 0, 1)$ *be a semiring and* $\varphi \in K\langle\langle T_\Sigma\rangle\rangle$. *If* φ *is recognizable by a wta over the semiring* K, *then* φ *is recognizable by a pwta over the semiring* K.

Proof. Let $M = (Q, \Sigma, K, F, \delta)$ be a wta over K such that $\varphi = \varphi_M$. We construct the pwta $M' = (Q, \Sigma, K, F', \delta')$ over K such that, for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m, q \in Q$, we have $\delta'_m(\sigma)_{q_1 \ldots q_m, q} = \lambda x_1, \ldots, x_m.(x_1 \odot \ldots \odot x_m \odot \delta_m(\sigma)_{q_1 \ldots q_m, q})$ and $F'_q = \lambda x.(x \odot F_q)$. It should be clear that $\varphi_M = \varphi_{M'}$. □

Now we give an example of a tree series that is recognizable by a pwta over the semiring $\underline{\mathbb{N}}$.

Example 7.3. Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$, and consider the pwta $M = (\{q\}, \Sigma, \underline{\mathbb{N}}, F, \delta)$ where $\delta_0(\alpha)_{\varepsilon,q} = 2$, $\delta_1(\gamma)_{q,q} = \lambda x.(x \cdot x)$, and $F_q = \lambda x.x$. Then it should be clear that, for every $n \geq 0$,

$$(\varphi_M, \gamma^n(\alpha)) = 2^{2^n}.$$

Lemma 7.4. The tree series φ_M appearing in Example 7.3 cannot be recognized by any wta over $\underline{\mathbb{N}}$.

Proof. Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $M' = (Q, \Sigma, \underline{\mathbb{N}}, F', \delta')$ be an arbitrary wta over $\underline{\mathbb{N}}$. Let us give an upper bound for $(\varphi_{M'}, \gamma^n(\alpha))$ for $n \geq 0$. For this, let

$$m = \max\{\delta'_1(\gamma)_{p,q}, \delta'_0(\alpha)_{\varepsilon,q}, F'_q \mid p, q \in Q\}.$$

Then, it can be proved by a straightforward induction that, for every $q \in Q$ and $n \geq 0$,

$$h_{\delta'}(\gamma^n(\alpha))_q \leq |Q|^n m^{n+1},$$

where $|Q|$ is the cardinality of Q. The induction step for $n \geq 1$ works as follows:

$$\begin{aligned}
h_{\delta'}(\gamma^n(\alpha))_q &= \overline{\delta'_1(\gamma)}(h_{\delta'}(\gamma^{n-1}(\alpha)))_q \\
&= \textstyle\sum_{p \in Q} h_{\delta'}(\gamma^{n-1}(\alpha))_p \cdot \delta'_1(\gamma)_{p,q} \\
&\leq \textstyle\sum_{p \in Q} |Q|^{n-1} m^n m \\
& \text{(by the I.H. and } \delta'_1(\gamma)_{p,q} \leq m) \\
&= |Q|^n m^{n+1}.
\end{aligned}$$

Then $(\varphi_{M'}, \gamma^n(\alpha)) = \sum_{q \in Q} h_{\delta'}(\gamma^n(\alpha))_q \cdot F'_q \leq \sum_{q \in Q} |Q|^n m^{n+1} m = |Q|^{n+1} m^{n+2}$. Therefore, M' cannot recognize φ_M because $(\varphi_M, \gamma^n(\alpha)) > (\varphi_{M'}, \gamma^n(\alpha))$ for a sufficiently large n. □

Theorem 7.5. *There is a tree series that is recognizable by a pwta over the semiring* $\underline{\mathbb{N}}$ *but not recognizable by any wta over the semiring* $\underline{\mathbb{N}}$.

Proof. It follows from Example 7.3 and Lemma 7.4. □

Finally, we show an interesting property of pwta over linearly ordered semirings in which the operations are monotonic. For some input tree s, it is possible that a pwta computes coefficients $a_1, \ldots, a_m$ for the direct subtrees $s_1, \ldots, s_m$ of s and the coefficient $h_\delta(s)_q$ is smaller than $\min\{a_1, \ldots, a_m\}$. On the contrary, a wta can compute the coefficient $h_\delta(s)_q$ which is either (due to monotonicity of K) at least as big as $\max\{a_1, \ldots, a_m\}$ or can ignore the $a_1, \ldots, a_m$ by putting $h_\delta(s)_q = 0$; then, however, it cannot compute any other value different from 0.

8 Weighted tree automata over M-monoids

A set Ω of functions over some set K is compatible with Σ if, for every $m \geq 0$, $\Sigma^{(m)} \neq \emptyset$ implies $\Omega^{(m)} \neq \emptyset$. Throughout this section we mean that the M-monoid $(K, \oplus, 0, \Omega)$ is such that Ω is compatible with Σ. Moreover, we assume that $\mathrm{id}_K \in \Omega^{(1)}$ and $0^m \in \Omega^{(m)}$ for every $m \geq 0$ satisfying that $\Sigma^{(m)} \neq \emptyset$.

A *weighted tree automaton (for short: wta) over an M-monoid* is a tuple $M = (Q, \Sigma, K, F, \mu)$, where Q is a finite set, Σ is a ranked alphabet, $(K, \oplus, 0, \Omega)$ is a multioperator monoid, $F = (F_q \mid q \in Q)$ is a Q-family of functions $F_q \in \Omega^{(1)}$, and μ is a family $(\mu_m : \Sigma^{(m)} \to (\Omega^{(m)})^{Q^m \times Q} \mid m \geq 0)$ of functions.

For every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, we define the mapping $\overline{\mu_m(\sigma)} : (K^Q)^m \to K^Q$ in the following way. For every $v_1, \ldots, v_m \in K^Q$ and $q \in Q$,

$$\overline{\mu_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} \mu_m(\sigma)_{q_1 \ldots q_m, q}((v_1)_{q_1}, \ldots, (v_m)_{q_m}).$$

The Σ-*algebra associated with* M is the Σ-algebra $M_{alg} = (K^Q, \overline{\Sigma}_\mu)$ where $\overline{\Sigma}_\mu = (\overline{\mu_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$, and the *initial* Σ-*homomorphism to* M_{alg} is denoted by $h_\mu : T_\Sigma \to K^Q$.

The tree series recognized by M is the tree series $\varphi_M \in A\langle\langle T_\Sigma \rangle\rangle$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} F_q(h_\mu(s)_q).$$

Definition 8.1. Let $(K, \oplus, 0, \Omega)$ be an M-monoid. A tree series $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ is *recognizable by wta over* K if there is a wta $M = (Q, \Sigma, K, F, \mu)$ over K such that $\varphi = \varphi_M$.

First we show that if we consider wta over semiring M-monoids, i.e., in particular, the operations of Ω are multilinear, then they are equally powerful as wta over the involved semiring.

Definition 8.2. Let $M = (Q, \Sigma, K, F, \delta)$ be a wta over the commutative semiring $(K, \oplus, \odot, 0, 1)$ and $N = (Q, \Sigma, K, F', \mu)$ a wta over the semiring M-monoid $(K, \oplus, \odot, 0, 1, \Omega)$. We say that M and N are *related* if

- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m, q \in Q$ we have that

$$\mu_m(\sigma)_{q_1 \ldots q_m, q}(\underbrace{1, \ldots, 1}_{m}) = \delta_m(\sigma)_{q_1 \ldots q_m, q},$$

- and for every $q \in Q$, we have that $F'_q(1) = F_q$.

Lemma 8.3. Let M and N be as in Definition 8.2. Then $h_\mu = h_\delta$ and $\varphi_M = \varphi_N$.

Proof. Let $m \geq 0$, $\sigma \in \Sigma^{(m)}$, $s_1, \ldots, s_m \in T_\Sigma$, and $s = \sigma(s_1, \ldots, s_m)$. Then

$$\begin{aligned}
h_\delta(\sigma(s_1, \ldots, s_m))_q &= \textstyle\bigoplus_{q_1, \ldots, q_m \in Q} h_\delta(s_1)_{q_1} \odot \ldots \odot h_\delta(s_m)_{q_m} \odot \delta_m(\sigma)_{q_1 \ldots q_m, q} \\
&= \textstyle\bigoplus_{q_1, \ldots, q_m \in Q} \left(\bigodot_{1 \leq i \leq m} h_\delta(s_i)_{q_i} \right) \odot \delta_m(\sigma)_{q_1 \ldots q_m, q} \\
&\quad \text{(by commutativity)} \\
&= \textstyle\bigoplus_{q_1, \ldots, q_m \in Q} \left(\bigodot_{1 \leq i \leq m} h_\delta(s_i)_{q_i} \right) \odot \mu_m(\sigma)_{q_1 \ldots q_m, q}(\underbrace{1, \ldots, 1}_{m}) \\
&= \textstyle\bigoplus_{q_1, \ldots, q_m \in Q} \mu_m(\sigma)_{q_1 \ldots q_m, q}(h_\delta(s_1)_{q_1}, \ldots, h_\delta(s_m)_{q_m}) \\
&\quad \text{(by multilinearity)} \\
&= h_\mu(\sigma(s_1, \ldots, s_m))_q.
\end{aligned}$$

Moreover, for every $s \in T_\Sigma$, we have that $(\varphi_M, s) = \bigoplus_{q \in Q} h_\delta(s)_q \odot F_q = \bigoplus_{q \in Q} h_\delta(s)_q \odot F'_q(1) = \bigoplus_{q \in Q} F'_q(h_\delta(s)_q) = \bigoplus_{q \in Q} F'_q(h_\mu(s)_q) = (\varphi_N, s)$. □

Theorem 8.4. *Let $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$. Then, φ is recognizable by a wta over the semiring M-monoid $(K, \oplus, \odot, 0, 1, \Omega)$ if, and only if φ is recognizable by a wta over the commutative semiring $(K, \oplus, \odot, 0, 1)$.*

Proof. Let $N = (Q, \Sigma, K, F', \mu)$ be a wta over the semiring M-monoid $(K, \oplus, \odot, 0, 1, \Omega)$. Then consider the wta $M = (Q, \Sigma, K, F, \delta)$ over the commutative semiring $(K, \oplus, \odot, 0, 1)$ where

- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m, q \in Q$ we define

$$\delta_m(\sigma)_{q_1 \ldots q_m, q} = \mu_m(\sigma)_{q_1 \ldots q_m, q}(\underbrace{1, \ldots, 1}_{m}),$$

- and for every $q \in Q$ we define $F_q = F'_q(1)$.

Then, M and N are related in the sense of Definition 8.2, and by Lemma 8.3 we have that $\varphi_M = \varphi_N$.

Vice versa, let φ be recognizable by some wta $M = (Q, \Sigma, K, F, \delta)$ over the commutative semiring $(K, \oplus, \odot, 0, 1)$. Then we construct the wta $N = (Q, \Sigma, K, F', \mu)$ over the semiring M-monoid $(K, \oplus, \odot, 0, 1, \Omega)$ where

- $\Omega = \{\omega_{m,k} \mid 0 \leq m \leq \max \Sigma, k \in K\}$, where $\max \Sigma = \max\{r \mid \Sigma^{(r)} \neq \emptyset\}$, and the operation $\omega_{m,k} : K^m \to K$ is defined for every $k_1, \ldots, k_m \in K$ by $\omega_{m,k}(k_1, \ldots, k_m) = \bigodot_{1 \leq i \leq m} k_i \odot k$ (note that $\omega_{m,k}$ is multilinear)

- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m, q \in Q$ we define $\mu_m(\sigma)_{q_1 \cdots q_m, q} = \omega_{m,k}$ where $k = \delta_m(\sigma)_{q_1 \ldots, q_m, q}$, and
- for every $q \in Q$ we define $F'_q \in \Omega^{(1)}$ by $F'_q = \omega_{1,F_q}$.

Again it is clear that M and N are related in the sense of Definition 8.2 and by Lemma 8.3 we have that $\varphi_M = \varphi_N$. ☐

Next we relate pwta and wta over M-monoids. Since polynomials are special functions, it can easily be shown that tree series recognizable by pwta over a semiring K are also recognizable by a wta over a suitable M-monoid. In order to find this M-monoid, we associate an M-monoid $\underline{P}(K)$ with every semiring K.

Definition 8.5. Let Σ be a ranked alphabet and $(K, \oplus, \odot, 0, 1)$ a semiring. The M-monoid $\underline{P}(K) = (K, \oplus, 0, \Omega)$ is defined as follows. For every $m \geq 0$, let $\Omega^{(m)} = P_m(K)$ if $\Sigma^{(m)} \neq \emptyset$ and $\Omega^{(m)} = \emptyset$ otherwise.

Thus the following result is trivial.

Theorem 8.6. *Let K be a semiring and $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$. Then φ is recognizable by a pwta over the semiring K if, and only if φ is recognizable by a wta over the M-monoid $\underline{P}(K)$.*

Now we give an M-monoid and an example of a tree series that is recognizable by a wta over that M-monoid. For this, let $\underline{\mathbb{N}}_{\text{exp}} = (\mathbb{N}, +, 0, \Omega)$ be the M-monoid, where $+$ is the addition over $\mathbb{N}$, $\Omega^{(0)} = \{0, 1\}$ and $\Omega^{(1)} = \{0^1, \mathrm{id}_{\mathbb{N}}, \lambda x.2^x\}$.

Example 8.7. Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$, and consider the wta $M = (\{q\}, \Sigma, \underline{\mathbb{N}}_{\text{exp}}, F, \delta)$ where $\delta_0(\alpha)_{\varepsilon,q} = 1$, $\delta_1(\gamma)_{q,q} = \lambda x.2^x$, and $F_q = \lambda x.x$. Then it should be clear that, for every $n \geq 0$,

$$(\varphi_M, \gamma^n(\alpha)) = \exp(2, n),$$

where $\exp(m, 0) = 1$ and $\exp(m, n+1) = m^{\exp(m,n)}$.

Next we are going to show that φ_M cannot be recognized by any ptwa over the semiring $\underline{\mathbb{N}}$.

Lemma 8.8. The tree series φ_M appearing in Example 8.7 cannot be recognized by any pwta over the semiring $\underline{\mathbb{N}}$.

Proof. We prove by contradiction. Therefore, let us assume that there is a pwta $N = (Q, \Sigma, \underline{\mathbb{N}}, F', \delta')$ such that $\varphi_M = \varphi_N$. Let

$$H = \{\delta'_1(\gamma)_{p,q}, F'_q \mid p, q \in Q\},\ f = \sum_{g \in H} g \text{ and } a = \max\{\delta'_0(\alpha)_{\varepsilon,q} \mid q \in Q\}.$$

In the rest of the proof we use without reference the fact that unary polynomials over $\mathbb{N}$ are monotonic, i.e, for every $g \in P_1(\underline{\mathbb{N}})$ and $a, b \in \mathbb{N}$, if $a \leq b$, then $g(a) \leq g(b)$. Moreover, we use that, for every $H' \subseteq H$, $f' = \sum_{g \in H'} g$ and $a \in \mathbb{N}$, we have $f'(a) \leq f(a)$.

First we prove the following two statements.

<u>*Statement 1.*</u> For every $n \geq 0$ and $q \in Q$, the approximation $h_{\delta'}(\gamma^n(\alpha))_q \leq f^n(a)$ holds.

Proof of Statement 1. We prove by induction on n. In case $n = 0$, we get

$$h_{\delta'}(\gamma^0(\alpha))_q = h_{\delta'}(\alpha)_q = \delta'_0(\alpha)_{\varepsilon,q} \le a = f^0(a).$$

Assume Statement 1 holds for $n-1$. Then

$$\begin{aligned} h_{\delta'}(\gamma^n(\alpha))_q &= (\overline{\delta'_1(\gamma)}(h_{\delta'}(\gamma^{n-1}(\alpha)))_q \\ &= \textstyle\sum_{p\in Q} \delta'_1(\gamma)_{p,q}(h_{\delta'}(\gamma^{n-1}(\alpha))_p) \\ &\le \textstyle\sum_{p\in Q} \delta'_1(\gamma)_{p,q}(f^{n-1}(a)), \\ &\ \text{(by the induction hypothesis)} \\ &= (\textstyle\sum_{p\in Q} \delta'_1(\gamma)_{p,q})(f^{n-1}(a)), \\ &\le f(f^{n-1}(a)) = f^n(a). \end{aligned}$$

Statement 2. For every $n \ge 0$, $(\varphi_N, \gamma^n(\alpha)) \le f^{n+1}(a)$.

Proof of Statement 2. We can calculate as follows.

$$\begin{aligned} (\varphi_N, \gamma^n(\alpha)) &= \textstyle\sum_{q\in Q} F'_q(h_{\delta'}(\gamma^n(\alpha))_q) \\ &\le \textstyle\sum_{q\in Q} F'_q(f^n(a)) \\ &\ \text{(by Statement 1)} \\ &= (\textstyle\sum_{q\in Q} F'_q)(f^n(a)) \\ &\le f(f^n(a)) = f^{n+1}(a). \end{aligned}$$

This finishes the proof of Statement 2.

Now, by $\varphi_M = \varphi_N$ we obtain that,

$$\exp(2,n) = (\varphi_M, \gamma^n(\alpha)) = (\varphi_N, \gamma^n(\alpha)) \le f^{n+1}(a),$$

which is a contradiction. □

Theorem 8.9. *There is a tree series which is recognizable by a wta over the M-monoid $\underline{\mathbb{N}}_{\exp}$ and not recognizable by a pwta over $\underline{\mathbb{N}}$.*

Proof. The statement follows from Example 8.7 and Lemma 8.8. □

Roughly speaking, the previous theorem says that, by using exponentiation rather than multiplication, we can compute other tree series than by pwta over the usual semiring $\underline{\mathbb{N}}$. This is not surprising, because we have achieved the different power by restricting the operations of the underlying algebra to polynomials. But once allowing to change the underlying algebra freely, we might also ask the following questions: let K be an M-monoid and φ a tree series, which is recognizable by a wta over K. Then, is there a semiring K' such that $K \subseteq K'$ and φ is recognizable by a wta over K'? A partial answer was given in [Mal05] for deterministic wta. A wta $(Q, \Sigma, K, F, \delta)$ over an M-monoid is *deterministic* if for every $m \ge 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m \in Q$, there is at most one $q \in Q$ such that $\delta_m(\sigma)_{q_1\ldots q_m,q} \ne 0_m$.

Theorem 8.10. *(cf. Corollary 1 of [Mal05]). Let K be a distributive M-monoid and φ be a tree series which is recognizable by a deterministic wta over K. Then there is a semiring K' such that $K \subseteq K'$ and φ is recognizable by a wta over K'.*

Clearly, the semiring K' is rather sophisticated: it simulates the application of an operation to its m arguments in a stepwise fashion.

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