Inclusion Diagrams for Classes of Deterministic Bottom-up Tree-to-Tree-Series Transformations

Andreas Maletti*

Faculty of Computer Science, Dresden University of Technology
D–01062 Dresden, Germany. E-mail: maletti@tcs.inf.tu-dresden.de

Abstract

In this paper we investigate the relationship between classes of tree-to-tree-series (for short: t-ts) and o-tree-to-tree-series (for short: o-t-ts) transformations computed by restricted deterministic bottom-up weighted tree transducers (for short: deterministic bu-w-tt). Essentially, deterministic bu-w-tt are deterministic bottom-up tree series transducers [EFV02, FV03a, FV03b, FGV04], but the former are defined over monoids whereas the latter are defined over semirings and only use the multiplicative monoid thereof. In particular, the common restrictions of non-deletion, linearity, totality, and homomorphism [Eng75] can equivalently be defined for deterministic bu-w-tt.

Using well-known results of classical tree transducer theory (cf., e.g., [Eng75, Fül91]) and also new results on deterministic bu-w-tt, we order classes of t-ts and o-t-ts transformations computed by restricted deterministic bu-w-tt by set inclusion. More precisely, for every commutative monoid we completely specify the inclusion relation of the classes of t-ts and o-t-ts transformations for all sensible combinations of restrictions by means of inclusion diagrams.

1 Introduction

Bottom-up tree series transducers [Kui99, EFV02, FV03a, FGV04] were introduced as a generalization of bottom-up tree transducers [Rou70, Tha70, Eng75] and bottom-up weighted tree automata [Sei94, Kui97a, Boz99]. Bottom-up weighted tree automata have been applied to code selection in compilers [FSW94] and tree pattern matching [Sei92]. Moreover, a rich theory of bottom-up tree transducers was developed (cf. [Eng75, Bak79, Eng82, GS84, GS97, NP92, CDG+97] as seminal or survey papers and monographs) during the seventies, whereas bottom-up weighted tree automata just recently received more attention (e.g., [Sei92, Sei94, Kui97a, Bor03, BV03, DPV03, DV03, ÉK03]).

In [EFV02, FV03a, FV03b, FGV04] several generalizations of well-known theorems of the theory of tree transducers have been proved for bottom-up tree series transducers, e.g.,

- the generalization of the decomposition of the class of bottom-up tree transformations (cf. Theorem 5.7 of [EFV02] and Page 220 of [Eng75]); in its turn the result of [Eng75] generalizes the decomposition of gsm-mappings as proved in [Niv68];
- the generalization of (some) composition hierarchy results for bottom-up tree transformation classes (cf. Theorem 6.24 of [FGV04] and Corollary 8.13(iii) of [GS84]);
- the generalization of the equivalence of a rewrite semantics and the initial algebra semantics for bottom-up tree transducers (cf. Theorem 5.10 of [FV03b] and Lemma 5.6 of [Eng75]).

Roughly speaking, a bottom-up tree series transducer is a bottom-up tree transducer in which the transitions carry a weight; a weight is an element of some semiring. The rewrite semantics works as follows. Along a successful computation on some input tree, the weights of the involved transitions are combined by means of the semiring multiplication; if there is more than one successful computation for some pair of input- and output-trees, the weights of these computations are combined by means of the semiring addition.

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In this paper we deal with deterministic bottom-up tree series transducers. In this case, for every input tree there is at most one successful computation (cf. Proposition 3.12 of [EFV02]) and thus the semiring addition is irrelevant. Hence we base our investigations on so-called deterministic bottom-up weighted tree transducers (for short: deterministic bu-w-tt) over some multiplicative monoid. Essentially, these are deterministic bottom-up tree series transducers over some semiring of which only the multiplicative part is used. More formally, a deterministic bu-w-tt is a tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$, wherein Q is a finite set of states, Σ and Δ are ranked alphabets of input and output symbols, respectively, $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ is a monoid with an absorbing element $\mathbf{0}$, $F \subseteq Q$ is a set of final states, $\delta = (\, \delta^k_\sigma : Q^k \longrightarrow Q \,)_{k\in\mathbb{N},\, \sigma\in\Sigma^{(k)}}$ is a Σ-indexed family of transition mappings, and $\mu = (\, \mu^k_\sigma : Q^k \longrightarrow A[T_\Delta(X_k)] \,)_{k\in\mathbb{N},\, \sigma\in\Sigma^{(k)}}$ is a Σ-indexed family of output mappings. Therein $T_\Delta(X_k)$ denotes the set of all Δ-trees indexed by variables of $X_k = \{x_1, \ldots, x_k\}$ and $A[T_\Delta(X_k)]$ denotes the set of all monomials over A and $T_\Delta(X_k)$, i.e., the set of all mappings $\varphi : T_\Delta(X_k) \longrightarrow A$ which map at most one output tree $t \in T_\Delta(X_k)$ to a monoid element different from $\mathbf{0}$. Using pure substitution $\longleftarrow$ and o-substitution $\overset{o}{\longleftarrow}$ of tree series [EFV02, FV03a] in order to substitute monomials into a monomial, we can define $\widehat{\delta} : T_\Sigma \longrightarrow Q$ as the unique homomorphism from the initial Σ-algebra T_Σ to the Σ-algebra $(Q, (\, \delta^k_\sigma \,)_{k\in\mathbb{N}, \sigma\in\Sigma^{(k)}})$. Moreover, for every modifier $\mathrm{mod} \in \{\varepsilon, o\}$ we can define the mapping $\widehat{\mu}_{\mathrm{mod}} : T_\Sigma \longrightarrow A[T_\Delta]$ by

$$\widehat{\mu}_{\mathrm{mod}}(\sigma(s_1, \ldots, s_k)) = \mu^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k)) \overset{\mathrm{mod}}{\longleftarrow} (\widehat{\mu}_{\mathrm{mod}}(s_1), \ldots, \widehat{\mu}_{\mathrm{mod}}(s_k)).$$

The mod-t-ts transformation $\tau^{\mathrm{mod}}_M : T_\Sigma \longrightarrow A[T_\Delta]$ computed by M is then defined to be

$$\tau^{\mathrm{mod}}_M(s) = \begin{cases} \widehat{\mu}_{\mathrm{mod}}(s) & \text{, if } \widehat{\delta}(s) \in F \\ \widetilde{\mathbf{0}} & \text{, otherwise} \end{cases},$$

where $\widetilde{\mathbf{0}}$ denotes the monomial which maps every output tree to $\mathbf{0}$. Thus a deterministic bu-w-tt either produces no output (i.e., $\widetilde{\mathbf{0}}$) or a single output tree t weighted with a monoid element a, i.e., the monomial $a\,t$. Deterministic bu-w-tt over the (multiplicative) monoid $\mathbb{Z}_2 = (\{0,1\}, \cdot, 1)$ with the absorbing element 0 essentially are deterministic bottom-up tree transducers (cf. Section 4 of [EFV02]).

In the same way as for deterministic bottom-up tree transducers or deterministic bottom-up tree series transducers, we can also define restrictions for deterministic bu-w-tt, e.g., the restrictions of non-deletion, linearity, totality, and homomorphism (cf., e.g., [Eng75]). The class of mod-t-ts transformations computed by deterministic bu-w-tt obeying the restrictions π (e.g., being a non-deleting homomorphism) over the monoid $\mathcal{A}$ is denoted by $\pi\text{–BOT}^{\mathrm{mod}}(\mathcal{A})$. Usually we abbreviate the restrictions by their first letter, e.g., h abbreviates homomorphism, and use juxtaposition of the letters to denote a combination of restrictions, e.g., hn for non-deleting homomorphism.

Our main results are in the inclusion diagrams contained in Section 4 (cf. Theorem 4.8, Theorem 4.17, Theorem 4.20, Theorem 4.23, and Theorem 4.25). Specifically, we can conclude that

- the monoid $\mathbb{Z}_2$ is (up to isomorphism) the only monoid $\mathcal{A}$ such that for every combination π of restrictions the equality $\pi\text{–BOT}^o(\mathcal{A}) = \pi\text{–BOT}(\mathcal{A})$ holds (cf. Corollary 4.6), and
- idempotent monoids $\mathcal{A}$ are the only monoids where $\text{hn–BOT}^o(\mathcal{A}) = \text{hn–BOT}(\mathcal{A})$ holds (cf. Corollary 4.16).

In the following let us consider combinations π of restrictions which do not contain the homomorphism restriction. It turns out that

- for every monoid $\mathcal{A}$ we have $\pi\text{–BOT}(\mathcal{A}) = \pi\text{–BOT}^o(\mathcal{A})$, if both the non-deletion and linearity restriction are present in π (cf. Theorem 5.5 of [FV03a] and Observation 4.4),
- for every periodic and commutative monoid $\mathcal{A}$ we have $\pi\text{–BOT}^o(\mathcal{A}) \subseteq \pi\text{–BOT}(\mathcal{A})$, whenever the non-deletion restriction is present in π (cf. Corollary 4.12),

- for every periodic and commutative monoid $\mathcal{A}$ we have $\pi\text{–BOT}(\mathcal{A}) \subseteq \pi\text{–BOT}^o(\mathcal{A})$, whenever the linearity restriction is present in π (cf. Corollary 4.12),
- for every periodic, commutative, and regular monoid $\mathcal{A}$ we have $\pi\text{–BOT}(\mathcal{A}) \subseteq \pi\text{–BOT}^o(\mathcal{A})$ independently of non-deletion or linearity (cf. Corollary 4.18), and
- for every periodic and commutative group $\mathcal{A}$ we have $\pi\text{–BOT}^o(\mathcal{A}) = \pi\text{–BOT}(\mathcal{A})$ independently of non-deletion or linearity (cf. Corollary 4.24).

In the remaining cases for commutative monoids $\mathcal{A}$ and combinations π of restrictions we have that $\pi\text{–BOT}^o(\mathcal{A})$ and $\pi\text{–BOT}(\mathcal{A})$ are incomparable with respect to set inclusion. In particular, if the monoid $\mathcal{A}$ is non-periodic, then for every combination π of restrictions not containing both the non-deletion and linearity restriction we obtain the incomparability of $\pi\text{–BOT}^o(\mathcal{A})$ and $\pi\text{–BOT}(\mathcal{A})$ (cf. Lemma 4.7).

This paper is structured as follows. Section 2 reviews the relevant basic mathematical notions and notations, in particular partial orders, trees and bottom-up tree transducers, monoids and semirings, and substitutions of formal tree series. Section 3 recalls the definition of deterministic bottom-up tree series transducers from [EFV02], introduces deterministic bu-w-tt along with the outlined restrictions. Moreover we relate the notions of deterministic bottom-up tree series transducer, deterministic bu-w-tt, and deterministic tree transducer. Finally, Section 4 details the inclusion diagrams obtained for the various subclasses of t-ts and o-t-ts transformations computed by restricted deterministic bu-w-tt. The inclusion diagrams will be complete in the sense that we present an inclusion diagram for every commutative monoid with an absorbing element $\mathbf{0}$.

2 Preliminaries

In this section we present some basic notions and notations required in the sequel. The first subsection recalls partial orders [DP02] and associated notions. Words, trees, and tree transducers [MS97, GS84, GS97] are considered in the second subsection, whereas the third subsection is dedicated to algebraic structures and, in particular, monoids [Jac85, Jac89] and semirings [Kui97b, HW98, Gol99]. Finally, the section is concluded by the presentation of formal tree series [BR82, Kui97b] and tree series substitution [EFV02, FV03a].

2.1 Partial orders

The set $\{0, 1, 2, \ldots\}$ of all non-negative integers is denoted by $\mathbb{N}$, and the set $\{1, 2, \ldots\}$ of all positive integers is denoted by $\mathbb{N}_+$. For every two integers $i, j \in \mathbb{N}$ the subset $\{ k \in \mathbb{N} \mid i \le k \le j \}$ is abbreviated by the interval $[i, j]$. In particular, we use the shorthand $[j]$ instead of $[1, j]$. Recall that $\operatorname{card}(S)$ denotes the *cardinality*, i.e., the number of elements, of a finite set S, hence $\operatorname{card}([j]) = j$. The *power set* of a set S is the set of all its subsets, i.e., $\mathcal{P}(S) = \{ S' \mid S' \subseteq S \}$, and the set of all finite subsets is $\mathcal{P}_{\mathrm{f}}(S) = \{ S' \subseteq S \mid S' \text{ is finite} \}$. Finally we write $f : S_1 \longrightarrow S_2$ for a total mapping from the non-empty set S_1 into the non-empty set S_2. The *range of* f is then defined to be the set $\{ f(s_1) \mid s_1 \in S_1 \}$.

Given a non-empty set S, a binary relation $\le \, \subseteq S \times S$ is called *partial order (on S)*, if $\le$ is (i) *reflexive*, i.e., for every element $s \in S$ we have $s \le s$, (ii) *anti-symmetric*, i.e., for every two elements $s_1, s_2 \in S$ the facts $s_1 \le s_2$ and $s_2 \le s_1$ imply $s_1 = s_2$, and (iii) *transitive*, i.e., for every three elements $s_1, s_2, s_3 \in S$ with $s_1 \le s_2$ and $s_2 \le s_3$ also $s_1 \le s_3$ holds.

A partial order $\le \, \subseteq S \times S$, which fulfils for every two elements $s_1, s_2 \in S$ the condition that $s_1 \le s_2$ or $s_2 \le s_1$, is said to be a *total order*. Contrary, the fact that neither $s_1 \le s_2$ nor $s_2 \le s_1$ (or equivalently: s_1 and s_2 are *incomparable*) is expressed as $s_1 \bowtie s_2$. As usual the *strict order* $< \, \subseteq S \times S$ is derived from the partial order $\le$ by setting $s_1 < s_2$, if and only if $s_1 \le s_2$ and $s_1 \ne s_2$. Moreover, we define the *covering relation* $\lessdot \, \subseteq S \times S$ derived from the partial order $\le$ by setting $s_1 \lessdot s_2$, if $s_1 < s_2$ and for every $s \in S$ the condition $s_1 \le s < s_2$ implies $s = s_1$. Whenever $s_1 \lessdot s_2$ we say that s_1 *is covered by* s_2.

Finite partial orders can be visualized by means of HASSE *diagrams* [DP02]. A HASSE diagram is a (directed, acyclic, and unlabeled) graph $G = (S, \lessdot)$ with the set S of vertices and the set $\lessdot$ of edges, i.e., there is a directed edge from vertex $s_1 \in S$ to vertex $s_2 \in S$, if and only if $s_1 \lessdot s_2$. In pictorial expressions the vertices are displayed by naming the element of S and the edges are drawn as line segments connecting vertices, where we assume that all edges are directed upwards and a line segment is only supposed to intersect with a vertex, if the vertex is either its starting or ending point. An *inclusion diagram* [FV98] is a HASSE diagram for a set $\mathfrak{S}$ which is partially ordered by set inclusion $\subseteq$; thus the elements of $\mathfrak{S}$ are again sets.

Finally, a binary relation $\sim \subseteq S \times S$ is said to be an *equivalence relation*, if $\sim$ is (i) reflexive, (ii) transitive, and (iii) *symmetric*, i.e., for every two elements $s_1, s_2 \in S$ the property $s_1 \sim s_2$ implies $s_2 \sim s_1$. The *equivalence class* of $s \in S$ (with respect to $\sim$) is the set $[s]_\sim = \{\, s' \in S \mid s \sim s' \,\}$.

2.2 Words, trees, and bottom-up tree transducers

By a *word of length* $n \in \mathbb{N}$ we mean an element of the n-fold Cartesian product $S^n = S \times \cdots \times S$ of a set S. The set of all words over S is denoted by S^*, where the particular element $() \in S^0$, called the *empty word*, is displayed as ε, and the *length of a word* $w \in S^*$ is denoted by $|w|$; thus $|\varepsilon| = 0$.

Every non-empty and finite set S is called *alphabet*, of which elements are termed *symbols*. A *ranked alphabet* is defined to be a pair (Σ, rk), of which Σ is an alphabet and $\mathrm{rk} : \Sigma \longrightarrow \mathbb{N}$ is a total mapping associating to every symbol of Σ its *rank*. For every $n \in \mathbb{N}$ we use $\Sigma^{(n)}$ to denote the set of symbols having rank n, i.e., $\Sigma^{(n)} = \{\, \sigma \in \Sigma \mid \mathrm{rk}(\sigma) = n \,\}$. In the following we will usually assume the rk-mapping to be implicitly given, identify (Σ, rk) with Σ, and specify the ranked alphabet by listing the elements of Σ with their ranks put in parentheses as superscripts as, for example, in $\{\sigma^{(2)}, \alpha^{(0)}\}$. Moreover, we generally suppose that $\Sigma^{(0)} \neq \emptyset$ for apparent reasons.

In the following let Σ be a ranked alphabet and $X = \{\, x_i \mid i \in \mathbb{N}_+ \,\}$ be a fixed countable set of (formal) variables. The set of (finite, labeled, and ordered) Σ-*trees indexed by* $V \subseteq X$, denoted by $T_\Sigma(V)$, is inductively defined to be the smallest set T such that (i) $V \cup \Sigma^{(0)} \subseteq T$ and (ii) for every $k \in \mathbb{N}_+$, symbol $\sigma \in \Sigma^{(k)}$, and k elements $s_1, \ldots, s_k \in T$ also $\sigma(s_1, \ldots, s_k) \in T$. The set T_Σ of *ground* trees is an abbreviation for $T_\Sigma(\emptyset)$. Moreover, given a tree $s \in T_\Sigma(V)$ and a unary symbol $\gamma \in \Sigma^{(1)}$, we abbreviate the tree

$$\underbrace{\gamma(\gamma(\cdots(\gamma(s))\cdots))}_{n\text{-times } \gamma}$$

simply by $\gamma^n(s)$. Note that $\gamma^0(s) = s$.

The number of occurrences of a given variable or symbol $z \in V \cup \Sigma$ in a Σ-tree $s \in T_\Sigma(V)$ indexed by V is denoted by $|s|_z$. For every integer $n \in \mathbb{N}$ we denote the set $\{x_1, \ldots, x_n\} \subset X$ by the shorthand X_n (note that $X_0 = \emptyset$). Given an integer $n \in \mathbb{N}$, a Σ-tree $s \in T_\Sigma(X_n)$ indexed by X_n, and trees $t_1, \ldots, t_n \in T_\Sigma(V)$, the expression $s[t_1, \ldots, t_n]$ denotes the result of replacing (in parallel) for every index $i \in [n]$ every occurrence of x_i in the tree s by the tree t_i, i.e., $x_i[t_1, \ldots, t_n] = t_i$ for every index $i \in [n]$ and $(\sigma(s_1, \ldots, s_k))[t_1, \ldots, t_n] = \sigma(s_1[t_1, \ldots, t_n], \ldots, s_k[t_1, \ldots, t_n])$ for every $k \in \mathbb{N}$, symbol $\sigma \in \Sigma^{(k)}$, and k trees $s_1, \ldots, s_k \in T_\Sigma(X_n)$. Moreover, for tree languages $L, L_1, \ldots, L_k \subseteq T_\Sigma$ we use $L[L_1, \ldots, L_k] = \bigcup_{s \in L, t_1 \in L_1, \ldots, t_k \in L_k} s[t_1, \ldots, t_k]$.

Let $Y \subset X$ be a finite subset of X and let $s \in T_\Sigma(X)$ be a Σ-tree indexed by X. The tree s is called *non-deleting in* Y (likewise *linear in* Y), if every variable $y \in Y$ occurs at least once, i.e., $1 \leq |s|_y$, (likewise at most once, i.e., $|s|_y \leq 1$) in the tree s. We recursively define the standard mappings $\mathrm{size}, \mathrm{height} : T_\Sigma(V) \longrightarrow \mathbb{N}_+$ by the following equalities:

- for every tree $s \in V \cup \Sigma^{(0)}$ we have $\mathrm{size}(s) = 1 = \mathrm{height}(s)$,
- for every integer $k \in \mathbb{N}_+$, symbol $\sigma \in \Sigma^{(k)}$, and k trees $s_1, \ldots, s_k \in T_\Sigma(V)$ we have

$$\mathrm{size}(\sigma(s_1, \ldots, s_k)) = 1 + \sum_{i \in [k]} \mathrm{size}(s_i) \qquad \text{and} \qquad \mathrm{height}(\sigma(s_1, \ldots, s_k)) = 1 + \max_{i \in [k]} \mathrm{height}(s_i).$$

Let Σ be a ranked alphabet in which just one symbol is non-nullary, i.e., $\bigcup_{n\in\mathbb{N}_+}\Sigma^{(n)}=\{\sigma\}$. The set of *fully balanced (and symmetric) trees (over Σ)* is defined to be the smallest subset $T\subseteq T_\Sigma$ such that $\Sigma^{(0)}\subseteq T$, and given a fully balanced tree $s\in T$, the tree $\sigma(s,\ldots,s)\in T$ is fully balanced. Note that if $\mathrm{card}(\Sigma^{(0)})=1$, then the height of a fully balanced tree already characterizes the tree uniquely. This property certainly is not fulfilled for general Σ-trees over an arbitrary ranked alphabet Σ, but for each given height $n\in\mathbb{N}_+$ there exist only finitely many Σ-trees of T_Σ having height n. Likewise this property holds for the size which is stated in the next observation.

2.1 Observation (Finite equivalence classes)
Let Σ be a ranked alphabet. Moreover, let $\equiv_{\mathrm{size}}\subseteq T_\Sigma\times T_\Sigma$ be the equivalence relation defined for every two trees $s_1,s_2\in T_\Sigma$ by $s_1\equiv_{\mathrm{size}}s_2$, if and only if $\mathrm{size}(s_1)=\mathrm{size}(s_2)$. Then for every tree $s\in T_\Sigma$ the equivalence class $[s]_{\equiv_{\mathrm{size}}}$ is finite. □

Finally, we shortly recall the concept of a deterministic bottom-up tree transducer [Rou70, Tha70, Eng75, GS84] (splitting up a rule into its state behavior and the computed output in an obvious way). A *deterministic bottom-up tree transducer* is a tuple $M=(Q,\Sigma,\Delta,F,\delta,\mu)$, where Q and $F\subseteq Q$ are finite sets of *states* and *final states*, respectively, Σ and Δ are the *input* and *output ranked alphabets*, respectively, $\delta=(\,\delta^k_\sigma:\ Q^k\longrightarrow Q\,)_{k\in\mathbb{N},\sigma\in\Sigma^{(k)}}$ is a family of *transition mappings*, and $(\,\mu^k_\sigma:\ Q^k\longrightarrow\mathcal{P}_{\mathrm{f}}(T_\Delta(X_k))\,)_{k\in\mathbb{N},\sigma\in\Sigma^{(k)}}$ is a family of *output mappings*. Additionally, for every integer $k\in\mathbb{N}$, input symbol $\sigma\in\Sigma^{(k)}$, and k states $q_1,\ldots,q_k\in Q$ we require $\mathrm{card}(\mu^k_\sigma(q_1,\ldots,q_k))\le 1$. The semantics of deterministic bottom-up tree transducers is defined inductively as follows. Let $\widehat{\delta}:T_\Sigma\longrightarrow Q$ be the mapping with $\widehat{\delta}(\sigma(s_1,\ldots,s_k))=\delta^k_\sigma(\widehat{\delta}(s_1),\ldots,\widehat{\delta}(s_k))$ for every integer $k\in\mathbb{N}$, input symbol $\sigma\in\Sigma^{(k)}$, and k trees $s_1,\ldots,s_k\in T_\Sigma$. Further let

$$\widehat{\mu}:\ T_\Sigma\longrightarrow\mathcal{P}_{\mathrm{f}}(T_\Delta)\quad\text{with}\quad\widehat{\mu}(\sigma(s_1,\ldots,s_k))=\mu^k_\sigma(\widehat{\delta}(s_1),\ldots,\widehat{\delta}(s_k))[\widehat{\mu}(s_1),\ldots,\widehat{\mu}(s_k)].$$

The *tree transformation computed by M* is the mapping $\tau_M:\ T_\Sigma\longrightarrow\mathcal{P}_{\mathrm{f}}(T_\Delta)$ defined by

$$\tau_M(s)=\{\,t\in\widehat{\mu}(s)\mid\widehat{\delta}(s)\in F\,\}.$$

Note that $\mathrm{card}(\tau_M(s))\le 1$ for every input tree $s\in T_\Sigma$. The class of tree transformations computable by deterministic bottom-up tree transducers will be denoted by d–$\mathrm{BOT}_{\mathrm{tt}}$.

2.3 Monoids and semirings

A *monoid* is an algebraic structure $\mathcal{A}=(A,\otimes,\mathbf{1})$ consisting of a *carrier (set)* A together with a binary operation $\otimes:\ A^2\longrightarrow A$ and a constant element $\mathbf{1}\in A$, such that the operation $\otimes$ is *associative*, i.e., for every three elements $a_1,a_2,a_3\in A$ the equality $a_1\otimes(a_2\otimes a_3)=(a_1\otimes a_2)\otimes a_3$ is satisfied, and the constant element $\mathbf{1}$ is the *unit element* with respect to operation $\otimes$, i.e., for every element $a\in A$ we demand $\mathbf{1}\otimes a=a=a\otimes\mathbf{1}$. Further, the monoid $\mathcal{A}$ is said to be *commutative*, if for every two elements $a_1,a_2\in A$ the equality $a_1\otimes a_2=a_2\otimes a_1$ is fulfilled. The monoid $\mathcal{A}$ possesses an *absorbing element* $\mathbf{0}\in A$, if for every $a\in A$ the equality $a\otimes\mathbf{0}=\mathbf{0}=\mathbf{0}\otimes a$ holds. If an absorbing element exists, then it is necessarily unique. Moreover, it can be adjoined to every monoid not possessing an absorbing element. To show this, let $(A,\otimes,\mathbf{1})$ be a monoid and $\mathbf{0}\notin A$ be a new element. Then $(A\cup\{\mathbf{0}\},\odot,\mathbf{1})$ with $a_1\odot a_2=a_1\otimes a_2$, if $a_1,a_2\in A$ and otherwise $a_1\odot a_2=\mathbf{0}$, is a monoid with an absorbing element, namely $\mathbf{0}$. We denote a monoid $(A,\odot,\mathbf{1})$ possessing the absorbing element $\mathbf{0}$ by $(A,\odot,\mathbf{1},\mathbf{0})$. For the sake of simplicity we assume that for no monoid considered, the element $\mathbf{1}$ is an absorbing element, i.e., we ignore the trivial monoid with the singleton carrier set.

Let $\mathcal{A}=(A,\otimes,\mathbf{1})$ be a monoid. As usual, for every element $a\in A$ and integer $n\in\mathbb{N}$ we denote by a^n the n-fold product $a\otimes\cdots\otimes a$ and set $a^0=\mathbf{1}$. Further, given an integer $n\in\mathbb{N}$ and a family $(\,a_i\,)_{i\in[n]}$ of elements $a_i\in A$, we also use the *product (notation)* $\prod_{i\in[n]}a_i=a_1\otimes\cdots\otimes a_n$, where the order is determined by the total order $1<2<\cdots$ on the index set. Note that $\prod_{i\in[0]}a_i=\mathbf{1}$. Next we define some common properties of monoids. The monoid $\mathcal{A}$ is said to be

- *finite*, if A is finite,
- *idempotent*, if for every element $a \in A$ we have $a \otimes a = a$,
- *periodic*, if for every element $a \in A$ there exist non-negative integers $i, j \in \mathbb{N}$ such that $i \neq j$ and $a^i = a^j$.
- *regular*, if for every element $a \in A$ there exists an element $a' \in A$, also called *a weak inverse of* a, such that $a \otimes a' \otimes a = a$, and
- *a group*, if for every element $a \in A$ there exists an element $a' \in A$, also called *the inverse of* a, such that $a \otimes a' = \mathbf{1} = a' \otimes a$.

We denote groups by $(A, \otimes, (\cdot)^{-1}, \mathbf{1})$, where $(\cdot)^{-1} : A \longrightarrow A$ maps each element to its (unique) inverse. Furthermore, we say that a monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ with an absorbing $\mathbf{0}$ is a group (with an absorbing zero) and denote this by $(A, \odot, (\cdot)^{-1}, \mathbf{1}, \mathbf{0})$, if for every element $a \in A \setminus \{\mathbf{0}\}$ there exists an inverse element. The following observation collects some trivial interrelations between the aforementioned properties.

2.2 Observation (Interrelations of the properties)
Let $\mathcal{A} = (A, \otimes, \mathbf{1})$ be a monoid. We observe the following implications between properties of $\mathcal{A}$.

(i) Finiteness implies periodicity.

(ii) Idempotency implies periodicity and regularity.

(iii) If $\mathcal{A}$ is a group, then $\mathcal{A}$ is also regular and for every element $a \in A$ the equality $a = a^2$ implies $a = \mathbf{1}$. □

Important monoids possessing an absorbing element include

- the multiplicative *monoid of the non-negative integers* $\mathbb{N} = (\mathbb{N}, \cdot, 1, 0)$ with the common operation of multiplication,
- the additive *group of the integers* $\mathbb{Z}_\infty = (\{\mathbb{Z} \cup \{+\infty\}, +, 0, (+\infty))$ with the usual addition on integers $\mathbb{Z}$ extended to $(+\infty)$ such that $(+\infty)$ is an absorbing element,
- the multiplicative group $\mathbb{Z}_2 = (\{0, 1\}, \cdot, 1, 0)$ with multiplication modulo 2,
- the multiplicative group $\mathbb{Z}_3 = (\{0, 1, 2\}, \cdot, 1, 0)$ with multiplication modulo 3,
- the multiplicative monoid $\mathbb{Z}_4 = (\{0, 1, 2, 3\}, \cdot, 1, 0)$ with multiplication modulo 4,
- the multiplicative monoid $\mathbb{Z}_6 = (\{0, 1, 2, 3, 4, 5\}, \cdot, 1, 0)$ with multiplication modulo 6,
- the max-*monoid over the reals* $\mathbb{R}_{\max} = (\mathbb{R} \cup \{+\infty, -\infty\}, \max, (-\infty), (+\infty))$ with the standard maximum operation on the reals $\mathbb{R}$, and
- the *language monoid* $\mathbb{L}_S = (\mathcal{P}(S^*), \circ, \{\varepsilon\}, \emptyset)$ for some alphabet S with concatenation of words lifted to sets of words as multiplication.

The properties of the introduced monoids are summarized in Table 1, where we assume that S is a non-trivial alphabet, i.e., $1 < \mathrm{card}(S)$, otherwise $\mathbb{L}_S$ is commutative.

By a *semiring (with one and absorbing zero)* we mean an algebraic structure $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with the operations of *addition* $\oplus : A^2 \longrightarrow A$ and *multiplication* $\odot : A^2 \longrightarrow A$, of which $(A, \oplus, \mathbf{0})$, also called the *additive monoid*, and $(A, \odot, \mathbf{1}, \mathbf{0})$, also called the multiplicative monoid, are monoids. Additionally, the former monoid is required to be commutative, the latter possesses $\mathbf{0}$ as an absorbing element, and the monoids are connected via the *distributivity laws*, i.e., for every three elements $a_1, a_2, a_3 \in A$ the equalities $a_1 \odot (a_2 \oplus a_3) = (a_1 \odot a_2) \oplus (a_1 \odot a_3)$ and

monoid	commutative	finite	idempotent	periodic	regular	group
$\mathbb{N}$	yes	NO	NO	NO	NO	NO
$\mathbb{Z}_\infty$	yes	NO	NO	NO	yes	yes
$\mathbb{Z}_2$	yes	yes	yes	yes	yes	yes
$\mathbb{Z}_3$	yes	yes	NO	yes	yes	yes
$\mathbb{Z}_4$	yes	yes	NO	yes	NO	NO
$\mathbb{Z}_6$	yes	yes	NO	yes	yes	NO
$\mathbb{R}_{\max}$	yes	NO	yes	yes	yes	NO
$\mathbb{L}_S$	NO	NO	NO	NO	NO	NO

Table 1: Various monoids and their properties.

$(a_1 \oplus a_2) \odot a_3 = (a_1 \odot a_3) \oplus (a_2 \odot a_3)$ hold. A *commutative semiring* $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is defined to be a semiring, in which the monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ is commutative.

In semirings we use the product notation of the multiplicative monoid and the *sum (notation)* $\sum_{i\in I} a_i$ for every index set I such that only finitely many $a_i \in A$ with $i \in I$ are different from $\mathbf{0}$. Note that the order is obviously irrelevant due to commutativity and note further that $\sum_{i\in[0]} a_i = \mathbf{0}$. By convention, we assume that multiplication has a higher (binding) priority than addition, e.g., we read $a_1 \oplus a_2 \odot a_3$ as $a_1 \oplus (a_2 \odot a_3)$. Examples of semirings can be found, for example, in [HW98, Gol99].

2.3 Observation (Not every multiplicative monoid is suitable for a semiring)
There exists a monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ with an absorbing $\mathbf{0}$ such that there does not exist a semiring $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$.

Proof. We firstly provide the operation table of such a monoid $\mathcal{A} = (\{\mathbf{0}, \mathbf{1}, a, b\}, \odot, \mathbf{1}, \mathbf{0})$ which is even commutative.

$\odot$	$\mathbf{0}$	$\mathbf{1}$	a	b
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	a	b
a	$\mathbf{0}$	a	a	b
b	$\mathbf{0}$	b	b	a

Now suppose there exists a commutative monoid $(A, \oplus, \mathbf{0})$ such that $(\{\mathbf{0}, \mathbf{1}, a, b\}, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring. Consider the sum $\mathbf{1} \oplus b$.

<u>Case 1:</u> Let $\mathbf{1} \oplus b \in \{\mathbf{1}, a\}$. Then by distributivity $a \odot (\mathbf{1} \oplus b) = a \oplus b = a$, but $b \odot (\mathbf{1} \oplus b) = b \oplus a = b$. Hence $a \oplus b \neq b \oplus a$ which is contradictory.

<u>Case 2:</u> Let $\mathbf{1} \oplus b = b$. Then again by distributivity $a \odot (\mathbf{1} \oplus b) = a \oplus b = b$, but $b \odot (\mathbf{1} \oplus b) = b \oplus a = a$. Hence $b \oplus a \neq a \oplus b$ which is contradictory.

<u>Case 3:</u> Let $\mathbf{1} \oplus b = \mathbf{0}$. Consider the sum

$$(\mathbf{1} \oplus b) \oplus a = a \neq \mathbf{1} = \mathbf{1} \oplus a \odot \mathbf{0} = \mathbf{1} \oplus a \odot (\mathbf{1} \oplus b) = \mathbf{1} \oplus a \oplus b = \mathbf{1} \oplus (b \oplus a),$$

which is a contradiction to associativity. ■

However, we can always embed the multiplicative monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ into a semiring as follows. Let $\perp \notin A$ be a new element and let $A' = A \cup \{\perp\}$. Further, define the operations $\oplus, \otimes : A' \times A' \longrightarrow A'$ for every $a_1, a_2 \in A'$ by

$$a_1 \oplus a_2 = \begin{cases} \mathbf{0} & \text{, if } a_1, a_2 \in A \\ a_1 & \text{, if } a_2 = \perp \\ a_2 & \text{, otherwise} \end{cases} \quad \text{and} \quad a_1 \otimes a_2 = \begin{cases} a_1 \odot a_2 & \text{, if } a_1, a_2 \in A \\ \perp & \text{, otherwise} \end{cases}.$$

Then $(A', \oplus, \otimes, \perp, \mathbf{1})$ is a semiring (with a new zero).

2.4 Formal tree series

Let Δ be a ranked alphabet and additionally $V \subseteq X$ be a subset of variables. Every total mapping $\varphi : T_\Delta(V) \longrightarrow A$ from Δ-trees indexed by V into a non-empty set A is called *formal tree series (over Δ, V, and A)*. We use $A\langle\langle T_\Delta(V)\rangle\rangle$ to denote the set of all formal tree series over Δ, V, and A. Given a tree $t \in T_\Delta(V)$, we usually write (φ, t), termed the *coefficient* of t, instead of $\varphi(t)$ and $\sum_{t \in T_\Delta(V)} (\varphi, t)\, t$ instead of the tree series φ, in order to follow the established conventions. For example,

$$\sum_{t \in T_\Delta(V)} \mathrm{size}(t)\, t$$

is the tree series, which associates to every tree its size.

Let $(A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid with an absorbing $\mathbf{0}$ and $\varphi \in A\langle\langle T_\Delta(V)\rangle\rangle$ be a tree series. The *support* of φ is defined to be the set $\mathrm{supp}(\varphi) = \{\, t \in T_\Delta(V) \mid (\varphi, t) \neq \mathbf{0}\,\}$. Whenever the set $\mathrm{supp}(\varphi)$ is finite, we say that φ is a *polynomial*, and moreover, a polynomial φ is said to be a *monomial*, if $\mathrm{card}(\mathrm{supp}(\varphi)) \leq 1$. The set of all monomial (likewise polynomial) formal tree series (over Δ, V, and A) is denoted by $A[T_\Delta(V)]$ (likewise $A\langle T_\Delta(V)\rangle$). A tree series $\varphi \in A\langle\langle T_\Delta(V)\rangle\rangle$ is said to be *boolean*, if for every tree $t \in T_\Delta(V)$ the coefficient obeys $(\varphi, t) \in \{\mathbf{0}, \mathbf{1}\}$. Provided a subset $L \subseteq T_\Delta(V)$ we define the *characteristic tree series of L* by

$$(\mathrm{char}(L), t) = \begin{cases} \mathbf{1} & \text{, if } t \in L \\ \mathbf{0} & \text{, otherwise} \end{cases}$$

for every tree $t \in T_\Delta(V)$. Note that $\mathrm{char}(L)$ is boolean and $\mathrm{char}(L) \in A\langle T_\Delta(V)\rangle$ if and only if $L \in \mathcal{P}_{\mathrm{f}}(T_\Delta(V))$. Moreover, $\mathrm{char}(L) \in A[T_\Delta(V)]$ if and only if $L \in \mathcal{P}_{\mathrm{f}}(T_\Delta(V))$ and $\mathrm{card}(L) \leq 1$.

In addition, if there is an element $a \in A$ such that for every tree $t \in T_\Delta(V)$ the coefficient $(\varphi, t) = a$ is constant, then the tree series φ is said to be *constant* and we use $\widetilde{a}$ to abbreviate such a tree series φ. Hence a monomial φ obeys either $\varphi = \widetilde{\mathbf{0}}$ or $\mathrm{card}(\mathrm{supp}(\varphi)) = 1$, thus $\varphi = a\, t$ for some monoid element $a \in A$ and tree $t \in T_\Delta(V)$.

Provided that $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is even a semiring, then we can define the sum of two tree series $\psi_1, \psi_2 \in A\langle\langle T_\Delta(V)\rangle\rangle$ pointwise for every tree $t \in T_\Delta(V)$ to be $(\psi_1 \oplus \psi_2, t) = (\psi_1, t) \oplus (\psi_2, t)$. Tree substitution can then be generalized to tree languages [ES77, ES78] as well as tree series over semirings. Following the IO-substitution approach, the common definition of tree series substitution found, for example, in [EFV02] lets $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring, $n \in \mathbb{N}$ be an integer, $\varphi \in A\langle T_\Delta(X_n)\rangle$ be a tree series, and $(\psi_1, \ldots, \psi_n) \in A\langle T_\Delta(V)\rangle^n$ be an n-tuple of tree series. *(Pure) substitution* of the tuple $(\psi_1, \ldots, \psi_n)$ *into* the tree series φ, denoted by $\varphi \longleftarrow (\psi_1, \ldots, \psi_n)$, is then defined by

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \mathrm{supp}(\varphi), \\ (\forall i \in [n]) \colon t_i \in \mathrm{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i) \Big)\, t[t_1, \ldots, t_n].$$

Irrespective of the number of occurrences of a formal variable x_i for some $i \in [n]$, the coefficient (ψ_i, t_i) is taken into account exactly once, even if the variable does not appear at all in the tree t. This particularity led to the introduction of a different notion of substitution defined in [FV03a] as follows.

$$\varphi \overset{o}{\longleftarrow} (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \mathrm{supp}(\varphi), \\ (\forall i \in [n]) \colon t_i \in \mathrm{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i)^{|t|_{x_i}} \Big)\, t[t_1, \ldots, t_n]$$

This notion of substitution, called *o-substitution*, takes the coefficient (ψ_i, t_i) into account as often as the corresponding formal variable x_i appears in the tree t. However, both notions are defined only for formal tree series over semirings. Next we will restrict the substitutions to monomials and thereby obtain notions of substitutions also defined for monoids.

Let $(A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid, $\varphi \in A[T_\Delta(X_n)]$ be a monomial, $(\psi_1, \ldots, \psi_n) \in A[T_\Delta(V)]^n$ be a n-tuple of monomials, and mod $\in \{\varepsilon, o\}$ be a modifier. The mod-*substitution of* the tuple $(\psi_1, \ldots, \psi_n)$ *into* the monomial φ, denoted by $\varphi \overset{\text{mod}}{\longleftarrow}_\star (\psi_1, \ldots, \psi_n)$, is defined for every $n+1$ elements $a, a_1, \ldots, a_n \in A \setminus \{\mathbf{0}\}$, tree $t \in T_\Delta(X_n)$, and n trees $t_1, \ldots, t_n \in T_\Delta(V)$ by the following axioms.

(i) $\varphi \overset{\text{mod}}{\longleftarrow}_\star () = \varphi$,

(ii) $\widetilde{\mathbf{0}} \overset{\text{mod}}{\longleftarrow}_\star (\psi_1, \ldots, \psi_n) = \widetilde{\mathbf{0}}$ and $\varphi \overset{\text{mod}}{\longleftarrow}_\star (\psi_1, \ldots, \psi_{i-1}, \widetilde{\mathbf{0}}, \psi_{i+1}, \ldots, \psi_n) = \widetilde{\mathbf{0}}$ for every $i \in [n]$,

(iii) $a\, t \longleftarrow_\star (a_1\, t_1, \ldots, a_n\, t_n) = (a \odot \prod_{i \in [n]} a_i)\, t[t_1, \ldots, t_n]$, and

(iii') $a\, t \overset{o}{\longleftarrow}_\star (a_1\, t_1, \ldots, a_n\, t_n) = (a \odot \prod_{i \in [n]} a_i^{|t|_{x_i}})\, t[t_1, \ldots, t_n]$.

This way (i), (ii), and (iii) characterize pure substitution on monomials, whereas (i), (ii), and (iii') characterize o-substitution on monomials. It is easily seen using Proposition 3.4 of [FV03a], that these are really the restrictions of the respective notions of substitution defined for semirings $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ to their multiplicative monoid $(A, \odot, \mathbf{1}, \mathbf{0})$, i.e.,

$$\varphi \overset{\text{mod}}{\longleftarrow} (\psi_1, \ldots, \psi_n) = \varphi \overset{\text{mod}}{\longleftarrow}_\star (\psi_1, \ldots, \psi_n).$$

Henceforth we will drop the star from the substitution on monomials.

Finally, we mention that in [Kui99] a notion of substitution based on the OI-substitution approach [ES77, ES78] is introduced. There the number of occurrences of a certain formal variable is taken into account as well. In this paper we only deal with the IO-substitution approach.

3 Deterministic bottom-up weighted tree transducers

In this section we will firstly recall the notion of a deterministic bottom-up tree series transducer [EFV02, FV03a]. Then we will present another model called deterministic bottom-up weighted tree transducer (abbreviated deterministic bu-w-tt), and show that deterministic bu-w-tt over the multiplicative monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ of a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ are equivalent to deterministic bottom-up tree series transducers over $\mathcal{A}$. The main advantage of deterministic bu-w-tt is the fact that they are defined over a monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ only and hence that we can deal with more general algebraic structures (cf. Observation 2.3). We present the necessary definitions in a compact style and refer the reader to [EFV02, FV03a] for an elaborated introduction into general tree series transducers and weighted tree transducers.

Before we proceed with the definition of deterministic bottom-up tree series transducers, we recall some basic notions concerning matrices. Let I and J be countable *index sets* and let S be a set of *entries*. An $(I \times J)$-*matrix over* S is a mapping $K : I \times J \longrightarrow S$. The set of all matrices over S with indices of $I \times J$ is denoted by $S^{I \times J}$. The element $K(i,j)$ is called the (i,j)-*entry* of the matrix K and also written as $K_{i,j}$. If it is understood that the matrix K is a *row-vector* or *column-vector* (i.e., I or J is a singleton set, respectively), then we generally omit the element of the singleton set when indexing elements of the matrix K. Accordingly, we write, for example, K^I instead of $K^{I \times \{1\}}$, whenever we do not want to stress that the matrix K is a column-vector.

Given a finite set Q of states, input and output ranked alphabet Σ and Δ, respectively, and a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, a *deterministic bottom-up tree representation (over Q, Σ, Δ, and A)* is a family $(\mu_k)_{k \in \mathbb{N}}$ of mappings, where for every $k \in \mathbb{N}$ the mapping μ_k has type

$$\mu_k : \Sigma^{(k)} \longrightarrow A[T_\Delta(X)]^{Q \times Q^k}.$$

Moreover, for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k-tuple of states $w \in Q^k$ there exists at most one state $q \in Q$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$. A *deterministic bottom-up tree series transducer (over Σ and Δ)* is defined as a six-tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, where

- Q and $F \subseteq Q$ are non-empty, finite sets of *states* and *final states*, respectively,
- Σ and Δ are the *input* and *output ranked alphabet*, respectively; both disjoint to Q;
- $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, and
- μ is a deterministic bottom-up tree representation over Q, Σ, Δ, and A.

For every modifier mod $\in \{\varepsilon, o\}$, integer $k \in \mathbb{N}$, and input symbol $\sigma \in \Sigma^{(k)}$ the deterministic bottom-up tree representation μ induces a mapping $\overline{\mu_k(\sigma)}^{\text{mod}} : (A\langle T_\Delta\rangle^Q)^k \longrightarrow A\langle T_\Delta\rangle^Q$ defined componentwise for every state $q \in Q$ and k vectors $R_1, \ldots, R_k \in A\langle T_\Delta\rangle^Q$ by

$$\overline{\mu_k(\sigma)}^{\text{mod}}(R_1, \ldots, R_k)_q = \sum_{(q_1,\ldots,q_k)\in Q^k} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \overset{\text{mod}}{\longleftarrow} \big((R_1)_{q_1}, \ldots, (R_k)_{q_k}\big).$$

Note that $(A\langle T_\Delta\rangle^Q, (\overline{\mu_k(\sigma)}^{\text{mod}})_{k\in\mathbb{N},\sigma\in\Sigma^{(k)}})$ defines a Σ-algebra, and T_Σ is the initial Σ-algebra. Thus there exists a unique homomorphism $h_\mu^{\text{mod}} : T_\Sigma \longrightarrow A\langle T_\Delta\rangle^Q$, which is defined for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k trees $s_1, \ldots, s_k \in T_\Sigma$ by

$$h_\mu^{\text{mod}}(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}^{\text{mod}}(h_\mu^{\text{mod}}(s_1), \ldots, h_\mu^{\text{mod}}(s_k)).$$

In fact, it can easily be proved by structural induction that for every input tree $s \in T_\Sigma$ we have $h_\mu^{\text{mod}}(s) \in A[T_\Delta]^Q$, hence we can replace the set $A\langle T_\Delta\rangle^Q$ by $A[T_\Delta]^Q$ in the types of $\overline{\mu_k(\sigma)}^{\text{mod}}$ and h_μ^{mod}. Finally, the mod-*tree-to-tree-series transformation*, abbreviated mod-t-ts transformation, computed by the deterministic bottom-up tree series transducer M is $\tau_M^{\text{mod}} : T_\Sigma \longrightarrow A[T_\Delta]$ specified for every input tree $s \in T_\Sigma$ by $\tau_M^{\text{mod}}(s) = \sum_{q\in F} h_\mu^{\text{mod}}(s)_q$.

3.1 Definition (Deterministic bottom-up weighted tree transducer)
A *deterministic bottom-up weighted tree transducer* (over $\mathcal{A}$), abbreviated deterministic bu-w-tt in the following, is defined as a tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ where

- Q and $F \subseteq Q$ are finite and non-empty sets of *states* and *final states*, respectively,
- Σ and Δ are the *input* and *output ranked alphabet*, respectively; both disjoint to Q;
- $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ is a monoid with an absorbing element $\mathbf{0}$,
- $\delta = (\,\delta_\sigma^k : Q^k \longrightarrow Q\,)_{k\in\mathbb{N},\sigma\in\Sigma^{(k)}}$ is a family of *state transition mappings*, and
- $\mu = (\,\mu_\sigma^k : Q^k \longrightarrow A[T_\Delta(X_k)]\,)_{k\in\mathbb{N},\sigma\in\Sigma^{(k)}}$ is a family of *output mappings*. □

The deterministic bu-w-tt M is *boolean*, if for every integer $k \in \mathbb{N}$ and input symbol $\sigma \in \Sigma^{(k)}$ every monomial in the range of μ_σ^k is boolean. We will also make use of the following syntactic restrictions of deterministic bu-w-tt. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ be a deterministic bu-w-tt; we say that M is

- *non-deleting* (likewise *linear*), if for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k states $q_1, \ldots, q_k \in Q$ the variable $x \in X_k$ appears at least once (likewise at most once), i.e., $1 \le |t|_x$ (likewise $|t|_x \le 1$), in any tree $t \in \text{supp}(\mu_\sigma^k(q_1, \ldots, q_k))$,
- *total*, if $F = Q$ and for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k states $q_1, \ldots, q_k \in Q$ we have $\mu_\sigma^k(q_1, \ldots, q_k) \neq \widetilde{\mathbf{0}}$, and
- a *homomorphism*, if M is total and Q is a singleton.

In case M is a deterministic homomorphism bu-w-tt, we will just say that M is a homomorphism bu-w-tt. Finally, we should assign a formal semantics to deterministic bu-w-tt. In fact, we define two different semantics, namely the tree-to-tree-series transformation, abbreviated t-ts transformation, and the o-tree-to-tree-series transformation, abbreviated o-t-ts transformation. Both are defined in the very same manner except for the type of substitution being used.

3.2 Definition (Semantics of deterministic bu-w-tt)
Let mod $\in \{\varepsilon, o\}$ be a modifier and $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ be a deterministic bu-w-tt over the monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$. For every input tree $s \in T_\Sigma$ we define the mappings $\widehat{\delta} : T_\Sigma \longrightarrow Q$ and $\widehat{\mu}_{\text{mod}} : T_\Sigma \longrightarrow A[T_\Delta]$ by structural recursion as follows. For every integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k subtrees $s_1, \ldots, s_k \in T_\Sigma$ we have $\widehat{\delta}(\sigma(s_1, \ldots, s_k)) = \delta^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k))$ and

$$\widehat{\mu}_{\text{mod}}(\sigma(s_1, \ldots, s_k)) = \mu^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k)) \xleftarrow{\text{mod}} (\widehat{\mu}_{\text{mod}}(s_1), \ldots, \widehat{\mu}_{\text{mod}}(s_k)).$$

The mod-*tree-to-tree-series transformation computed by* M is the mapping $\tau^{\text{mod}}_M : T_\Sigma \longrightarrow A[T_\Delta]$ specified for every input tree $s \in T_\Sigma$ by

$$\tau^{\text{mod}}_M(s) = \begin{cases} \widehat{\mu}_{\text{mod}}(s) & \text{, if } \widehat{\delta}(s) \in F \\ \widetilde{\mathbf{0}} & \text{, otherwise} \end{cases} .$$

□

3.3 Example (A deterministic bu-w-tt computing the size)
The deterministic bu-w-tt $M_{\text{size}} = (\{\star\}, \Sigma, \Sigma, \mathbb{Z}_\infty, \{\star\}, \delta, \mu)$ with input and output ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, state transition mappings $\delta = (\delta^2_\sigma, \delta^0_\alpha)$, and output mappings $\mu = (\mu^2_\sigma, \mu^0_\alpha)$ is defined by

$$\delta^2_\sigma(\star, \star) = \delta^0_\alpha() = \star \quad , \quad \mu^2_\sigma(\star, \star) = 1\,\sigma(x_1, x_2) \quad \text{, and} \quad \mu^0_\alpha() = 1\,\alpha.$$

We observe that for every input tree $s \in T_\Sigma$ we have $\tau_{M_{\text{size}}}(s) = \tau^o_{M_{\text{size}}}(s) = \text{size}(s)\ s$. Moreover, M_{size} is a linear and non-deleting homomorphism bu-w-tt, which is not boolean. □

In the sequel we investigate the computational power of various subclasses of deterministic bu-w-tt and compare their computational power by means of set inclusion. The next definition establishes shorthands for such classes of mod-t-ts transformations also taking the two different notions of substitution into account.

3.4 Definition (Classes of tree-to-tree-series transformations)
Let mod $\in \{\varepsilon, o\}$ and $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid. We define several classes of mod-t-ts transformations $\tau : T_\Sigma \longrightarrow A[T_\Delta]$ computable by restricted deterministic bu-w-tt over $\mathcal{A}$ as shown in Table 2.

Formally speaking, let Pref $= \{\text{n}, \text{l}, \text{t}, \text{h}\}$ be a set of abbreviations standing for non-deleting, linear, total, and homomorphism, respectively. Moreover, let $r \subseteq$ Pref. The class dr–BOT$^{\text{mod}}(\mathcal{A})$ denotes the class of all mod-t-ts transformations $\tau : T_\Sigma \longrightarrow A[T_\Delta]$ such that there exists a deterministic bu-w-tt $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ with $\tau^{\text{mod}}_M = \tau$ and M obeys all the restrictions abbreviated in r. Henceforth, we will omit the set braces and the separating comma and just list the letters in r. We say that r is a prefix.

Note that all sensible combinations of abbreviations are listed in Table 2 together with their corresponding set of prefixes, and we generally omit the d and the prefix t (standing for deterministic and total) in case the prefix h (standing for homomorphism) is present, because homomorphism tree transducers are deterministic and total by definition. Finally we define the set $\Pi = \{\text{d}, \text{dn}, \text{dl}, \text{dt}, \text{h}, \text{dnl}, \text{dnt}, \text{hn}, \text{dlt}, \text{hl}, \text{dnlt}, \text{hnl}\}$ of sensible combinations and the restrictions $\Pi_r = \{\, \pi \in \Pi \mid r \in \pi \,\}$ for every $r \in$ Pref. □

We note that all the restrictions and classes have been defined for deterministic bottom-up tree series transducers [EFV02, FV03a] as well. Next we establish relations between deterministic bu-w-tt, deterministic bottom-up tree series transducers, and deterministic bottom-up tree transducers.

Notation	set of prefixes	denotes the class of mod-t-ts transformations computable by ... deterministic bu-w-tt over $\mathcal{A}$
d–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\emptyset$	unrestricted
dn–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{n}\}$	non-deleting
dl–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{l}\}$	linear
dt–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{t}\}$	total
h–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{t},\mathrm{h}\}$	homomorphism
dnl–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{n},\mathrm{l}\}$	non-deleting and linear
dnt–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{n},\mathrm{t}\}$	non-deleting and total
hn–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{n},\mathrm{t},\mathrm{h}\}$	non-deleting homomorphism
dlt–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{l},\mathrm{t}\}$	linear and total
hl–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{l},\mathrm{t},\mathrm{h}\}$	linear homomorphism
dnlt–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{n},\mathrm{l},\mathrm{t}\}$	non-deleting, linear, and total
hnl–BOT$^{\mathrm{mod}}(\mathcal{A})$	$\{\mathrm{n},\mathrm{l},\mathrm{t},\mathrm{h}\}$	non-deleting and linear homomorphism

Table 2: Various classes of mod-tree-to-tree-series transformations.

Firstly, let us show that deterministic bu-w-tt over multiplicative monoids of some semiring compute the same class of mod-t-ts transformations as deterministic bottom-up tree series transducers. Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring, $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \mu_1)$ be a deterministic bottom-up tree series transducer, and $M_2 = (Q_2, \Sigma, \Delta, (A, \odot, \mathbf{1}, \mathbf{0}), F_2, \delta_2, \mu_2)$ be a deterministic bu-w-tt over the multiplicative monoid of $\mathcal{A}$. The devices M_1 and M_2 are called *related*, if $Q_1 = Q_2$, $F_1 = F_2$, and for every integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and $k + 1$ states $q, q_1, \dots, q_k \in Q$ we have

$$(\mu_1)_k(\sigma)_{q,(q_1,\dots,q_k)} \neq \widetilde{\mathbf{0}} \quad \text{implies} \quad (\delta_2)^k_\sigma(q_1, \dots, q_k) = q \text{ and } (\mu_2)^k_\sigma(q_1, \dots, q_k) = (\mu_1)_k(\sigma)_{q,(q_1,\dots,q_k)},$$

as well as $(\mu_1)_k(\sigma)_{(\delta_2)^k_\sigma(q_1,\dots,q_k),(q_1,\dots,q_k)} = (\mu_2)^k_\sigma(q_1, \dots, q_k)$. A straightforward induction on the structure of the input tree $s \in T_\Sigma$ then shows for every modifier $\mathrm{mod} \in \{\varepsilon, o\}$ that

$$(\widehat{\mu_2})_{\mathrm{mod}}(s) = h^{\mathrm{mod}}_{\mu_1}(s)_{\widehat{\delta_2}(s)}$$

and thus $\tau^{\mathrm{mod}}_{M_1}(s) = \tau^{\mathrm{mod}}_{M_2}(s)$ for related M_1 and M_2. Note that M_1 obeys the restrictions of $\pi \in \Pi$, if and only if M_2 obeys the restrictions of π.

3.5 Observation (Deterministic bu-w-tt and bottom-up tree series transducers)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring. Then for every $\pi \in \Pi$ and modifier $\mathrm{mod} \in \{\varepsilon, o\}$ we have

$$\pi\text{–BOT}^{\mathrm{mod}}(\mathcal{A}) = \pi\text{–BOT}^{\mathrm{mod}}((A, \odot, \mathbf{1}, \mathbf{0})),$$

where π–BOT$^{\mathrm{mod}}(\mathcal{A})$ denotes the class of all mod-t-ts transformations computable by bottom-up tree series transducers obeying all the restrictions of π (cf. [EFV02, FV03a]). □

Secondly, we transfer the obvious relationship between deterministic bottom-up tree transducers on the one hand and deterministic bottom-up tree series transducers over the Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ on the other hand (cf. Corollary 4.7 of [EFV02] and Corollary 5.9 of [FV03a]) to the corresponding relationship between deterministic bottom-up tree transducers and deterministic bu-w-tt over $\mathbb{Z}_2$. Let $S = \{ L \in \mathcal{P}_{\mathrm{f}}(T_\Delta) \mid \mathrm{card}(L) \leq 1 \}$ and $\sim \subseteq \mathbb{Z}_2[T_\Delta] \times S$ be the relation defined by $\varphi \sim L$, if and only if $L = \mathrm{supp}(\varphi)$. Indeed the relation $\sim$ is a bijection. Consequently, for every $\tau_1 : T_\Sigma \longrightarrow \mathbb{Z}_2[T_\Delta]$ and $\tau_2 : T_\Sigma \longrightarrow S$ let $\tau_1 \sim \tau_2$, if and only if for every input tree $s \in T_\Sigma$ we have $\tau_1(s) \sim \tau_2(s)$. Moreover, let $\sim$ also be defined on classes of mappings in the obvious way.

3.6 Observation (Deterministic bu-w-tt and bottom-up tree transducers)
For every $\pi \in \Pi$ and modifier $\mathrm{mod} \in \{\varepsilon, o\}$ we have π–BOT$^{\mathrm{mod}}(\mathbb{Z}_2) \sim \pi$–BOT$_{\mathrm{tt}}$, where π–BOT$_{\mathrm{tt}}$ denotes the class of all tree transformations computable by bottom-up tree transducers obeying all the restrictions of π (cf. [Eng75]).

Proof. In the same spirit as $\sim$ a relation between deterministic bottom-up tree transducers and deterministic bu-w-tt over the group $\mathbb{Z}_2$ can be established (cf. Corollary 4.7 of [EFV02]). More precisely, a deterministic bottom-up tree transducer $M_1 = (Q_1, \Sigma, \Delta, F_1, \delta_1, \mu_1)$ is *related to* a deterministic bu-w-tt $M_2 = (Q_2, \Sigma, \Delta, \mathbb{Z}_2, F_2, \delta_2, \mu_2)$, if $Q_1 = Q_2$, $F_1 = F_2$, $\delta_1 = \delta_2$, and for every integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k states $q_1, \ldots, q_k \in Q$ the following condition holds.

$$(\mu_1)^k_\sigma(q_1, \ldots, q_k) = \operatorname{supp}((\mu_2)^k_\sigma(q_1, \ldots, q_k)).$$

Note that for every combination $\pi \in \Pi$ we have that M_1 obeys the restrictions of π, if and only if M_2 obeys them. Moreover, if M_1 and M_2 are related, then $\tau_{M_1} \sim \tau^{\mathrm{mod}}_{M_2}$ (cf. Corollary 4.7 of [EFV02] and Corollary 5.9 of [FV03a]). The proof of the last statement is straightforward and left to the reader. ∎

Thus deterministic bottom-up tree transducers and deterministic bu-w-tt over the group $\mathbb{Z}_2$ are equally powerful, which allows us to treat deterministic bottom-up tree transducers as if they were deterministic bu-w-tt over the group $\mathbb{Z}_2$ in order to have a unique presentation.

3.7 Corollary (Corollary of Observation 3.6)
For every combination $\pi \in \Pi$ we have π–$\mathrm{BOT}^o(\mathbb{Z}_2) = \pi$–$\mathrm{BOT}(\mathbb{Z}_2)$. □

4 Inclusion diagrams

In this section we investigate the relation between classes of t-ts and o-t-ts transformations computed by deterministic bu-w-tt with respect to set inclusion. We derive several inclusion diagrams displaying the relationships given certain properties of the underlying monoid. Firstly, let us state the well-known inclusion diagram for deterministic bu-w-tt over the group $\mathbb{Z}_2$, i.e., for deterministic bottom-up tree transducers. Figure 1 displays the inclusion diagram for all classes of t-ts and o-t-ts transformations defined in Definition 3.4 (for $\mathcal{A} = \mathbb{Z}_2$). In order to present a concise diagram, we shorten the denotation of the classes from π–$\mathrm{BOT}^{\mathrm{mod}}(\mathcal{A})$ to just π^{mod} for every combination $\pi \in \Pi$ and $\mathrm{mod} \in \{\varepsilon, o\}$. Moreover, we use $\pi^{=}$ to express that π–$\mathrm{BOT}^o(\mathcal{A}) = \pi$–$\mathrm{BOT}(\mathcal{A})$.

Secondly, let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative monoid with at least three elements. In Subsection 4.1 we derive some statements which hold for every such monoid $\mathcal{A}$. Thirdly, we will consider the case that $\mathcal{A}$ is non-periodic (cf. Subsection 4.2). Subsection 4.3 is dedicated to periodic, but non-regular monoids $\mathcal{A}$. Automatically such a monoid $\mathcal{A}$ is non-idempotent and no group with an absorbing element by Observation 2.2. The next case handled in Subsection 4.4 additionally assumes that $\mathcal{A}$ is regular, but still not idempotent and no group with an absorbing element. Thereafter, we consider the case in which $\mathcal{A}$ is idempotent. This again excludes the case that $\mathcal{A}$ is a group with an absorbing element, which is finally taken care of in Subsection 4.6.

4.1 Theorem (The inclusion diagram for the group $\mathbb{Z}_2$)
Figure 1 is the inclusion diagram of the displayed classes of t-ts and o-t-ts transformations over $\mathbb{Z}_2$ ordered by set inclusion.

Proof. The equalities are concluded from Corollary 3.7 and all the inclusions hold by definition. Finally, the following four statements are sufficient to prove strictness and incomparability.

(i)	dnlt–$\mathrm{BOT}(\mathbb{Z}_2) \not\subseteq$ h–$\mathrm{BOT}(\mathbb{Z}_2)$,	(ii)	dnl–$\mathrm{BOT}(\mathbb{Z}_2) \not\subseteq$ dt–$\mathrm{BOT}(\mathbb{Z}_2)$,
(iii)	hn–$\mathrm{BOT}(\mathbb{Z}_2) \not\subseteq$ dl–$\mathrm{BOT}(\mathbb{Z}_2)$,	(iv)	hl–$\mathrm{BOT}(\mathbb{Z}_2) \not\subseteq$ dn–$\mathrm{BOT}(\mathbb{Z}_2)$.

The non-inclusions (i) and (ii) are trivial and the non-inclusions (iii) and (iv) are due to Theorem 3.3 of [Fül91]. ∎

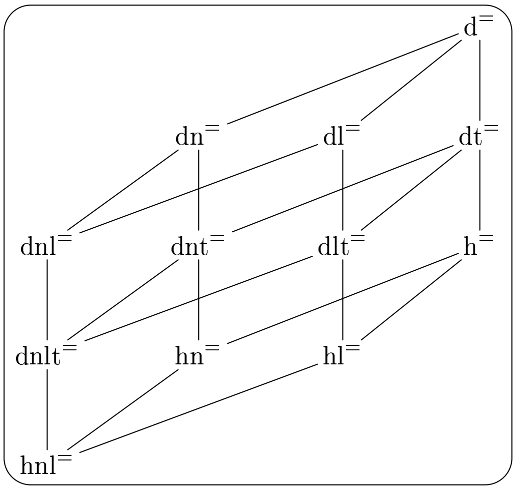

Figure 1: Inclusion diagram for the group $\mathbb{Z}_2$.

4.1 Results for arbitrary monoids

In this subsection we derive some statements which hold irrespective of the underlying monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$. Firstly, we show how to use the results of the inclusion diagram in Figure 1 in order to obtain incomparability results for classes of t-ts and o-t-ts transformations over monoids $\mathcal{A}$ different from $\mathbb{Z}_2$. Roughly speaking, we show that all non-inclusions present in Figure 1 are preserved in the transition from $\mathbb{Z}_2$ to $\mathcal{A}$. This is mainly due to the fact that $\mathbb{Z}_2$ is a submonoid (with absorbing $\mathbf{0}$) of $\mathcal{A}$. Hence we take a counterexample in $\mathbb{Z}_2$, i.e., a mod_1-t-ts transformation τ which is in the class $\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathbb{Z}_2)$, but not in class $\pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$ for some modifiers $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ and $\pi_1, \pi_2 \in \Pi$, and then prove that τ is also a counterexample for the inclusion $\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, i.e., τ is trivially in $\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$ because $\mathbb{Z}_2$ is a submonoid of $\mathcal{A}$, but still not in $\pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

4.2 Lemma (Lifting lemma)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid and $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ be two modifiers. Furthermore let $\pi_1 \in \Pi_{\mathrm{t}}$ and $\pi_2 \in \Pi$ be two prefixes.

If $\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathbb{Z}_2) \not\subseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$, then also $\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

Proof. Let $\tau \in \pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathbb{Z}_2) \setminus \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$ be a mod_1-t-ts transformation, hence there exists a deterministic bu-w-tt M' obeying the restrictions π_1 such that $\tau = \tau_{M'}^{\mathrm{mod}_1}$. Apparently, $\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathbb{Z}_2) \subseteq \pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$, because $\mathbb{Z}_2$ is a submonoid with an absorbing $\mathbf{0}$ of $\mathcal{A}$. Thus there exists a total deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π_1 such that $\tau_{M_1}^{\mathrm{mod}_1} = \tau$. Note that $\widehat{\mu_1}_{\mathrm{mod}_1}(s) \neq \widetilde{\mathbf{0}}$ for every input tree $s \in T_\Sigma$.

Now we prove by contradiction that $\tau \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$. Assume that $\tau \in \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, i.e., there exists a deterministic bu-w-tt $M_2 = (Q_2, \Sigma, \Delta, \mathcal{A}, F_2, \delta_2, \mu_2)$ obeying the restrictions of π_2 such that $\tau_{M_2}^{\mathrm{mod}_2} = \tau$. The remaining proof first shows that there also exists a boolean deterministic bu-w-tt M'' obeying the restrictions of π_2 such that $\tau_{M''}^{\mathrm{mod}_2} = \tau$. The final step then shows that the existence of M'' would yield that $\tau \in \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$ contrary to the fact that $\tau \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$. Hence $\tau \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

We construct a boolean deterministic bu-w-tt $M'' = (Q_2, \Sigma, \Delta, \mathcal{A}, F_2, \delta_2, \mu'')$ obeying the restrictions π_2 and $\tau_{M''}^{\mathrm{mod}_2} = \tau_{M_2}^{\mathrm{mod}_2} = \tau$ as follows. Let $\mu'' = (\,(\mu'')_\sigma^k\,)_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}}$ and for every integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k states $q_1, \ldots, q_k \in Q_2$ let

$$(\mu'')_\sigma^k(q_1, \ldots, q_k) = \mathrm{char}(\mathrm{supp}((\mu_2)_\sigma^k(q_1, \ldots, q_k))).$$

Obviously, M'' is boolean and obeys the restrictions of π_2. For our subgoal it remains to show that $\tau_{M''}^{\mathrm{mod}_2} = \tau_{M_2}^{\mathrm{mod}_2}$. Therefore we obviously have to prove that $\widehat{\mu''}_{\mathrm{mod}_2}(s) = \widehat{\mu_2}_{\mathrm{mod}_2}(s)$ for every input tree $s \in T_\Sigma$. We perform induction over the structure of the input tree s.

Induction base: The induction base is included in the induction step using $k = 0$.

Induction step: Let $s = \sigma(s_1, \ldots, s_k)$ for some integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and input subtrees $s_1, \ldots, s_k \in T_\Sigma$. We distinguish two separate cases.

(i) Let $i \in [k]$ be an index such that $\widehat{\mu 2}_{\mathrm{mod}_2}(s_i) = \widetilde{\mathbf{0}}$ or $(\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) = \widetilde{\mathbf{0}}$. Then $\tau^{\mathrm{mod}_2}_{M_2}(s) = \widetilde{\mathbf{0}}$, but contrary $\tau^{\mathrm{mod}_2}_{M_2}(s) = \tau^{\mathrm{mod}_1}_{M_1}(s) \neq \widetilde{\mathbf{0}}$ because M_1 is total.

(ii) Assume that for every index $i \in [k]$ we have $\widehat{\mu 2}_{\mathrm{mod}_2}(s_i) \neq \widetilde{\mathbf{0}}$ and $(\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) = at$ for some monoid element $a \in A \setminus \{\mathbf{0}\}$ and output tree $t \in T_\Delta(X_k)$. By induction hypothesis also $\widehat{\mu 2}_{\mathrm{mod}_2}(s_i) = \widehat{\mu''}_{\mathrm{mod}_2}(s_i)$ holds, and consequently, $\widehat{\mu 2}_{\mathrm{mod}_2}(s_i) = \mathbf{1}\, t_i$ for some output tree $t_i \in T_\Delta$ because M'' is boolean. Then

$$\begin{aligned}
&\widehat{\mu 2}_{\mathrm{mod}_2}(\sigma(s_1, \ldots, s_k)) \\
&\quad= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu 2}_{\mathrm{mod}_2}(s_1), \ldots, \widehat{\mu 2}_{\mathrm{mod}_2}(s_k)) \\
&\quad= a\, t \xleftarrow{\mathrm{mod}_2} (\mathbf{1}\, t_1, \ldots, \mathbf{1}\, t_k) \\
&\quad= a\, t[t_1, \ldots, t_k].
\end{aligned}$$

Since $\tau^{\mathrm{mod}_1}_{M_1}(s) \neq \widetilde{\mathbf{0}}$ we conclude that $\tau^{\mathrm{mod}_2}_{M_2}(s) = \widehat{\mu 2}_{\mathrm{mod}_2}(s)$. Further, M_1 is boolean, so also $\widehat{\mu 2}_{\mathrm{mod}_2}(s)$ is boolean and we continue with

$$\begin{aligned}
&\widehat{\mu 2}_{\mathrm{mod}_2}(\sigma(s_1, \ldots, s_k)) \\
&\quad= a\, t[t_1, \ldots, t_k] \\
&\quad= \mathbf{1}\, t[t_1, \ldots, t_k] \\
&\quad= \mathbf{1}\, t \xleftarrow{\mathrm{mod}_2} (\mathbf{1}\, t_1, \ldots, \mathbf{1}\, t_k) \\
&\quad= (\mu'')^k_\sigma(\widehat{\delta''}(s_1), \ldots, \widehat{\delta''}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu''}_{\mathrm{mod}_2}(s_1), \ldots, \widehat{\mu''}_{\mathrm{mod}_2}(s_k)) \\
&\quad= \widehat{\mu''}_{\mathrm{mod}_2}(\sigma(s_1, \ldots, s_k)).
\end{aligned}$$

Hence there also exists a boolean deterministic bu-w-tt M'' obeying the restrictions of π_2 such that $\tau^{\mathrm{mod}_2}_{M''} = \tau$. Immediately, we obtain that $M = (Q_2, \Sigma, \Delta, \mathbb{Z}_2, F_2, \delta_2, \mu'')$ is a deterministic bu-w-tt obeying all the restrictions of π_2 over $\mathbb{Z}_2$ such that $\tau^{\mathrm{mod}_2}_{M} = \tau$. However, this is contradictory to the assumption, because τ was chosen such that $\tau \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$, which finally proves the lemma. ■

Thus we can derive non-inclusion for classes of t-ts and o-t-ts transformations over the monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ simply by observing non-inclusion for the respective classes of t-ts and o-t-ts transformations over the group $\mathbb{Z}_2$. Roughly speaking, these latter non-inclusions are based solely on a deficiency in the tree output component of one class. For example, for any mod $\in \{\varepsilon, o\}$ the mod-t-ts transformation which maps each input tree s to a fully balanced binary tree of the same height with whatever non-zero cost cannot be computed by a linear deterministic bu-w-tt. In order to generate the fully balanced binary trees one definitely needs the copying of output trees. Another example is totality. The mod-t-ts transformation which maps every input tree to $\widetilde{\mathbf{0}}$ obviously cannot be computed by a total deterministic bu-w-tt.

The following corollary presents the conclusions of Figure 1 and Lemma 4.2. Moreover, it adds the missing case of totality, which is straightforward using the remark of the previous paragraph.

4.3 Corollary (Corollary of the Lifting lemma)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid and $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ be two modifiers. For every two combinations $\pi_1, \pi_2 \in \Pi$ such that there exists a prefix $r \in \mathrm{Pref}$ which occurs in π_2 but not in π_1, i.e., $r \in \pi_2 \setminus \pi_1$, we have

$$\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}).$$

Proof. We distinguish two cases.

(i) Let $r \neq$ t. Apparently, $r \notin \pi_1 \cup \{t\}$, so let $\pi_1' = \pi_1 \cup \{\mathrm{t}\}$. From Figure 1, we can check that π_1'–BOT($\mathbb{Z}_2$) $\not\subseteq$ π_2–BOT($\mathbb{Z}_2$) and with the help of Lemma 4.2 we conclude $\pi_1'\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$. Trivially, $\pi_1'\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$, hence

$$\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}).$$

(ii) Let $r =$ t. Moreover, let $\Sigma = \{\alpha^{(0)}\}$ be a ranked alphabet. We construct the linear and non-deleting deterministic bu-w-tt $M = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta, \mu)$ with transition mappings $\delta = (\delta^0_\alpha)$ and output mappings $\mu = (\mu^0_\alpha)$ specified by $\delta^0_\alpha() = \star$ and $\mu^0_\alpha() = \widetilde{\mathbf{0}}$. Apparently, $\tau_M^{\mathrm{mod}_1} \in \pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$ and $\tau_M^{\mathrm{mod}_1} \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, because t $\in \pi_2$. Hence

$$\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}).$$

■

Due to the previous corollary we can restrict our attention to the comparison of classes of t-ts transformations and of the corresponding classes of o-t-ts transformations. As a first comparison we restate the equality of the classes of t-ts and o-t-ts transformations for all restrictions which contain both the non-deletion as well as the linearity restriction. This equality was shown for tree series transducers in [FV03a] and the proof required for deterministic bu-w-tt is analogous.

4.4 Observation (cf. Theorem 5.5 of [FV03a])
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid. Then $\pi\text{–BOT}^o(\mathcal{A}) = \pi\text{–BOT}(\mathcal{A})$ for every $\pi \in \{\mathrm{dnl}, \mathrm{dnlt}, \mathrm{hnl}\}$.

Proof. We leave the actual proof to the reader, but note that

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_k) = \varphi \overset{o}{\longleftarrow} (\psi_1, \ldots, \psi_k)$$

for every integer $k \in \mathbb{N}$, ranked alphabet Δ, monomial $\varphi \in A[T_\Delta(X_k)]$, and k monomials $\psi_1, \ldots, \psi_k \in A[T_\Delta]$ such that every tree $t \in \mathrm{supp}(\varphi)$ is non-deleting and linear in X_k (cf. Proposition 3.10(a) of [FV03a]). ■

The final result of this subsection shows two non-inclusion results. Essentially, we prove that the classes of t-ts transformations and o-t-ts transformations computed by linear homomorphism bu-w-tt are incomparable. Due to the inclusion diagram presented in Figure 1, we cannot prove this result for every monoid with absorbing element, but rather we require that the monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ has at least three elements, i.e., $\mathbf{0} \neq \mathbf{1}$ and $\mathcal{A}$ is not isomorphic to $\mathbb{Z}_2$.

4.5 Lemma (Linear homomorphisms)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid and $A \setminus \{\mathbf{0}, \mathbf{1}\} \neq \emptyset$. Then

$$\mathrm{hl\text{–}BOT}(\mathcal{A}) \not\subseteq \mathrm{h\text{–}BOT}^o(\mathcal{A}) \quad \text{and} \quad \mathrm{hl\text{–}BOT}^o(\mathcal{A}) \not\subseteq \mathrm{h\text{–}BOT}(\mathcal{A}).$$

Proof. Firstly, we prove the former statement. We choose an arbitrary element $a \in A \setminus \{\mathbf{0}, \mathbf{1}\}$. Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$ be a ranked alphabet and $M_1 = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the linear homomorphism bu-w-tt with $\delta_1 = ((\delta_1)^1_\gamma, (\delta_1)^0_\alpha, (\delta_1)^0_\beta)$ and $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha, (\mu_1)^0_\beta)$ specified by

$$(\delta_1)^1_\gamma(\star) = (\delta_1)^0_\alpha() = (\delta_1)^0_\beta() = \star \quad , \quad (\mu_1)^1_\gamma(\star) = \mathbf{1}\,\alpha \quad , \quad (\mu_1)^0_\alpha() = a\,\alpha \quad , \quad (\mu_1)^0_\beta() = \mathbf{1}\,\beta.$$

Let $\tau = \tau_{M_1}$. Clearly, $\tau \in \mathrm{hl\text{–}BOT}(\mathcal{A})$, and moreover, $\tau(\gamma(\alpha)) = a\,\alpha$ and $\tau(\gamma(\beta)) = \mathbf{1}\,\alpha$.

Now let us prove that $\tau \notin \mathrm{h\text{–}BOT}^o(\mathcal{A})$. We prove this statement by contradiction, so assume that there exists a homomorphism bu-w-tt $M_2 = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$ such that $\tau^o_{M_2} = \tau$. Trivially, $\delta_2 = \delta_1$ and $\mu_2 = ((\mu_2)^1_\gamma, (\mu_2)^0_\alpha, (\mu_2)^0_\beta)$ with

$$(\mu_2)^1_\gamma(\star) = c\,t \quad , \quad (\mu_2)^0_\alpha() = a\,\alpha \quad , \quad (\mu_2)^0_\beta() = \mathbf{1}\,\beta$$

for some monoid element $c \in A$ and output tree $t \in T_\Sigma(X_1)$. Moreover, we readily observe $t = \alpha$, otherwise $\mathrm{supp}(\tau^o_{M_2}(\gamma(\beta))) \neq \{\alpha\}$. Consequently, $\tau^o_{M_2}(\gamma(\alpha)) = \tau^o_{M_2}(\gamma(\beta)) = c\,\alpha$. Thus we obtain the contradiction $a = \mathbf{1}$ and conclude that $\tau \notin \text{h–BOT}^o(\mathcal{A})$.

To show the latter statement, i.e., $\text{hl–BOT}^o(\mathcal{A}) \not\subseteq \text{h–BOT}(\mathcal{A})$, let $\tau^o = \tau^o_{M_1}$. Obviously, $\tau^o \in \text{hl–BOT}^o(\mathcal{A})$, and moreover, $\tau^o(\gamma(\alpha)) = \tau^o(\gamma(\beta)) = \mathbf{1}\,\alpha$. Let us prove that $\tau^o \notin \text{h–BOT}(\mathcal{A})$. We prove this statement by contradiction, so suppose that there exists a homomorphism bu-w-tt $M_3 = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta_3, \mu_3)$ such that $\tau_{M_3} = \tau^o$. Trivially, we observe that $\delta_3 = \delta_1$ and $\mu_3 = ((\mu_3)^1_\gamma, (\mu_3)^0_\alpha, (\mu_3)^0_\beta)$ with

$$(\mu_3)^1_\gamma(\star) = c\,t \quad , \quad (\mu_3)^0_\alpha() = a\,\alpha \quad , \quad (\mu_3)^0_\beta() = \mathbf{1}\,\beta$$

for some monoid element $c \in A$ and output tree $t \in T_\Sigma(X_1)$. Moreover, we again readily observe $t = \alpha$, else $\mathrm{supp}(\tau_{M_3}(\gamma(\beta))) \neq \{\alpha\}$. Consequently,

$$\tau_{M_3}(\gamma(\alpha)) = (c \odot a)\,\alpha = \mathbf{1}\,\alpha = c\,\alpha = \tau_{M_3}(\gamma(\beta)),$$

which yields $c = \mathbf{1}$ and hence also $a = \mathbf{1}$. This is contrary to the assumption that $a \in A \setminus \{\mathbf{0}, \mathbf{1}\}$. Thus we conclude that $\tau^o \notin \text{h–BOT}(\mathcal{A})$. ∎

In particular, the former lemma also proves that the classes of t-ts and o-t-ts transformations computed by homomorphism bu-w-tt are incomparable for all monoids different from $\mathbb{Z}_2$.

4.6 Corollary (Corollary of Lemma 4.5)
If and only if for every $\pi \in \Pi$ the equality $\pi\text{–BOT}(\mathcal{A}) = \pi\text{–BOT}^o(\mathcal{A})$ holds, then $\mathcal{A} = \mathbb{Z}_2$.

Proof. The equality in $\mathbb{Z}_2$ is shown in Theorem 4.1 and Lemma 4.5 proves the incomparability of $\text{hl–BOT}^o(\mathcal{A})$ and $\text{hl–BOT}(\mathcal{A})$ in all other monoids. ∎

However, without additional information about the monoid we are unable to prove further comparability or incomparability results. Hence we will consider monoids with certain properties in subsequent subsections. The properties will be chosen such that we obtain an inclusion diagram for every commutative monoid.

4.2 Non-periodic monoids

In this subsection we show that for non-periodic monoids almost all classes of t-ts and o-t-ts transformations (except the ones containing both the non-deletion and linearity restriction) computed by restricted deterministic bu-w-tt are incomparable with respect to set inclusion. An example of a non-periodic monoid is the multiplicative monoid of the non-negative integers $\mathbb{N}$. To be precise we even show that

$$\pi\text{–BOT}(\mathcal{A}) \not\subseteq \text{d–BOT}^o(\mathcal{A}) \quad \text{and} \quad \pi\text{–BOT}^o(\mathcal{A}) \not\subseteq \text{d–BOT}(\mathcal{A})$$

for every $\pi \in \{\text{hn}, \text{hl}\}$ and non-periodic monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$.

The general idea of the proof is the following. Let $a \in A$ be an element such that $a^i \neq a^j$, whenever $i \neq j$ where $i, j \in \mathbb{N}$. We construct a homomorphism bu-w-tt M_1, which computes a t-ts transformation τ in which arbitrarily large powers of the element $a \in A$ occur as weights in the range. Since every deterministic bu-w-tt M_2, which also computes τ but as a o-t-ts transformation, has only finitely many states, it must permit at least one final state q which accepts infinitely many input trees. We then show that it is impossible to encode enough information into this state in order to predict the cost of an input tree constructed from two subtrees accepted by q. The second statement is shown using a similar approach.

4.7 Lemma (Incomparability in non-periodic monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a non-periodic monoid. For every restriction $\pi \in \{\text{hn}, \text{hl}\}$ and two modifiers $\text{mod}_1, \text{mod}_2 \in \{\varepsilon, o\}$ with $\text{mod}_1 \neq \text{mod}_2$ we have

$$\pi\text{–BOT}^{\text{mod}_1}(\mathcal{A}) \not\subseteq \text{d–BOT}^{\text{mod}_2}(\mathcal{A}).$$

Proof. Let us prove the statement by case analysis on π. Case 1 considers the case where $\pi = \mathrm{hl}$ and Case 2 supposes $\pi = \mathrm{hn}$.

Case 1: Since $\mathcal{A}$ is non-periodic, there exists a monoid element $a \in A$ such that for every two integers $i, j \in \mathbb{N}$ we have $a^i = a^j$, if and only if $i = j$. Further let $\Delta = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Gamma = \{\gamma_1{}^{(1)}, \gamma_2{}^{(1)}, \alpha^{(0)}\}$ be ranked alphabets. We construct the linear homomorphism bu-w-tt $M_1 = (\{\star\}, \Gamma, \Delta, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ with transition mappings $\delta_1 = ((\delta_1)^1_{\gamma_1}, (\delta_1)^1_{\gamma_2}, (\delta_1)^0_\alpha)$ and output mappings $\mu_1 = ((\mu_1)^1_{\gamma_1}, (\mu_1)^1_{\gamma_2}, (\mu_1)^0_\alpha)$ specified by

$$(\delta_1)^1_{\gamma_1}(\star) = (\delta_1)^1_{\gamma_2}(\star) = (\delta_1)^0_\alpha() = \star \quad , \quad (\mu_1)^1_{\gamma_1}(\star) = a\,\gamma(x_1) \quad , \quad (\mu_1)^1_{\gamma_2}(\star) = (\mu_1)^0_\alpha() = \mathbf{1}\,\alpha.$$

Moreover, we define the mapping $l_1 : T_\Gamma \longrightarrow \mathbb{N}$ recursively for every $t \in T_\Gamma$ as follows.

$$l_1(\gamma_1(t)) = l_1(t) + 1 \quad \text{and} \quad l_1(\gamma_2(t)) = l_1(\alpha) = 0.$$

Note that M_1 computes the t-ts transformation $\tau_{M_1} : T_\Gamma \longrightarrow A[T_\Delta]$ mapping every input tree $s \in T_\Gamma$ to the monomial $a^{|s|_{\gamma_1}}\,\gamma^{l_1(s)}(\alpha)$, and the o-t-ts transformation $\tau^o_{M_1} : T_\Gamma \longrightarrow A[T_\Delta]$ mapping s to the monomial $a^{l_1(s)}\,\gamma^{l_1(s)}(\alpha)$.

Next, we prove that $\tau^{\mathrm{mod}_1}_{M_1} \notin \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, which yields $\text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A})$. For a contradiction we assume that there exists a deterministic bu-w-tt $M = (Q, \Gamma, \Delta, \mathcal{A}, F, \delta, \mu)$ such that $\tau^{\mathrm{mod}_2}_M = \tau^{\mathrm{mod}_1}_{M_1}$.

We observe that for every input tree $s \in T_\Gamma$ we have that $\widetilde{\mathbf{0}} \neq \tau^{\mathrm{mod}_1}_{M_1}(s)$, and consequently, $\tau^{\mathrm{mod}_2}_M(s) = \widehat{\mu}_{\mathrm{mod}_2}(s)$ as well as $\widehat{\delta}(s) \in F$. (Note that if $a^n = \mathbf{0}$ for some $n \in \mathbb{N}$, then $a^n = a^{n+1}$ which contradicts to our assumption) Next we prove that there are a state $q \in F$ and two trees $s_1, s_2 \in T_\Gamma$ such that $\widehat{\delta}(s_1) = q = \widehat{\delta}(s_2)$ and $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$ and $l_1(s_1) \neq l_1(s_2)$. Therefore we let $\Gamma' = \{\gamma_1{}^{(1)}, \alpha^{(0)}\} \subset \Gamma$, hence $T_{\Gamma'} \subseteq T_\Gamma$. We show that s_1 and s_2 can actually be chosen from $T_{\Gamma'}$. Clearly, there exist a state $q \in F$ and an infinite set $S \subseteq T_{\Gamma'}$ such that $q = \widehat{\delta}(s)$ for every $s \in S$, because Q is finite whereas $T_{\Gamma'}$ is infinite. For every tree $s \in S$ we have $\mathrm{size}(s) = |s|_{\gamma_1} + 1 = l_1(s) + 1$, because $S \subseteq T_{\Gamma'}$. In Observation 2.1 we have observed that $[s]_{\equiv_{\mathrm{size}}}$ is finite for every $s \in S$, hence by the pigeon-hole principle there must exist $s_1, s_2 \in S$ such that $\mathrm{size}(s_1) \neq \mathrm{size}(s_2)$, i.e., $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$ and $l_1(s_1) \neq l_1(s_2)$.

Hence we can safely assume that there exist a state $q \in F$ and trees $s_1, s_2 \in T_\Gamma$ such that $\widehat{\delta}(s_1) = q = \widehat{\delta}(s_2)$ and $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$ and $l_1(s_1) \neq l_1(s_2)$. Since

$$\mathrm{supp}(\tau^{\mathrm{mod}_1}_{M_1}(\gamma_2(s_1))) = \mathrm{supp}(\tau^{\mathrm{mod}_1}_{M_1}(\gamma_2(s_2))) = \{\alpha\},$$

and

$$\tau^{\mathrm{mod}_2}_M(\gamma_2(s_i)) = \widehat{\mu}_{\mathrm{mod}_2}(\gamma_2(s_i)) = \mu^1_{\gamma_2}(q) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu}_{\mathrm{mod}_2}(s_i))$$

for every $i \in [2]$, we have $\mu^1_{\gamma_2}(q) \neq \widetilde{\mathbf{0}}$, and thereby, $\mu^1_{\gamma_2}(q) = a'\,t$ for some non-zero semiring element $a' \in A \setminus \{\mathbf{0}\}$ and output tree $t \in T_\Delta(X_1)$. Next we prove that $t = \alpha$. Since $\tau^{\mathrm{mod}_2}_M = \tau^{\mathrm{mod}_1}_{M_1}$ we have that $\mathrm{supp}(\tau^{\mathrm{mod}_1}_{M_1}(s_i)) = \mathrm{supp}(\widehat{\mu}_{\mathrm{mod}_2}(s_i)) = \mathrm{supp}(\tau^{\mathrm{mod}_2}_M(s_i)) = \{\gamma^{l_1(s_i)}(\alpha)\}$. Then

$$\begin{aligned}
\alpha &= \mathrm{supp}(\tau^{\mathrm{mod}_1}_{M_1}(\gamma_2(s_i))) = \mathrm{supp}(\tau^{\mathrm{mod}_2}_M(\gamma_2(s_i))) \\
&= \mathrm{supp}(\mu^1_{\gamma_2}(q) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu}_{\mathrm{mod}_2}(s_i))) = t[\mathrm{supp}(\widehat{\mu}_{\mathrm{mod}_2}(s_i))] \\
&= t[\gamma^{l_1(s_i)}(\alpha)].
\end{aligned}$$

Now using $l_1(s_1) \neq l_1(s_2)$ we conclude $|t|_{x_1} = 0$, thus finally, $t = \alpha$.

We obtain for every integer $i \in [2]$

$$\tau^{\mathrm{mod}_2}_M(\gamma_2(s_i)) \;=\; a'\,\alpha \xleftarrow{\mathrm{mod}_2} (\tau^{\mathrm{mod}_1}_{M_1}(s_i)) \;=\; \begin{cases} (a' \odot a^{l_1(s_i)})\,\alpha & \text{, if } \mathrm{mod}_2 = \varepsilon \\ a'\,\alpha & \text{, if } \mathrm{mod}_2 = o \end{cases}.$$

Recall now that $\text{mod}_1 \neq \text{mod}_2$ and $\tau_{M_1}(\gamma_2(s_i)) = a^{|s_i|_{\gamma_1}}\,\alpha$ and $\tau^{o}_{M_1}(\gamma_2(s_i)) = a^{l_1(\gamma_2(s_i))}\,\alpha = \mathbf{1}\,\alpha$. Hence for every $i \in [2]$ we derive the equation

$$\begin{array}{rcll} a' \odot a^{l_1(s_i)} = \mathbf{1} & = & (\tau^{o}_{M_1}(\gamma_2(s_i)), \alpha) & \text{, if } \text{mod}_2 = \varepsilon \\ a' = a^{|s_i|_{\gamma_1}} & = & (\tau_{M_1}(\gamma_2(s_i)), \alpha) & \text{, if } \text{mod}_2 = o \end{array}.$$

In case $\text{mod}_2 = o$ this yields a contradiction outright, because $a' = a^{|s_1|_{\gamma_1}} = a^{|s_2|_{\gamma_1}}$, which apparently is contradictory due to $a^{|s_1|_{\gamma_1}} \neq a^{|s_2|_{\gamma_1}}$ by $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$. Finally, in the other case, i.e., $\text{mod}_2 = \varepsilon$, we effectively have $\mathbf{1} = a' \odot a^{l_1(s_1)} = a' \odot a^{l_1(s_2)}$. Now let $y_1 = \min(l_1(s_1), l_1(s_2))$, $y_2 = \max(l_1(s_1), l_1(s_2))$, and $d = y_2 - y_1$. Obviously, $y_1 \neq y_2$ and thereby $d \neq 0$ by $l_1(s_1) \neq l_1(s_2)$. We consider

$$\mathbf{1} = a' \odot a^{y_2} = a' \odot a^{y_1+d} = a' \odot a^{y_1} \odot a^{d} = \mathbf{1} \odot a^{d} = a^{d},$$

however $\mathbf{1} = a^0 = a^d$, if and only if $0 = d$, which is a contradiction. Irrespective of mod_2 we have thus proved that there is no deterministic bu-w-tt M having the property that $\tau^{\text{mod}_1}_{M_1} = \tau^{\text{mod}_2}_{M}$. Thus $\tau^{\text{mod}_1}_{M_1} \notin \text{d–BOT}^{\text{mod}_2}(\mathcal{A})$.

<u>Case 2:</u> Since $\mathcal{A}$ is non-periodic, there exists a monoid element $a \in A$ such that for every two integers $i, j \in \mathbb{N}$ we have $a^i = a^j$, if and only if $i = j$. Further let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta = \{\gamma^{(1)}, \alpha^{(0)}\}$ be ranked alphabets. We construct the non-deleting homomorphism bu-w-tt $M_2 = (\{\star\}, \Delta, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$ with transition mappings $\delta_2 = ((\delta_2)^1_\gamma, (\delta_2)^0_\alpha)$ and output mappings $\mu_2 = ((\mu_2)^1_\gamma, (\mu_2)^0_\alpha)$ defined by

$$(\delta_2)^1_\gamma(\star) = (\delta_2)^0_\alpha() = \star \quad , \quad (\mu_2)^1_\gamma(\star) = a\,\sigma(x_1, x_1) \quad , \quad (\mu_2)^0_\alpha() = a\,\alpha.$$

For every tree $s \in T_\Delta$ let $t_s \in T_\Sigma$ be the fully balanced output tree such that $\text{height}(t_s) = \text{height}(s)$. The t-ts transformation $\tau_{M_2} : T_\Delta \longrightarrow A[T_\Sigma]$ computed by M_2 maps the input tree s to the monomial $a^{\text{size}(s)}\,t_s$, whereas the o-t-ts transformation $\tau^{o}_{M_2} : T_\Delta \longrightarrow A[T_\Sigma]$ computed by M_2 maps the input tree s to the monomial $a^{\text{size}(t_s)}\,t_s$. Note that $\text{size}(t_s) = 2^{\text{size}(s)} - 1$.

Let us prove $\tau^{\text{mod}_1}_{M_2} \notin \text{d–BOT}^{\text{mod}_2}(\mathcal{A})$, thereby showing $\text{hn–BOT}^{\text{mod}_1}(\mathcal{A}) \not\subseteq \text{d–BOT}^{\text{mod}_2}(\mathcal{A})$. To derive a contradiction assume that there exists a deterministic bu-w-tt $M = (Q, \Delta, \Sigma, \mathcal{A}, F, \delta, \mu)$ such that $\tau^{\text{mod}_2}_{M} = \tau^{\text{mod}_1}_{M_2}$.

We again observe that for every input tree $s \in T_\Delta$ we have $\widetilde{\mathbf{0}} \neq \tau^{\text{mod}_1}_{M_2}(s)$, and consequently, $\tau^{\text{mod}_2}_{M}(s) = \widehat{\mu}_{\text{mod}_2}(s)$ as well as $\widehat{\delta}(s) \in F$. Moreover, T_Δ is infinite. In contrast M has only a finite set of final states F; hence there must exist a final state $q \in F$ and input trees $s_1, s_2 \in T_\Delta$ with $q = \widehat{\delta}(s_i)$ and $s_1 \neq s_2$ such that $t_{s_i} \in \text{supp}(\widehat{\mu}_{\text{mod}_2}(s_i))$ for $i \in [2]$. Since $s_1 \neq s_2$ we also have $\text{size}(s_1) \neq \text{size}(s_2)$ and $t_{s_1} \neq t_{s_2}$.

Apparently, $\widehat{\mu}_{\text{mod}_2}(\gamma(s_i)) = \mu^1_\gamma(q) \xleftarrow{\text{mod}_2} (\tau^{\text{mod}_1}_{M_2}(s_i))$, and furthermore, also $\tau^{\text{mod}_1}_{M_2}(\gamma(s_i)) \neq \widetilde{\mathbf{0}}$, hence $\widehat{\delta}(\gamma(s_i)) \in F$ and $\mu^1_\gamma(q) \neq \widetilde{\mathbf{0}}$. Let $\mu^1_\gamma(q) = a'\,t$ for some non-zero monoid element $a' \in A \setminus \{\mathbf{0}\}$ and output tree $t \in T_\Sigma(X_1)$.

Next we observe that $t = \sigma(x_1, x_1)$, which can easily be proved by contradiction as follows. Assume that $t \neq \sigma(x_1, x_1)$. Then for some index $j \in [2]$ the tree $t[t_{s_j}]$ is not fully balanced or its height is not $1 + \text{height}(t_{s_j})$, because $t_{s_1} \neq t_{s_2}$. Hence we obtain for every integer $i \in [2]$

$$\tau^{\text{mod}_2}_{M}(\gamma(s_i)) \;=\; a'\,\sigma(x_1, x_1) \;\xleftarrow{\text{mod}_2}\; (\tau^{\text{mod}_1}_{M_2}(s_i)) \;=\; \begin{cases} (a' \odot a^{\text{size}(t_{s_i})})\,\sigma(t_{s_i}, t_{s_i}) & \text{, if } \text{mod}_2 = \varepsilon \\ (a' \odot a^{2\cdot\text{size}(s_i)})\,\sigma(t_{s_i}, t_{s_i}) & \text{, if } \text{mod}_2 = o \end{cases}.$$

However, recall that $\tau_{M_2}(\gamma(s_i)) = a^{\text{size}(s_i)+1}\,\sigma(t_{s_i}, t_{s_i})$ and $\tau^{o}_{M_2}(\gamma(s_i)) = a^{2\cdot\text{size}(t_{s_i})+1}\,\sigma(t_{s_i}, t_{s_i})$. Hence for every $i \in [2]$ we derive the equation

$$\begin{array}{rcll} a' \odot a^{\text{size}(t_{s_i})} = a^{2\cdot\text{size}(t_{s_i})+1} & = & (\tau^{o}_{M_2}(\gamma(s_i)), \sigma(t_{s_i}, t_{s_i})) & \text{, if } \text{mod}_2 = \varepsilon \\ a' \odot a^{2\cdot\text{size}(s_i)} = a^{\text{size}(s_i)+1} & = & (\tau_{M_2}(\gamma(s_i)), \sigma(t_{s_i}, t_{s_i})) & \text{, if } \text{mod}_2 = o \end{array}.$$

For every $i \in [2]$ we let $y_i = \text{size}(t_{s_i})$, if $\text{mod}_2 = \varepsilon$, whereas we let $y_i = \text{size}(s_i)$ in case $\text{mod}_2 = o$. Note that in both cases $y_1 \neq y_2$. We continue with the equations

$$\begin{array}{llllll} a^{y_1+2\cdot y_2+1} & = a' \odot a^{y_2} \odot a^{y_1} & = a' \odot a^{y_1} \odot a^{y_2} & = a^{2\cdot y_1+y_2+1} & , \text{if } \text{mod}_2 = \varepsilon \\ a^{y_1+2\cdot y_2+1} & = a' \odot a^{2\cdot y_1} \odot a^{2\cdot y_2} & = a' \odot a^{2\cdot y_2} \odot a^{2\cdot y_1} & = a^{2\cdot y_1+y_2+1} & , \text{if } \text{mod}_2 = o \end{array}.$$

Thus in any case $a^{y_1+2\cdot y_2+1} = a^{2\cdot y_1+y_2+1}$. Since $a^i \neq a^j$ whenever $i \neq j$ for all $i, j \in \mathbb{N}$, we conclude $y_1 + 2 \cdot y_2 + 1 = 2 \cdot y_1 + y_2 + 1$ and thereby $y_1 = y_2$ which contradicts to $y_1 \neq y_2$. Consequently, irrespective of mod_2 we have proved that there is no deterministic bu-w-tt M having the property that $\tau_{M_2}^{\text{mod}_1} = \tau_M^{\text{mod}_2}$. Thus $\tau_{M_2}^{\text{mod}_1} \notin \text{d–BOT}^{\text{mod}_2}(\mathcal{A})$. ■

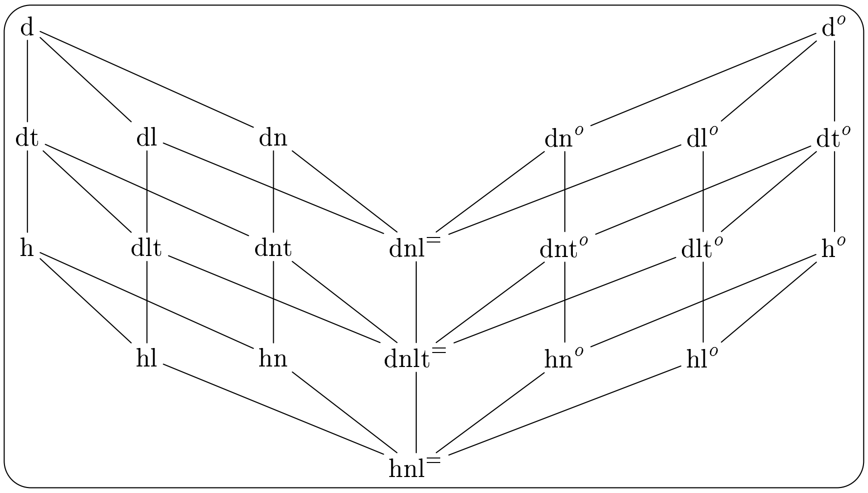

Figure 2: Inclusion diagram for non-periodic monoids.

Together with the results of Subsection 4.1 we can already derive the inclusion diagram (cf. Figure 2) for non-periodic monoids. We observe that the classes of t-ts and o-t-ts transformations are incomparable, whenever inclusion is not trivial by definition or given as a result of Observation 4.4.

4.8 Theorem (Non-periodic monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a non-periodic monoid with an absorbing element $\mathbf{0}$. Figure 2 is the inclusion diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.

Proof. All the inclusions are trivial and the equalities are due to Observation 4.4. Then the following six statements are sufficient to prove strictness and incomparability. For every two distinct modifiers $\text{mod}_1, \text{mod}_2 \in \{\varepsilon, o\}$, i.e., $\text{mod}_1 \neq \text{mod}_2$,

(i) $\text{dnlt–BOT}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\text{mod}_1}(\mathcal{A})$, (ii) $\text{dnl–BOT}(\mathcal{A}) \not\subseteq \text{dt–BOT}^{\text{mod}_1}(\mathcal{A})$,
(iii) $\text{hn–BOT}^{\text{mod}_1}(\mathcal{A}) \not\subseteq \text{dl–BOT}^{\text{mod}_1}(\mathcal{A})$, (iv) $\text{hl–BOT}^{\text{mod}_1}(\mathcal{A}) \not\subseteq \text{dn–BOT}^{\text{mod}_1}(\mathcal{A})$,
(v) $\text{hl–BOT}^{\text{mod}_1}(\mathcal{A}) \not\subseteq \text{d–BOT}^{\text{mod}_2}(\mathcal{A})$, (vi) $\text{hn–BOT}^{\text{mod}_1}(\mathcal{A}) \not\subseteq \text{d–BOT}^{\text{mod}_2}(\mathcal{A})$.

The non-inclusions (i) – (iv) are proved in Corollary 4.3, whereas non-inclusions (v) and (vi) follow from Lemma 4.7. ■

4.3 Periodic and commutative monoids

In this subsection we consider monoids which are periodic and commutative. For example, the monoid $\mathbb{Z}_4$ is periodic and commutative (without being regular). It is easily seen that in commutative and periodic monoids $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ the carrier set $\langle A' \rangle_\odot$ of the least submonoid with the absorbing element $\mathbf{0}$ generated from a finite set $A' \subseteq A$ is again finite. This property is essential in the core construction of this subsection, because it allows to keep track of the current weight in the states.

4.9 Observation (Periodicity and finitely generated submonoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative and periodic monoid. For every finite subset $A' \subseteq A$ we have that $\langle A' \rangle_\odot$ is finite.

Proof. We first observe that $\langle \emptyset \rangle_\odot = \{\mathbf{0}, \mathbf{1}\}$. Consequently, let $A' = \{a_1, \ldots, a_k\} \subseteq A$ for some $k \in \mathbb{N}_+$. Then

$$\langle A' \rangle_\odot = \{\, a_1^{i_1} \odot \cdots \odot a_k^{i_k} \mid i_1, \ldots, i_k \in \mathbb{N} \,\} = \{\, a_1^{i_1} \odot \cdots \odot a_k^{i_k} \mid i_1 \in [0, n_1], \ldots, i_k \in [0, n_k] \,\},$$

where for every $j \in [k]$ the integer $n_j \in \mathbb{N}$ is such that there exists another integer $m_j \in \mathbb{N}$ with $n_j < m_j$ and $a_j^{n_j} = a_j^{m_j}$. Hence $\langle A' \rangle_\odot$ is a finite set. ∎

Given a deterministic bu-w-tt computing a t-ts transformation τ, we construct another deterministic bu-w-tt computing τ as o-t-ts transformation. Moreover, most of the restrictions defined for deterministic bu-w-tt (namely non-deleting, linear, and total) are preserved by this construction. However, a homomorphism bu-w-tt might yield a non-homomorphism bu-w-tt, because the construction increases the state-space compared to the given bu-w-tt.

The next definition abstracts the central feature required to model one type of substitution with the help of the other. We encapsulate this feature in a family of mappings in order to be able to invoke the construction later under different premises. More precisely, in subsequent corollaries of the lemma we will prove that such a family of mappings exists provided that the monoid has certain properties, e.g., is a group.

4.10 Definition (Family of translation mappings)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid, $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ be a deterministic bu-w-tt, and $\mathrm{mod} \in \{\varepsilon, o\}$ be a modifier. In addition let $f_{M,\mathrm{mod}} = (\, f^k_{M,\mathrm{mod}} \,)_{k \in \mathbb{N}}$ be a family of mappings where for every integer $k \in \mathbb{N}$ we have

$$f^k_{M,\mathrm{mod}} \colon \Big(\bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q} \mathrm{supp}(\mu^k_\sigma(q_1, \ldots, q_k)) \Big) \times [k] \times A \longrightarrow A.$$

If f satisfies for every element $t \in \bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q} \mathrm{supp}(\mu^k_\sigma(q_1, \ldots, q_k))$, index $i \in [k]$, and monoid element $a \in A$ the statements

(i) $f^k_{M,\mathrm{mod}}(t, i, a) = \mathbf{0}$, if $a = \mathbf{0}$,

(ii) $f^k_{M,\mathrm{mod}}(t, i, a) \odot a^{|t|_{x_i}} = a$, if $\mathrm{mod} = \varepsilon$, and

(iii) $f^k_{M,\mathrm{mod}}(t, i, a) \odot a = a^{|t|_{x_i}}$, if $\mathrm{mod} = o$,

then f is called a *family of* mod-*translation mappings* for M. □

Let $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ be two modifiers. For every deterministic bu-w-tt M_1, for which there exists a family of mod_1-translation mappings, we can construct another deterministic bu-w-tt M_2 computing the mod_2-t-ts transformation $\tau^{\mathrm{mod}_2}_{M_2} = \tau^{\mathrm{mod}_1}_{M_1}$. Due to the periodicity and commutativity of the monoid $\mathcal{A}$ and the determinism of M_1 we can encode the current weight in the current state. This way we can define the weight of the transitions using the weight of the subcomputations.

4.11 Lemma (Periodic and commutative monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic and commutative monoid and $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ be two modifiers. Moreover, let $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ be a deterministic bu-w-tt obeying all the restrictions of $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. Whenever there exists a family of mod_1-translation mappings $f_{M_1,\mathrm{mod}_1} = (\, f^k_{M_1,\mathrm{mod}_1} \,)_{k \in \mathbb{N}}$, there also exists a deterministic bu-w-tt $M_2 = (Q_2, \Sigma, \Delta, \mathcal{A}, F_2, \delta_2, \mu_2)$ obeying the restrictions of π such that $\tau^{\mathrm{mod}_1}_{M_1} = \tau^{\mathrm{mod}_2}_{M_2}$.

Proof. If $\mathrm{mod}_1 = \mathrm{mod}_2$, then the statement becomes trivial. So it remains to prove the property for distinct mod_1 and mod_2. Let

$$C = \{\, ((\mu_1)^k_\sigma(q_1,\ldots,q_k), t) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q_1,\ldots,q_k \in Q, t \in \mathrm{supp}((\mu_1)^k_\sigma(q_1,\ldots,q_k)) \,\} \cup \{\mathbf{0}\}$$

be the finite set of monoid elements occuring in the monomials in the range of μ_1. Since $\mathcal{A}$ is periodic and commutative, we conclude that $\langle C\rangle_\odot$ is finite. We construct the bu-w-tt M_2 by setting the set Q_2 of states to $Q_2 = Q_1 \times \langle C\rangle_\odot$ and the set F_2 of final states to $F_2 = F_1 \times \langle C\rangle_\odot$. Moreover, let $k \in \mathbb{N}$ be an integer, $\sigma \in \Sigma^{(k)}$ be an input symbol, $q_1,\ldots,q_k \in Q_1$ be k states, and $a_1,\ldots,a_k \in \langle C\rangle_\odot$ be k submonoid elements. Now we define the submonoid element a and the monomial m as follows. If $(\mu_1)^k_\sigma(q_1,\ldots,q_k) = \widetilde{\mathbf{0}}$ or for some index $i \in [k]$ we have $a_i = \mathbf{0}$, then let $a = \mathbf{0}$ and $m = \widetilde{\mathbf{0}}$. Otherwise suppose that $(\mu_1)^k_\sigma(q_1,\ldots,q_k) = a_0\, t$ for some non-zero submonoid element $a_0 \in C \setminus \{\mathbf{0}\}$ and output tree $t \in T_\Delta(X_k)$ and let

$$a = \begin{cases} a_0 \odot a_1 \odot \cdots \odot a_k & \text{, if } \mathrm{mod}_1 = \varepsilon \\ a_0 \odot a_1^{|t|_{x_1}} \odot \cdots \odot a_k^{|t|_{x_k}} & \text{, if } \mathrm{mod}_1 = o \end{cases}$$

and $m = \big(f^k_{M_1,\mathrm{mod}_1}(t,1,a_1) \odot \cdots \odot f^k_{M_1,\mathrm{mod}_1}(t,k,a_k) \odot a_0\big)\, t$. Clearly, $a \in \langle C\rangle_\odot$, so we let

$$\begin{aligned} (\delta_2)^k_\sigma((q_1,a_1),\ldots,(q_k,a_k)) &= ((\delta_1)^k_\sigma(q_1,\ldots,q_k),a), \\ (\mu_2)^k_\sigma((q_1,a_1),\ldots,(q_k,a_k)) &= m. \end{aligned}$$

Obviously, M_2 is non-deleting (likewise linear and total, respectively), if M_1 is non-deleting (likewise linear and total, respectively). Let $s \in T_\Sigma$ be an input tree. Finally, suppose that $\widehat{\mu_1}_{\mathrm{mod}_1}(s) = a\, t$ for some submonoid element $a \in \langle C\rangle_\odot$ and output tree $t \in T_\Delta$. We show that the following equalities hold.

$$\widehat{\mu_2}_{\mathrm{mod}_2}(s) = \widehat{\mu_1}_{\mathrm{mod}_1}(s) \quad \text{and} \quad \widehat{\delta_2}(s) = (\widehat{\delta_1}(s), a).$$

Induction base: Let the input tree be $s = \alpha$ with $\alpha \in \Sigma^{(0)}$. Then

$$\widehat{\mu_2}_{\mathrm{mod}_2}(s) = (\mu_2)^0_\alpha() = (\mu_1)^0_\alpha() = \widehat{\mu_1}_{\mathrm{mod}_1}(s).$$

Moreover, $\widehat{\delta_2}(s) = (\delta_2)^0_\alpha() = ((\delta_1)^0_\alpha(), a') = (\widehat{\delta_1}(s), a')$ where

$$\begin{aligned} a' &= \begin{cases} \mathbf{0} & \text{, if } \mathrm{supp}((\mu_1)^0_\alpha()) = \emptyset \\ ((\mu_1)^0_\alpha, t') & \text{, if } \mathrm{supp}((\mu_1)^0_\alpha()) = \{t'\} \end{cases} \\ &= \begin{cases} \mathbf{0} & \text{, if } \mathrm{supp}(\widehat{\mu_1}_{\mathrm{mod}_1}) = \emptyset \\ (\widehat{\mu_1}_{\mathrm{mod}_1}, t') & \text{, if } \mathrm{supp}(\widehat{\mu_1}_{\mathrm{mod}_1}) = \{t'\} \end{cases} \\ &= a. \end{aligned}$$

Induction step: Let the input tree be $s = \sigma(s_1,\ldots,s_k)$ for some $k \in \mathbb{N}_+$, input symbol $\sigma \in \Sigma^{(k)}$, and input subtrees $s_1,\ldots,s_k \in T_\Sigma$. Then we have

$$\begin{aligned} \widehat{\mu_2}_{\mathrm{mod}_2}(s) &= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1),\ldots,\widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_2}_{\mathrm{mod}_2}(s_1),\ldots,\widehat{\mu_2}_{\mathrm{mod}_2}(s_k)) \\ &= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1),\ldots,\widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1),\ldots,\widehat{\mu_1}_{\mathrm{mod}_1}(s_k)). \end{aligned}$$

Let $\widehat{\mu_1}_{\mathrm{mod}_1}(s_i) = a_i\, t_i$ for some output tree $t_i \in T_\Delta$ and every index $i \in [k]$. By induction hypothesis we have further that $\widehat{\delta_2}(s_i) = (\widehat{\delta_1}(s_i), a_i)$.

<u>Case 1:</u> In the first case let $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1),\ldots,\widehat{\delta_1}(s_k)) = \widetilde{\mathbf{0}}$ or for some index $i \in [k]$ let $a_i = \mathbf{0}$. Then by construction we obtain $(\mu_2)^k_\sigma(\widehat{\delta_2}(s_1),\ldots,\widehat{\delta_2}(s_k)) = \widetilde{\mathbf{0}}$. Hence $\widehat{\mu_1}_{\mathrm{mod}_1}(s) = \widetilde{\mathbf{0}} = \widehat{\mu_2}_{\mathrm{mod}_2}(s)$.

Case 2: Let $a_0 \in C \setminus \{\mathbf{0}\}$ be a non-zero submonoid element and $t' \in T_\Delta(X_k)$ be an output tree such that $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = a_0\, t'$. We deduce

$$\begin{aligned}
\widehat{\mu_2}_{\mathrm{mod}_2}(s) &= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= (\mu_2)^k_\sigma((\widehat{\delta_1}(s_1), a_1), \ldots, (\widehat{\delta_1}(s_k), a_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= \Big(\prod_{i\in[k]} f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_0\Big)\, t' \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= \Big(\prod_{i\in[k]} f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_0 \odot a_1^{m_1} \odot \cdots \odot a_k^{m_k}\Big)\, t'[t_1, \ldots, t_k] \\
&= \Big(\prod_{i\in[k]} (f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_i^{m_i}) \odot a_0\Big)\, t'[t_1, \ldots, t_k]
\end{aligned}$$

where for every index $i \in [k]$ we let

$$m_i = \begin{cases} 1 & \text{, if } \mathrm{mod}_2 = \varepsilon \\ |t'|_{x_i} & \text{, if } \mathrm{mod}_2 = o \end{cases}.$$

Recall that our general assumption was $\mathrm{mod}_1 \neq \mathrm{mod}_2$, so we now distinguish two cases, in each of which we take a closer look at the product $f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_i^{m_i}$ for every index $i \in [k]$. Firstly, let $\mathrm{mod}_1 = \varepsilon$. Then $f^k_{M_1,\varepsilon}(t', i, a_i) \odot a_i^{|t'|_{x_i}} = a_i$ by Definition 4.10(ii). On the other hand, let $\mathrm{mod}_1 = o$. Immediately we obtain $f^k_{M_1,o}(t', i, a_i) \odot a_i = a_i^{|t'|_{x_i}}$ by Definition 4.10(iii). Hence we continue with

$$\begin{aligned}
\widehat{\mu_2}_{\mathrm{mod}_2}(s) &= \Big(\prod_{i\in[k]} (f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_i^{m_i}) \odot a_0\Big)\, t'[t_1, \ldots, t_k] \\
&= a_0\, t' \xleftarrow{\mathrm{mod}_1} (a_1\, t_1, \ldots, a_k\, t_k) \\
&= (\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) \xleftarrow{\mathrm{mod}_1} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= \widehat{\mu_1}_{\mathrm{mod}_1}(s).
\end{aligned}$$

This concludes the proof of the first property.

Let $\widehat{\mu_1}_{\mathrm{mod}_1}(s) = a\, t$ for some submonoid element $a \in \langle C\rangle_\odot$ and output tree $t \in T_\Delta$. Thus it remains to show that $\widehat{\delta_2}(s) = (\widehat{\delta_1}(s), a)$. In a straightforward manner we derive

$$\begin{aligned}
\widehat{\delta_2}(s) &= (\delta_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) &&= (\delta_2)^k_\sigma((\widehat{\delta_1}(s_1), a_1), \ldots, (\widehat{\delta_1}(s_k), a_k)) \\
&= ((\delta_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)), a') &&= (\widehat{\delta_1}(s), a'),
\end{aligned}$$

where $a' = \mathbf{0}$, if $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = \widetilde{\mathbf{0}}$ or for some index $i \in [k]$ we have $a_i = \mathbf{0}$. Hence $a' = a$. Otherwise let $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = a_0\, t'$ for some non-zero submonoid element $a_0 \in C \setminus \{\mathbf{0}\}$ and output tree $t' \in T_\Delta(X_k)$. Consequently,

$$a' = \begin{cases} a_0 \odot a_1 \odot \cdots \odot a_k & \text{, if } \mathrm{mod}_1 = \varepsilon \\ a_0 \odot a_1^{|t'|_{x_1}} \odot \cdots \odot a_k^{|t'|_{x_k}} & \text{, if } \mathrm{mod}_1 = o \end{cases}.$$

Hence $\widehat{\mu_2}_{\mathrm{mod}_2}(s) = \widehat{\mu_1}_{\mathrm{mod}_1}(s) = a'\, t'[t_1, \ldots, t_k]$ and $a = a'$, which concludes the proof of the statement. ■

The next corollary shows that in case we have a non-deleting (likewise linear) deterministic bu-w-tt, then we can specify a family of mod-translation mappings with $\mathrm{mod} = o$ (likewise $\mathrm{mod} = \varepsilon$) and then apply the previous lemma to obtain an inclusion result.

4.12 Corollary (Non-deletion and linearity in periodic and commutative monoids) Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic and commutative monoid and $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ be two modifiers. We have $\pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$ for every $\pi \in P$ where

$$P = \begin{cases} \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}} & \text{, if } \mathrm{mod}_1 = o \\ \Pi_{\mathrm{l}} \setminus \Pi_{\mathrm{h}} & \text{, if } \mathrm{mod}_1 = \varepsilon \end{cases}.$$

Proof. Trivially the statement holds, if $\mathrm{mod}_1 = \mathrm{mod}_2$. Thus assume that mod_1 and mod_2 are distinct.

Case 1: Let $\mathrm{mod}_1 = o$ and $\tau^o \in \pi\text{–BOT}^o(\mathcal{A})$ for some $\pi \in \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a non-deleting deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π such that $\tau^o_{M_1} = \tau^o$. Moreover, let $f_{M_1,o} = (\, f^k_{M_1,o}\,)_{k \in \mathbb{N}}$ be the family of mappings

$$f^k_{M_1,o} : \Big(\bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k)) \Big) \times [k] \times A \longrightarrow A$$

defined for every integer $k \in \mathbb{N}$, output tree $t \in \bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,o}(t, i, a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a^{|t|_{x_i} - 1} & \text{, otherwise} \end{cases}.$$

Each mapping $f^k_{M_1,o}(t, i, a)$ is well-defined, because by the non-deletion restriction we have $1 \leq |t|_{x_i}$ for every output tree $t \in \bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$ and index $i \in [k]$. Consequently, the exponent is non-negative in the definition of $f^k_{M_1,o}(t, i, a)$. Moreover, $f_{M_1,o}$ is trivially a family of o-translation mappings. Thus, due to Lemma 4.11, there exists a non-deleting deterministic bu-w-tt M_2 obeying the restrictions of π such that $\tau_{M_2} = \tau^o$. Hence $\pi\text{–BOT}^o(\mathcal{A}) \subseteq \pi\text{–BOT}(\mathcal{A})$ for every $\pi \in \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}}$.

Case 2: Secondly, let $\mathrm{mod}_1 = \varepsilon$ and $\tau \in \pi\text{–BOT}(\mathcal{A})$ for some $\pi \in \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a linear deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π such that $\tau_{M_1} = \tau$. Moreover, let $f_{M_1,\varepsilon} = (\, f^k_{M_1,\varepsilon}\,)_{k \in \mathbb{N}}$ be the family of mappings

$$f^k_{M_1,\varepsilon} : \Big(\bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k)) \Big) \times [k] \times A \longrightarrow A$$

defined for every integer $k \in \mathbb{N}$, output tree $t \in \bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,\varepsilon}(t, i, a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a^{1 - |t|_{x_i}} & \text{, otherwise} \end{cases}.$$

Each mapping $f^k_{M_1,\varepsilon}(t, i, a)$ is well-defined, because by the linearity restriction we have $|t|_{x_i} \leq 1$ for every output tree $t \in \bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$ and index $i \in [k]$. Consequently, the exponent is non-negative in the definition of $f^k_{M_1,\varepsilon}(t, i, a)$. Moreover, $f_{M_1,\varepsilon}$ is obviously a family of translation mappings. Thus there exists a linear deterministic bu-w-tt M_2 obeying the restrictions of π such that $\tau^o_{M_2} = \tau$ due to Lemma 4.11. Hence $\pi\text{–BOT}(\mathcal{A}) \subseteq \pi\text{–BOT}^o(\mathcal{A})$ for every $\pi \in \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}}$. ∎

These are all the non-trivial inclusion results we are able to prove without requiring further properties of the monoid. So it remains to show incomparability results similar to Lemma 4.7. We start by showing that as long as the monoid is not regular, there exists a non-deleting homomorphism bu-w-tt computing a t-ts transformation, which cannot be computed by a deterministic bu-w-tt as o-t-ts transformation. We finally note that periodicity is not even required for the proof, which is similar to the proof of the corresponding statement in non-periodic semirings (cf. Lemma 4.7).

4.13 Lemma (Non-deleting homomorphism bu-w-tt in non-regular monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative monoid which is not regular.

$$\text{hn–BOT}(\mathcal{A}) \not\subseteq \text{d–BOT}^o(\mathcal{A})$$

Proof. Since the monoid $\mathcal{A}$ is not regular, there exists an element $a \in A$ such that there is no $b \in A$ with $b \odot a^2 = a$. Let $M_1 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the homomorphism bu-w-tt specified by the input ranked alphabet $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}\}$, output ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, transition mappings $\delta_1 = ((\delta_1)^1_\gamma, (\delta_1)^0_\alpha)$, and output mappings $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha)$.

$$(\delta_1)^1_\gamma(\star) = (\delta_1)^0_\alpha() = \star \quad , \quad (\mu_1)^1_\gamma(\star) = \mathbf{1}\,\sigma(x_1, x_1) \quad , \quad (\mu_1)^0_\alpha() = a\,\alpha.$$

Clearly, M_1 is a non-deleting homomorphism bu-w-tt, so $\tau = \tau_{M_1} \in \text{hn–BOT}(\mathcal{A})$. For every input tree $s \in T_\Gamma$ let $t_s \in T_\Sigma$ be the fully balanced output tree such that the heights of the trees s and t_s are equal. An easy calculation yields that for every input tree $s \in T_\Gamma$ the equality $\tau(s) = a\,t_s$ holds.

Next we prove that $\tau \notin \text{d–BOT}^o(\mathcal{A})$. In order to derive a contradiction assume that there is a deterministic bu-w-tt $M_2 = (Q_2, \Gamma, \Sigma, \mathcal{A}, F_2, \delta_2, \mu_2)$ such that $\tau^o_{M_2} = \tau$. Since for every $s \in T_\Gamma$ it holds that $\tau(s) \neq \widetilde{\mathbf{0}}$ and M_2 has only a finite set Q_2 of states, there must exist a final state $q \in F_2$ such that for two distinct input trees $s_1, s_2 \in T_\Gamma$, i.e., $s_1 \neq s_2$, we have $\widehat{\delta_2}(s_1) = q = \widehat{\delta_2}(s_2)$. Consequently, $\tau^o_{M_2}(s_i) = \widehat{\mu_2}_o(s_i)$ for every index $i \in [2]$. Moreover, also $\widehat{\delta_2}(\gamma(s_i)) \in F$, hence

$$\tau^o_{M_2}(\gamma(s_i)) = \widehat{\mu_2}_o(\gamma(s_i)) = (\mu_2)^1_\gamma(\widehat{\delta_2}(s_i)) \stackrel{o}{\longleftarrow} (\widehat{\mu_2}_o(s_i)) = (\mu_2)^1_\gamma(q) \stackrel{o}{\longleftarrow} (\tau(s_i)).$$

Trivially, $(\mu_2)^1_\gamma(q) \neq \widetilde{\mathbf{0}}$, otherwise $\tau^o_{M_2}(\gamma(s_i)) = \widetilde{\mathbf{0}}$. Let $(\mu_2)^1_\gamma(q) = b\,t$ for some monoid element $b \in A$ and output tree $t \in T_\Sigma(X_1)$. Moreover, recall that $\tau(s_i) = a\,t_{s_i}$. We can readily conclude that $t = \sigma(x_1, x_1)$, else either $t[t_{s_1}]$ or $t[t_{s_2}]$ is not fully balanced or $\text{height}(t[t_{s_i}]) \neq \text{height}(s_i) + 1$ for some index $i \in [2]$. We continue with

$$\tau^o_{M_2}(\gamma(s_i)) = (\mu_2)^1_\gamma(q) \stackrel{o}{\longleftarrow} (\tau(s_i)) = b\,\sigma(x_1, x_1) \stackrel{o}{\longleftarrow} (a\,t_{s_i}) = (b \odot a^2)\,\sigma(t_{s_i}, t_{s_i}).$$

According to $\tau^o_{M_2} = \tau$, we also derive

$$\tau^o_{M_2}(\gamma(s_i)) = (b \odot a^2)\,\sigma(t_{s_i}, t_{s_i}) = a\,\sigma(t_{s_i}, t_{s_i}) = \tau(\gamma(s_i)).$$

Consequently, we should have $b \odot a^2 = a$, but the element a was chosen such that this is impossible. Thus we arrived at a contradiction which yields $\tau \notin \text{d–BOT}^o(\mathcal{A})$. ∎

Secondly, we show that there exists an o-t-ts transformation τ computed by a linear homomorphism bu-w-tt such that there exists no deterministic bu-w-tt computing τ as t-ts transformation unless $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ is actually a group with an absorbing element $\mathbf{0}$.

4.14 Lemma (Linear homomorphism bu-w-tt in non-group monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative monoid which is no group.

$$\text{hl–BOT}^o(\mathcal{A}) \not\subseteq \text{d–BOT}(\mathcal{A})$$

Proof. The monoid $\mathcal{A}$ is no group, hence there exists a monoid element $a \in A \setminus \{\mathbf{0}\}$, which cannot be inverted, i.e., there is no element $b \in A$ such that $b \odot a = \mathbf{1}$. Let $M_1 = (\{\star\}, \Gamma, \Gamma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the homomorphism bu-w-tt specified by the ranked alphabet $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}\}$, transition mappings $\delta_1 = ((\delta_1)^1_\gamma, (\delta_1)^0_\alpha)$, and output mappings $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha)$.

$$(\delta_1)^1_\gamma(\star) = (\delta_1)^0_\alpha() = \star \quad , \quad (\mu_1)^1_\gamma(\star) = \mathbf{1}\,\alpha \quad , \quad (\mu_1)^0_\alpha() = a\,\alpha.$$

Clearly, M_1 is a linear homomorphism bu-w-tt, thus $\tau^o = \tau^o_{M_1} \in \text{hl–BOT}^o(\mathcal{A})$. A straightforward calculation yields $\tau^o(\alpha) = a\,\alpha$ and for every other input tree $s \in T_\Gamma \setminus \{\alpha\}$ the equality $\tau^o(s) = \mathbf{1}\,\alpha$ holds.

Next we prove that $\tau^o \notin$ d–BOT($\mathcal{A}$). For a contradiction assume that there exists a deterministic bu-w-tt $M = (Q_2, \Gamma, \Gamma, \mathcal{A}, F_2, \delta_2, \mu_2)$ such that $\tau_{M_2} = \tau^o$. Obviously,

$$a\,\alpha = \tau^o(\alpha) = \tau_{M_2}(\alpha) = \widehat{\mu_2}(\alpha) = (\mu_2)^0_\alpha().$$

Since we also have $\tau^o(\gamma(\alpha)) = \mathbf{1}\,\alpha$ we immediately obtain

$$\begin{array}{rclclcl} \tau_{M_2}(\gamma(\alpha)) & = & \widehat{\mu_2}(\gamma(\alpha)) & = & (\mu_2)^1_\gamma(\widehat{\delta_2}(\alpha)) \longleftarrow (\widehat{\mu_2}(\alpha)) & = & (\mu_2)^1_\gamma((\delta_2)^0_\alpha()) \longleftarrow (a\,\alpha) \\ & = & b\,t \longleftarrow (a\,\alpha) & = & (b \odot a)\,t[\alpha] & & \end{array}$$

for some element $b \in A$ and output tree $t \in T_\Gamma(X_1)$. Moreover, we have that $(b \odot a)\,t[\alpha] = \mathbf{1}\,\alpha$, hence $b \odot a = \mathbf{1}$. Contrary, the element a was chosen such that such an element b does not exist. Thus we derived the desired contradiction and conclude $\tau^o \notin$ d–BOT($\mathcal{A}$). ■

We have already seen in Lemma 4.13 that the class of all t-ts transformations computed by non-deleting homomorphism bu-w-tt is not contained in the class of all o-t-ts transformations computed by deterministic bu-w-tt as long as the monoid $\mathcal{A}$ is not regular, i.e., hn–BOT($\mathcal{A}$) $\nsubseteq$ d–BOT^o($\mathcal{A}$). It is furthermore clear that the class of all o-t-ts transformations computed by non-deleting homomorphism bu-w-tt is properly contained in the class of all t-ts transformations computed by deterministic bu-w-tt due to Corollary 4.12 (on periodic and commutative monoids), i.e., hn–BOT^o($\mathcal{A}$) $\subseteq$ d–BOT($\mathcal{A}$). However, the relation between the class of o-t-ts transformations computed by non-deleting homomorphism bu-w-tt and the class of t-ts transformations computed by non-deleting homomorphism bu-w-tt is yet unsettled. The next lemma will solve this question for all non-idempotent monoids.

4.15 Lemma (Non-deleting homomorphism in non-idempotent monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a non-idempotent monoid.

$$\text{hn–BOT}^o(\mathcal{A}) \nsubseteq \text{h–BOT}(\mathcal{A})$$

Proof. Let $a \in A \setminus \{\mathbf{0}, \mathbf{1}\}$ be a monoid element such that $a \odot a \neq a$. Such an element exists due to the assumption that $\mathcal{A}$ is non-idempotent. Moreover, let $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ be ranked alphabets and $M_1 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the non-deleting homomorphism bu-w-tt with $\delta_1 = ((\delta_1)^1_\gamma, (\delta_1)^0_\alpha, (\delta_1)^0_\beta)$ and $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha, (\mu_1)^0_\beta)$ specified by

$$(\delta_1)^1_\gamma(\star) = (\delta_1)^0_\alpha() = (\delta_1)^0_\beta() = \star \quad , \quad (\mu_1)^1_\gamma(\star) = \mathbf{1}\,\sigma(x_1, x_1) \quad , \quad (\mu_1)^0_\alpha() = a\,\alpha \quad , \quad (\mu_1)^0_\beta() = \mathbf{1}\,\alpha.$$

Let $\tau^o = \tau^o_{M_1}$. Clearly, $\tau^o \in$ hn–BOT^o($\mathcal{A}$), and moreover, $\tau^o(\gamma(\alpha)) = a^2\,\sigma(\alpha, \alpha)$ as well as $\tau^o(\gamma(\beta)) = \mathbf{1}\,\sigma(\alpha, \alpha)$.

Now let us prove that $\tau^o \notin$ h–BOT($\mathcal{A}$). We prove this statement by contradiction, so assume that there exists a homomorphism bu-w-tt $M_2 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$ such that $\tau_{M_2} = \tau^o$. Trivially, $\delta_2 = \delta_1$ and $\mu_2 = ((\mu_2)^1_\gamma, (\mu_2)^0_\alpha, (\mu_2)^0_\beta)$ with

$$(\mu_2)^1_\gamma(\star) = c\,t \quad , \quad (\mu_2)^0_\alpha() = a\,\alpha \quad , \quad (\mu_2)^0_\beta() = \mathbf{1}\,\alpha$$

for some monoid element $c \in A$ and output tree $t \in T_\Sigma(X_1)$. Moreover, we readily observe $t = \sigma(x_1, x_1)$. Consequently, $\tau_{M_2}(\gamma(\alpha)) = (c \odot a)\,\sigma(\alpha, \alpha)$ and $\tau_{M_2}(\gamma(\beta)) = c\,\sigma(\alpha, \alpha)$. Thus we obtain the equalities $c = \mathbf{1}$ and $c \odot a = a^2$, which yield $a = a^2$. Contrary, $a \in A$ was chosen such that $a \neq a^2$. Thus we derived the desired contradiction and conclude that $\tau^o \notin$ h–BOT($\mathcal{A}$). ■

4.16 Corollary (Corollary of Lemma 4.15)
If and only if hn–BOT^o($\mathcal{A}$) = hn–BOT($\mathcal{A}$) then $\mathcal{A}$ is idempotent.

Proof. The equality in idempotent monoids is proved in Corollary 4.22 and Lemma 4.15 proves the inequality in all non-idempotent monoids. ■

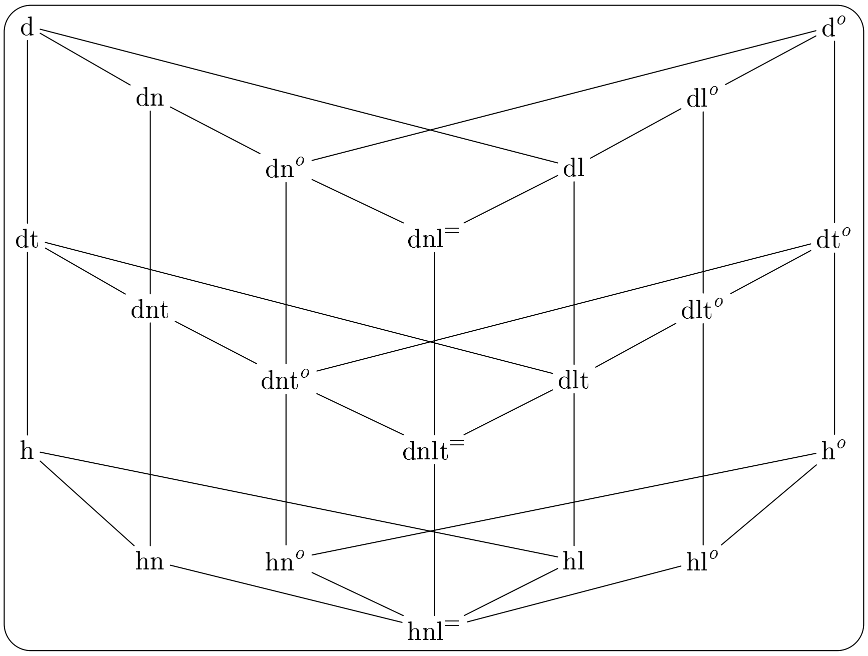

Figure 3: Inclusion diagram for periodic, commutative, and non-regular monoids.

Finally, we are able to present the inclusion diagram for periodic and commutative monoids $\mathcal{A}$, which are not regular. The latter restriction assures that $\mathcal{A}$ is also neither idempotent nor a group. Those cases will be handled in subsequent subsections.

4.17 Theorem (Periodic, commutative, and non-regular monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative, and non-regular monoid with an absorbing element $\mathbf{0}$. Figure 3 is the inclusion diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.

Proof. All the inclusions are either trivial or follow from Corollary 4.12, whereas the equalities are due to Observation 4.4. Then the following eight statements are sufficient to prove strictness and incomparability. For every modifier mod $\in \{\varepsilon, o\}$

(i)	$\text{dnlt–BOT}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\text{mod}}(\mathcal{A})$,	(ii)	$\text{dnl–BOT}(\mathcal{A}) \not\subseteq \text{dt–BOT}^{\text{mod}}(\mathcal{A})$,
(iii)	$\text{hn–BOT}^o(\mathcal{A}) \not\subseteq \text{dl–BOT}^o(\mathcal{A})$,	(iv)	$\text{hl–BOT}(\mathcal{A}) \not\subseteq \text{dn–BOT}(\mathcal{A})$,
(v)	$\text{hn–BOT}(\mathcal{A}) \not\subseteq \text{d–BOT}^o(\mathcal{A})$,	(vi)	$\text{hl–BOT}^o(\mathcal{A}) \not\subseteq \text{d–BOT}(\mathcal{A})$,
(vii)	$\text{hn–BOT}^o(\mathcal{A}) \not\subseteq \text{h–BOT}(\mathcal{A})$,	(viii)	$\text{hl–BOT}(\mathcal{A}) \not\subseteq \text{h–BOT}^o(\mathcal{A})$.

The non-inclusions (i) – (iv) are proved in Corollary 4.3, whereas we obtain non-inclusion (v) from Lemma 4.13, non-inclusion (vi) from Lemma 4.14, non-inclusion (vii) from Lemma 4.15, and non-inclusion (viii) from Lemma 4.5. ∎

4.4 Periodic, commutative, and regular monoids

In this subsection we consider monoids $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ which are periodic, commutative, and regular. An example of a periodic, commutative, and regular monoid which is not idempotent and neither a group is $\mathbb{Z}_6$. Specifically the regularity allows us to derive more inclusion results. The next corollary states this formally. Roughly speaking the classes of t-ts transformations become subsets of the corresponding classes of o-t-ts transformations, except for the classes bearing the homomorphism restriction.

4.18 Corollary (Periodic, commutative, and regular monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative, and regular monoid. Then for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$ we have $\pi\text{–BOT}(\mathcal{A}) \subseteq \pi\text{–BOT}^o(\mathcal{A})$.

Proof. Let $\tau \in \pi\text{–BOT}(\mathcal{A})$ for some $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π such that $\tau_{M_1} = \tau$. Moreover, let $f_{M_1,\varepsilon} = (\, f^k_{M_1,\varepsilon}\,)_{k\in\mathbb{N}}$ be the family of mappings

$$f^k_{M_1,\varepsilon} : \Big(\bigcup_{\sigma\in\Sigma^{(k)}, q_1,\dots,q_k\in Q_1} \operatorname{supp}((\mu_1)^k_\sigma(q_1,\dots,q_k))\Big) \times [k] \times A \longrightarrow A$$

defined for every integer $k \in \mathbb{N}$, output tree $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\dots,q_k\in Q_1} \operatorname{supp}((\mu_1)^k_\sigma(q_1,\dots,q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,\varepsilon}(t, i, a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a & \text{, if } a \neq \mathbf{0}, |t|_{x_i} = 0 \\ b^{|t|_{x_i}-1} & \text{, otherwise} \end{cases},$$

where $b \in A$ is such that $a^2 \odot b = a$. Such an element $b \in A$ exists for every $a \in A$ due to regularity.

Each mapping $f^k_{M_1,\varepsilon}(t, i, a)$ is well-defined, because in the case distinction every exponent is non-negative in the definition of $f^k_{M_1,\varepsilon}(t, i, a)$. Moreover, it is straightforward to prove that $f_{M_1,\varepsilon}$ is a family of translation mappings for M_1. Thus, due to Lemma 4.11, there exists a deterministic bu-w-tt M_2 obeying the restrictions π such that $\tau^o_{M_2} = \tau$. Hence $\pi\text{–BOT}(\mathcal{A}) \subseteq \pi\text{–BOT}^o(\mathcal{A})$ for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. ∎

Since we cannot apply Lemma 4.13 to show that the classes of t-ts and o-t-ts transformations computed by non-deleting homomorphism bu-w-tt are incomparable, but Lemma 4.15 already delivers one half, we establish the remaining half in the next lemma.

4.19 Lemma (Non-deleting homomorphisms in regular and non-idempotent monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative and regular, but non-idempotent monoid.

$$\text{hn–BOT}(\mathcal{A}) \not\subseteq \text{h–BOT}^o(\mathcal{A})$$

Proof. Since $\mathcal{A}$ is not idempotent, but regular, there exist monoid elements $a, b \in A \setminus \{\mathbf{0}, \mathbf{1}\}$ such that $a \neq a^2$ and $a^2 \odot b = a$. Let $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ be ranked alphabets and $M_1 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the non-deleting homomorphism bu-w-tt specified by

$$(\delta_1)^1_\gamma(\star) = (\delta_1)^0_\alpha() = \star \quad , \quad (\mu_1)^1_\gamma(\star) = a\,\sigma(x_1, x_1) \quad , \quad (\mu_1)^0_\alpha() = b\,\alpha.$$

Let $\tau = \tau_{M_1}$. Clearly, $\tau \in \text{hn–BOT}(\mathcal{A})$, and moreover, $\tau(\gamma(\alpha)) = (a \odot b)\,\sigma(\alpha, \alpha)$,

$$\tau(\gamma^2(\alpha)) = a\,\sigma(\sigma(\alpha,\alpha), \sigma(\alpha,\alpha)) \quad , \quad \tau(\gamma^3(\alpha)) = a^2\,\sigma(\sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha)), \sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha))).$$

Now let us prove that $\tau \notin \text{h–BOT}^o(\mathcal{A})$. We prove this statement by contradiction, so assume that there exists a homomorphism bu-w-tt $M_2 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$ such that $\tau^o_{M_2} = \tau$. Trivially, $\delta_2 = \delta_1$, $(\mu_2)^1_\gamma(\star) = c\,t$, and $(\mu_2)^0_\alpha() = b\,\alpha$ for some monoid element $c \in A$ and output tree $t \in T_\Sigma(X_1)$. Moreover, we readily observe $t = \sigma(x_1, x_1)$, otherwise $\operatorname{supp}(\tau^o_{M_2}(\gamma(\alpha))) \neq \{\sigma(\alpha,\alpha)\}$ or $\operatorname{supp}(\tau^o_{M_2}(\gamma^2(\alpha))) \neq \{\sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha))\}$. Hence $\tau^o_{M_2}(\gamma(\alpha)) = (b^2 \odot c)\,\sigma(\alpha,\alpha)$,

$$\begin{aligned} \tau^o_{M_2}(\gamma^2(\alpha)) &= (b^4 \odot c^3)\,\sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha)) \\ \tau^o_{M_2}(\gamma^3(\alpha)) &= (b^8 \odot c^7)\,\sigma(\sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha)),\sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha))). \end{aligned}$$

Thus we obtain the equalities

$$b^2 \odot c = a \odot b \quad , \quad b^4 \odot c^3 = a \quad , \quad b^8 \odot c^7 = a^2.$$

Now we compute as follows

$$a = b^4 \odot c^3 = (b^2 \odot c) \odot (b^2 \odot c) \odot c = (a \odot b) \odot (a \odot b) \odot c = (a^2 \odot b) \odot b \odot c = a \odot b \odot c$$

and $a^2 = b^8 \odot c^7 = (b^4 \odot c^3) \odot (b^4 \odot c^3) \odot c = a^2 \odot c$. Next we multiply the former equation with a which gives $a^2 = a^2 \odot b \odot c = a \odot c$ and the latter equation with b which yields $a = a^2 \odot b = a^2 \odot b \odot c = a \odot c$. Hence $a = a^2$, which is a contradiction, because a was chosen such that $a \neq a^2$. Thus we conclude that $\tau \notin$ h–BOT$^o(\mathcal{A})$. ■

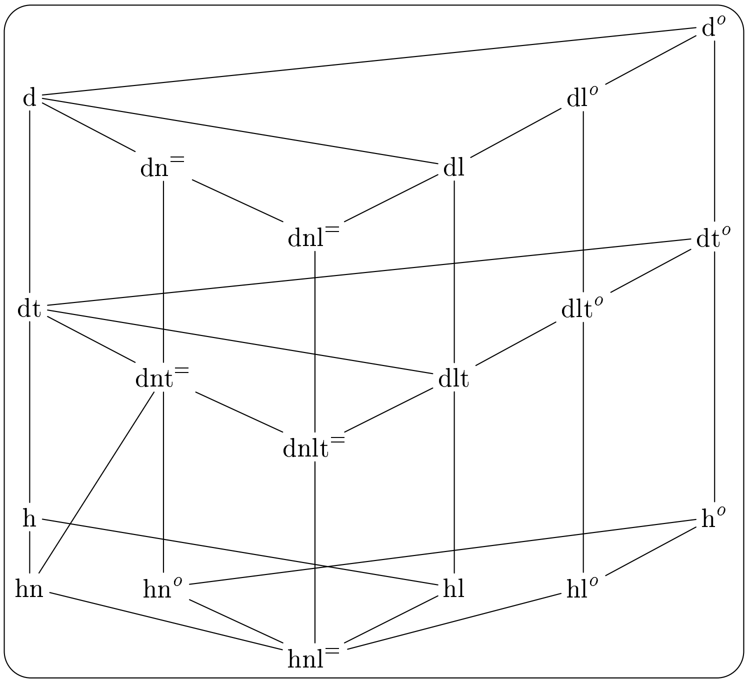

Figure 4: Inclusion diagram for periodic, commutative, and regular monoids, which are non-idempotent and no group.

At this point we have all the results necessary to derive the inclusion diagram for periodic, commutative, and regular monoids, which are neither idempotent nor groups.

4.20 Theorem (Periodic, commutative, and regular monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative, and regular monoid, which is non-idempotent and no group with an absorbing element $\mathbf{0}$. Figure 4 is the inclusion diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.

Proof. All the inclusions are either trivial or follow from Corollary 4.12 and Corollary 4.18. The equalities are due to Observation 4.4, Corollary 4.12 and Corollary 4.18. Then the following seven statements are sufficient to prove strictness and incomparability. For every two distinct modifiers $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$, i.e., $\mathrm{mod}_1 \neq \mathrm{mod}_2$,

(i) dnlt–BOT$(\mathcal{A}) \not\subseteq$ h–BOT$^{\mathrm{mod}_1}(\mathcal{A})$, (ii) dnl–BOT$(\mathcal{A}) \not\subseteq$ dt–BOT$^o(\mathcal{A})$,

(iii) hn–BOT$^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq$ dl–BOT$^o(\mathcal{A})$, (iv) hl–BOT$(\mathcal{A}) \not\subseteq$ dn–BOT$(\mathcal{A})$,

(v) hl–BOT$(\mathcal{A}) \not\subseteq$ h–BOT$^o(\mathcal{A})$, (vi) hl–BOT$^o(\mathcal{A}) \not\subseteq$ d–BOT$(\mathcal{A})$,

(vii) hl–BOT$^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq$ d–BOT$^{\mathrm{mod}_2}(\mathcal{A})$.

The non-inclusions (i) – (iv) are proved in Corollary 4.3, whereas non-inclusion (v) follows from Lemma 4.5, non-inclusion (vi) follows from Lemma 4.14, and non-inclusion (vii) follows from Lemma 4.15 and Lemma 4.19. ■

4.5 Commutative and idempotent monoids

This subsection is devoted to the study of commutative and idempotent monoids. The monoid $\mathbb{R}_{\max}$ is an example of such a monoid. Clearly, $a^n = a$ for every integer $n \in \mathbb{N}_+$ and element $a \in A$ of such a monoid. Hence we easily derive the following observation.

4.21 Observation (Substitution in idempotent monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be an idempotent monoid, $k \in \mathbb{N}$, and Δ be a ranked alphabet. For every non-deleting (in X_k) output tree $t \in T_\Delta(X_k)$, monoid element $a \in A$, and monomials $m_1, \ldots, m_k \in A[T_\Delta]$ we have that

$$a\, t \longleftarrow (m_1, \ldots, m_k) = a\, t \overset{o}{\longleftarrow} (m_1, \ldots, m_k). \qquad \square$$

4.22 Corollary (Corollary of Observation 4.21)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be an idempotent monoid. Then π–BOT$^o(\mathcal{A}) = \pi$–BOT$(\mathcal{A})$ for every $\pi \in \Pi_{\mathrm{n}}$. $\square$

These are indeed all the new results necessary to prove the inclusion diagram. Note that idempotent monoids are trivially regular and periodic, so we apply some of the results derived in Subsection 4.4.

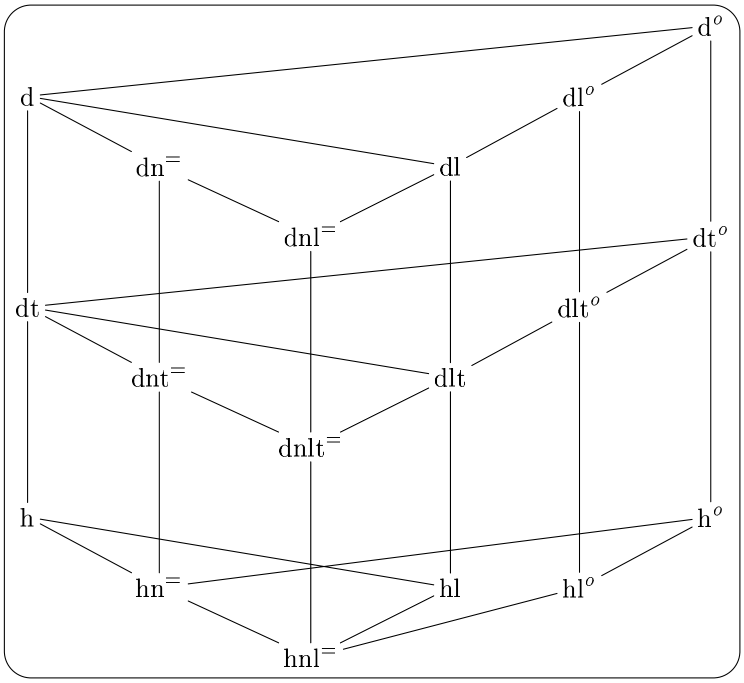

Figure 5: Inclusion diagram for commutative and idempotent monoids with at least three elements.

4.23 Theorem (Commutative and idempotent monoids)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative and idempotent monoid such that $A \setminus \{\mathbf{0}, \mathbf{1}\} \neq \emptyset$. Figure 5 is the inclusion diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.

Proof. All the inclusions are either trivial or follow from Corollary 4.18. The equalities are due to Observation 4.4 and Corollary 4.22. Then the following six statements are sufficient to prove strictness and incomparability. For every modifier mod $\in \{\varepsilon, o\}$

(i)	dnlt–BOT$(\mathcal{A}) \not\subseteq$ h–BOT$^{\mathrm{mod}}(\mathcal{A})$,	(ii)	dnl–BOT$(\mathcal{A}) \not\subseteq$ dt–BOT$^o(\mathcal{A})$,
(iii)	hn–BOT$(\mathcal{A}) \not\subseteq$ dl–BOT$^o(\mathcal{A})$,	(iv)	hl–BOT$(\mathcal{A}) \not\subseteq$ dn–BOT$(\mathcal{A})$,
(v)	hl–BOT$(\mathcal{A}) \not\subseteq$ h–BOT$^o(\mathcal{A})$,	(vi)	hl–BOT$^o(\mathcal{A}) \not\subseteq$ d–BOT$(\mathcal{A})$.

The non-inclusions (i) – (iv) are proved in Corollary 4.3, whereas non-inclusion (v) follows from Lemma 4.5 and non-inclusion (vi) follows from Lemma 4.14. ∎

4.6 Periodic and commutative groups

Finally, in this last subsection we consider periodic and commutative groups with an absorbing element $\mathbf{0}$. For example, the monoid $\mathbb{Z}_3$ fulfils all those restrictions. Note that all such monoids (except $\mathbb{Z}_2$) are non-idempotent. Due to the existence of inverses we can now easily derive a final corollary from Lemma 4.11.

4.24 Corollary (Periodic and commutative groups)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic and commutative group and $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$ be two modifiers. Then $\pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$ for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$.

Proof. The statement is trivial, if $\mathrm{mod}_1 = \mathrm{mod}_2$. Henceforth let mod_1 and mod_2 be distinct. Moreover, let $\tau \in \pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$ for some $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π such that $\tau_{M_1}^{\mathrm{mod}_1} = \tau$. Moreover, let $f_{M_1,\mathrm{mod}_1} = (\, f^k_{M_1,\mathrm{mod}_1}\,)_{k \in \mathbb{N}}$ be the family of mappings

$$f^k_{M_1,\mathrm{mod}_1} : \Big(\bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k)) \Big) \times [k] \times A \longrightarrow A$$

defined for every integer $k \in \mathbb{N}$, output tree $t \in \bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,\mathrm{mod}_1}(t, i, a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a^{1 - |t|_{x_i}} & \text{, if } a \neq \mathbf{0}, \mathrm{mod}_1 = \varepsilon \\ a^{|t|_{x_i} - 1} & \text{, if } a \neq \mathbf{0}, \mathrm{mod}_1 = o \end{cases}.$$

Each mapping $f^k_{M_1,\mathrm{mod}_1}(t, i, a)$ is trivially well-defined due to the existence of inverses. Moreover, it is straightforward to prove that f_{M_1,mod_1} is a family of mod_1-translation mappings. Thus there exists a deterministic bu-w-tt M_2 obeying the restrictions π such that $\tau_{M_2}^{\mathrm{mod}_2} = \tau$ due to Lemma 4.11. Hence $\pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$ for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. ∎

Since we demand that we have at least three elements, our group is non-idempotent which allows us to reuse some the results of earlier subsections. Finally, we present the last inclusion diagram.

4.25 Theorem (Periodic and commutative groups with at least three elements)
Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic and commutative group with an absorbing element $\mathbf{0}$ such that $A \setminus \{\mathbf{0}, \mathbf{1}\} \neq \emptyset$. Figure 6 is the inclusion diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.

Proof. All the inclusions are either trivial or follow from Corollary 4.24. The equalities are due to Observation 4.4 and Corollary 4.24. Then the following six statements are sufficient to prove strictness and incomparability. For every two distinct modifiers $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$, i.e., $\mathrm{mod}_1 \neq \mathrm{mod}_2$,

(i) $\text{dnlt–BOT}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\mathrm{mod}_1}(\mathcal{A})$,   (ii) $\text{dnl–BOT}(\mathcal{A}) \not\subseteq \text{dt–BOT}(\mathcal{A})$,

(iii) $\text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{dl–BOT}(\mathcal{A})$,   (iv) $\text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{dn–BOT}(\mathcal{A})$,

(v) $\text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\mathrm{mod}_2}(\mathcal{A})$,   (vi) $\text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

The non-inclusions (i) – (iv) are proved in Corollary 4.3, whereas non-inclusion (v) follows from Lemma 4.15 and Lemma 4.19 and non-inclusion (vi) follows from Lemma 4.5. ∎

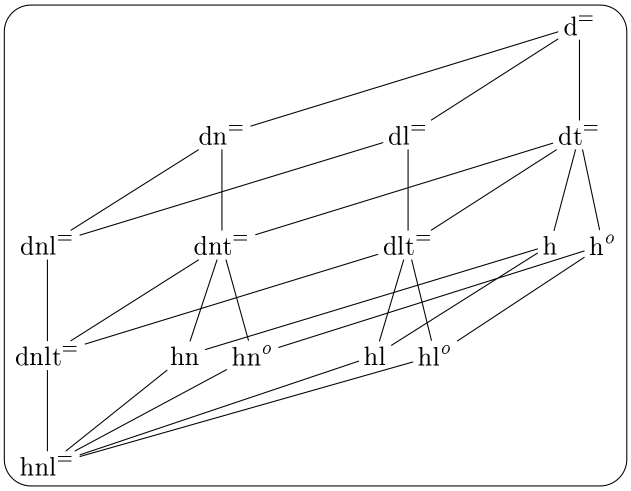

Figure 6: Inclusion diagram for periodic and commutative groups with an absorbing element **0** and at least three elements.

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