φ-Recognizable Formal Power Series are φ-Rational – An Alternative Proof

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Abstract: In this paper we show an alternative proof of the fact that φ-recognizable formal power series are φ-rational (cf. Lemma 3.5 of [DK02, DK03]). Our proof can be seen as an application of Tarjan's approach to solve algebraic path problems by separating the structural part from the algebraic part of the problem [Tar81].

1 Introduction

S.E. Kleene's famous result states that the classes of recognizable languages and of rational languages are equal [Kle56]. This result has been generalized to other classes of objects such as, e.g., formal power series [Sch61], formal power series in partially commuting variables [DG99], ω-languages [Büc60], infinitary formal power series [Püs03, DK02, DK03], tree languages [TW68], and formal tree series [Boz99, Kui97a, ÉK03, Pec03a, Pec03b, DPV03].

In this paper we focus on the class of φ-recognizable formal power series. A formal power series (for short: fps) is a mapping $S : \Delta^* \to A$ where Δ is an alphabet and A is the carrier set of a semiring $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$; the set of all such fps is denoted by $A\langle\langle\Delta^*\rangle\rangle$. Often an fps is also shown in the expansion form $S = \sum_{w\in\Sigma^*}(S, w)w$ where $(S, w) = S(w) \in A$ is the coefficient of w. For the theoretical background on fps, we refer the reader to [Eil74, SS78, KS86, BR88, Kui97b]. An fps S is recognizable if there is a *weighted automaton* $\mathcal{A}$ which accepts S (cf. Chapter VI, Section 6 of [Eil74]). Roughly speaking, $\mathcal{A}$ is a usual finite state automaton over strings, but with every transition of $\mathcal{A}$ a weight, i.e., an element of A, is associated. Then, given a word $w \in \Delta^*$, all the accepting runs of $\mathcal{A}$ on w are considered. For every such run, its weight is obtained by combining the weights of its transitions by means of the semiring multiplication $\odot_A$. If there is more than one run on w, then the weights of these runs are combined by using the semiring addition $\oplus_A$. Thereby $\mathcal{A}$ computes an fps $S(\mathcal{A}) : \Delta^* \to A$.

Now let $\varphi : A \to A$ be an endomorphism, i.e., a homomorphism from A to A; this will play the role of a discounting factor (cf. Section 8.2 of [GM84]). Then, along a run of $\mathcal{A}$ on w, higher and higher powers of φ are applied to the weights of the transitions, starting from φ^0 and ending in $\varphi^{|w|-1}$ where $|w|$ is the length of w. More precisely, if $\delta(q, a, q') \in A$ denotes the weight of the transition (q, a, q') and $w = a_1a_2 \ldots a_n$, then the weight of the run $r = (q_0, a_1, q_1)(q_1, a_2, q_2) \ldots (q_{n-1}, a_n, q_n)$ on w is

$$\delta_\varphi(r) = \delta(q_0, a_1, q_1) \odot_A \varphi(\delta(q_1, a_2, q_2)) \odot_A \ldots \odot_A \varphi^{n-1}(\delta(q_{n-1}, a_n, q_n)) \in A.$$

The fps accepted by $\mathcal{A}$ with discounting factor φ is denoted by $S_\varphi(\mathcal{A})$ and it is called *φ-recognizable*; the class of φ-recognizable fps is denoted by $A^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle$.

Weighted automata with discounting factor were introduced in [DK02, DK03]. There it was proved that φ-recognizable fps are exactly the φ-rational fps (cf. Theorem 3.6 of [DK02, DK03]; in fact, they also proved this equality for infinitary fps over the semiring $\underline{\mathbb{R}_{\max}}$ of real numbers with $\oplus_A = \max$ and $\odot_A = +$ and with discounting factors of the form $\varphi(x) = z \cdot x$ for some $0 \le z < 1$; cf. Theorem 6.7 of

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[DK02, DK03]). The class of rational fps is the smallest class of fps in $A\langle\langle\Delta^*\rangle\rangle$ containing all monomials $1_A w$ with $w \in \Delta^*$ and being closed under the rational operations sum $\oplus$, Cauchy-product $\odot$, Kleene-$*$ $(.)^*$ restricted to proper fps (i.e., those fps S such that $(S, \varepsilon) = 0_A$), and multiplication with coefficients in A (cf. Chapter VII of [Eil74]). The class $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ of *φ-rational fps* is obtained in the same way, but rather than using the Cauchy-product and the Kleene-$*$, the appropriate generalizations to the φ-Cauchy-product $\odot_\varphi$ and the φ-Kleene-$*$ $(.)^{*,\varphi}$ have to be used. Here, we focus on the implication that φ-recognizability implies φ-rationality (cf. Lemma 3.5 of [DK02, DK03]), i.e.,

$$A^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle \subseteq A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle.$$

In the proof of Lemma 3.5 of [DK02, DK03], the authors use the concept of generalized weighted automata, in which the transitions can be labeled by rational expressions. All their considerations and computations are done on the level of fps or elements of the semiring $\underline{A}$. This is typical for the proofs of almost all the Kleene results mentioned above (for formal languages cf. Lemma 7 of [Kle56], for fps cf. Section III.A.4 of [Sch61], for fps in partially commuting variables cf. Proposition 34 of [DG99], for tree languages cf. Theorem 9 of [TW68], and for formal tree series cf., e.g., Section 5 of [Boz99] or Lemma 5.1 of [DPV03]).

Here we would like to present an alternative approach for proving the inclusion $A^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle \subseteq A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$. Roughly speaking, our proof is on the level of sets of paths through the automaton and of (kind of) homomorphisms which map sets of paths to elements of $A\langle\langle\Delta^*\rangle\rangle$. This approach is an application of Tarjan's approach to solve algebraic path problems by separating the structural part from the algebraic part of the problem [Tar81]. Let us first briefly explain the algebraic path problem and then discuss Tarjan's approach.

Let A be a set and $G = (V, E, \lambda)$ be a finite, directed, and weighted graph where $V = \{1, \ldots, n\}$ is the set of vertices, $E \subseteq V \times V$ is the set of edges, and $\lambda : E \to A$ is a function that gives weights to the edges. The graph G is also called an A-weighted graph. For vertices $u, v \in V$, let the set of paths through G from u to v be denoted by $P_{u,v}$. Now assume that $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$ is a semiring. Then λ can be canonically extended to a path evaluation function $\widetilde{\lambda} : \{U \mid U \subseteq P_{u,v}, u, v \in V\} \to A$ by defining

$$\widetilde{\lambda}(\{p\}) = \lambda(u_0, u_1) \odot_A \lambda(u_1, u_2) \odot_A \ldots \odot_A \lambda(u_{m-1}, u_m)$$

for every path $p = (u_0, u_1, u_2, \ldots, u_{m-1}, u_m) \in P_{u,v}$ with $u = u_0$ and $v = u_m$ and

$$\widetilde{\lambda}(U) = \sum_{p \in U} \widetilde{\lambda}(\{p\})$$

for every set $U \subseteq P_{u,v}$ if this sum exists (note that the index set can be infinite and hence the sum needs not always exist). The *algebraic path problem* is the problem of calculating effectively, for every $u, v \in V$, the element $\widetilde{\lambda}(P_{u,v})$ (cf., e.g., Chapter IV, Definition 6.8 of [HW98], or Chapter 26.4 of [CLR90]). Depending on the semiring used, the algebraic path problem has many important instances (cf., e.g., Chapter IV, Example 6.7 of [HW98]), such as e.g.,

- using the boolean semiring $\underline{\mathbb{B}} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$, it is the problem of computing the reflexive, transitive closure of G,
- using the tropical semiring $\underline{Trop} = (\mathbb{N}_\infty, \min, +, \infty, 0)$, it is the shortest-path problem in G,
- using the semiring $\underline{Prob} = ([0, 1], \max, \cdot, 0, 1)$, it is the problem of computing the most-reliable connection in a communication-network, and
- using the semiring $\underline{Cap} = (\mathbb{R}^{\geq 0}_\infty, \max, \min, 0, \infty)$, it is the maximal-capacity problem.

In fact, it was B.A. Carré who found out that the appropriate algebraic structure for a unified framework, in which all such problems on graphs can be described, is the semiring structure [Car71] (also cf. [BC75]). Since then, many investigations have been done both in generalizing the framework of [Car71] and in showing efficient algorithms for solving the algebraic path problem (the original algorithms for the first two above mentioned instances can be found in the seminal papers [War62] and [Flo62], respectively). It is not the task of this paper to survey all the relevant literature of such investigations. Rather, we only refer the reader to [DP84] for an extended bibliography about shortest-path problems, to [Rot90, Huc92, The95] for survey papers and university theses, and to [Leh77] in which, for a very general class of semirings, so called closed semi-rings, an efficient algorithm (the so-called *Warshall-Floyd-Kleene algorithm*) for the computation of the iteration of the adjacency matrix of the graph G has been proved correct.

In [Tar81], R.E. Tarjan proposed an approach for solving the algebraic path problem by separating the structural part from the algebraic part of the problem. The *structural part* only considers sets of paths through the given graph G and the decompositions of these sets. Actually, Tarjan quotes the problem of computing, for every pair u, v of vertices, a path expression $e_{u,v}$, which represents the set $[\![e_{u,v}]\!] = P_{u,v}$ of all paths from u to v, as the most general path problem. This problem can be solved, e.g., by using the Warshall-Floyd-Kleene algorithm of [Leh77] in which $e_{u,v}$ is computed in the form of a rational expression, i.e., an expression which contains edges as elementary items and uses the rational operations union, concatenation, and Kleene-$*$ to build up more complex expressions. The *algebraic part* now is the proof that the canonical extension $\widetilde{\lambda}$ of the labeling function $\lambda : E \to A$, behaves like a homomorphism with respect to the rational operations. Then, $\widetilde{\lambda}(P_{u,v}) = \widetilde{\lambda}([\![e_{u,v}]\!])$ and the composition $[\![.]\!] \circ \widetilde{\lambda}$ can be "pulled over" the rational structure of $e_{u,v}$ thereby exposing a way how to compute $\widetilde{\lambda}(P_{u,v})$. In fact, Tarjan's approach has the flavour of initial algebra semantics [GT74, GTWW77].

In this paper we apply Tarjan's approach in order to obtain an alternative proof of inclusion $A^{\mathrm{rec},\varphi}\langle\!\langle \Delta^* \rangle\!\rangle \subseteq A^{\mathrm{rat},\varphi}\langle\!\langle \Delta^* \rangle\!\rangle$. For this purpose we have three tasks:

1. formulate the computation of the fps $S_\varphi(\mathcal{A})$ which is φ-accepted by some weighted automaton $\mathcal{A}$, as an algebraic path problem,
2. show that particular sets of paths have a unique decomposition which results in a rational representation, and
3. prove that the path evaluation function $\widetilde{\lambda}$ is compatible with this rational representation.

Let us discuss these tasks in more detail and start with Task 1. In fact, for every pair q, q' of states of $\mathcal{A}$, we consider the fps $S_\varphi(\mathcal{A})_{q,q'}$ which is φ-accepted by $\mathcal{A}$ with initial state q and final state q'. Now Task 1 means that we have to construct from $\mathcal{A}$ a weighted graph $G_{\mathcal{A}} = (V, E, \lambda)$ with $\lambda : E \to B$ for some set B, and we have to find a semiring $\underline{B}$ with carrier set B such that, for every pair q, q' of states of $\mathcal{A}$, we have that

$$S_\varphi(\mathcal{A})_{q,q'} = \widetilde{\lambda}(P_{q,q'}),$$

where $P_{q,q'}$ is the set of paths from q to q' in the graph $G_{\mathcal{A}}$ and $\widetilde{\lambda}$ is the canonical extension of λ. Clearly, V is the set of states of $\mathcal{A}$ (represented as set $\{1, \ldots, n\}$ of natural numbers) and the edges are derived from the transitions of $\mathcal{A}$; moreover, the edge labeling function λ has the type $E \to A\langle\!\langle \Delta^* \rangle\!\rangle$ and, for every q, q', it is defined by

$$\lambda(q, q') = \sum_{a \in \Delta} \delta(q, a, q')\ a.$$

(Note that, for every $w \notin \Delta$, we have $(\lambda(q, q'), w) = 0_A$.) The appropriate semiring $\underline{B}$ is the *semiring*

$$\underline{A^\varphi\langle\!\langle \Delta^* \rangle\!\rangle} = (A\langle\!\langle \Delta^* \rangle\!\rangle, \oplus, \odot_\varphi, \widetilde{0}_A, 1_A\varepsilon)$$

of skewed fps (cf. Lemma 3.1 of [DK03]) where $\widetilde{0}_A$ is the fps in $A\langle\!\langle \Delta^* \rangle\!\rangle$ which maps every word to 0_A. Task 1 is solved formally in our Lemma 5.4.

For Task 2 we analyse, for every pair q, q' of vertices of $G_{\mathcal{A}}$, the set $\widehat{P}^{(k)}_{q,q'}$ of nonempty paths p from q to q' for which the inner vertices of p are taken from the set $\{1, \ldots, k\}$. We prove that such sets have an unambiguous decomposition which results in a rational representation (cf. Lemma 6.4; also cf. the decomposition of slightly different path sets in the proof of Theorem 6.13 in Chapter IV of [HW98]). Then

$$P_{q,q'} = \widehat{P}^{(n)}_{q,q'} \cup Z_{q,q'}$$

where $Z_{q,q'} = \{\varepsilon\}$ if $q = q'$, and $\emptyset$ otherwise. Note that Task 2 does <u>not</u> depend on the labeling function λ and its canonical extension $\widetilde{\lambda}$. Rather, it only depends on the sets V and E of vertices and edges of $G_{\mathcal{A}}$.

For Task 3 we prove that, under certain conditions, $\widetilde{\lambda}$ behaves like a homomorphism, i.e., is compatible with respect to the rational operations on sets of paths (cf. Lemma 7.3). In fact, since the compatibility depends on a property of <u>sets of paths</u> (rather than on properties of path expressions), we do not use the concept of path expressions as in [Tar81] but directly work on sets of paths.

Then, after having solved Tasks 1 to 3, we obtain that $S_\varphi(\mathcal{A})_{q,q'} = \widetilde{\lambda}(P_{q,q'}) = \widetilde{\lambda}(\widehat{P}^{(n)}_{q,q'} \cup Z_{q,q'})$ and, due to compatibility, $\widetilde{\lambda}$ can be pulled over the rational structure of $\widehat{P}^{(n)}_{q,q'} \cup Z_{q,q'}$, thereby automatically exposing the rational structure of $\widetilde{\lambda}(P_{q,q'})$ and thus of $S_\varphi(\mathcal{A})_{q,q'}$ in the semiring $\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}$ (cf. Theorem 7.4).

Now, is there any advantage of applying Tarjan's approach to prove that recognizability implies rationality? Well, we have already observed that in the proofs of half of Kleene's theorem and of most of its generalizations the relevant argumentation and the calculations are done in some semiring $\underline{C}$ of semantic objects. The more complex the semantic objects are, the more complex the calculations in the proofs become. Using Tarjan's approach allows to factorize out from these proofs at least Task 2, i.e., the reasoning on the well-known dynamic programming equation

$$B^{(k)}_{q,q'} = B^{(k-1)}_{q,q'} \oplus_C B^{(k-1)}_{q,k} \odot_C (B^{(k-1)}_{k,k})^{*,C} \odot_C B^{(k-1)}_{k,q'},$$

which is inherent (more or less disguised) in all the above mentioned proofs (here the variables $B^{(k)}_{q,q'}$ represent semantic objects and $\oplus_C$, $\odot_C$, and $(.)^{*,C}$ are the rational operations in $\underline{C}$). For sets of paths, this equation reads as follows:

$$\widehat{P}^{(k)}_{q,q'} = \widehat{P}^{(k-1)}_{q,q'} \cup \widehat{P}^{(k-1)}_{q,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,q'},$$

which is more or less a folklore result (cf. Lemma 6.4(5)). What remains is Task 1 and Task 3 where, of course, the latter has to be done on the level of semantic objects. Thus, we believe that Tarjan's approach which separates the structural part (dealing with the dynamic programming equations on sets $\widehat{P}^{(k)}_{q,q'}$ of paths) from the algebraic part (dealing with the path evaluation function $\widetilde{\lambda}$) of the problem, can lead to a kind of proof economy when proving that recognizability implies rationality.

In Sections 2, 3, and 4 of this paper we briefly recall the concepts of fps, weighted automata, and rational fps, respectively. In Section 5 we formulate the computation of $S_\varphi(\mathcal{A})$ as an algebraic path problem. In Section 6 we deal with unambiguous decompositions of paths and in Section 7 we show that $\widetilde{\lambda}$ is compatible with the rational operations. Also we prove inclusion $A^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle \subseteq A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ there.

2 Formal power series

A *semiring* is an algebraic structure $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$ with two operations sum $\oplus_A$ and product $\odot_A$ such that $(A, \oplus_A, 0_A)$ is a commutative monoid, $(A, \odot_A, 1_A)$ is a monoid, and for every $a, b, c \in A$ the distributivity laws $(a \oplus_A b) \odot_A c = (a \odot_A c) \oplus_A (b \odot_A c)$ and $a \odot_A (b \oplus_A c) = (a \odot_A b) \oplus_A (a \odot_A c)$ hold; moreover, 0_A is an annihilator, i.e., $a \odot_A 0_A = 0_A \odot_A a = 0_A$. In the rest of this section $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$ denotes an arbitrary semiring and Δ an arbitrary alphabet.

Let M be an arbitrary set. A *formal power series* (for short: fps) over M and A is a mapping $S : M \to A$. For every $m \in M$, the element $S(m) \in A$ is called *coefficient* of m and it is denoted by (S, m). The *support of* S is defined as the set $supp(S) = \{m \in M \,|\, (S, m) \neq 0_A\}$. The fps S is called a *polynomial* if $supp(S)$ is finite and it is called a *monomial* if $supp(S)$ is a singleton or empty. A polynomial S is written as $a_1\, m_1 \oplus_A \ldots \oplus_A a_n\, m_n$, where $supp(S) = \{m_1, \ldots, m_n\}$ and, for every $1 \leq i \leq n$, $a_i = (S, m_i)$.

A family $(S_i \,|\, i \in I)$ of formal power series S_i over M and A is called *locally finite*, if for every $m \in M$, there are only finitely many $i \in I$ such that $(S_i, m) \neq 0_A$. In this case, we define the *sum* $\sum_{i \in I} S_i$ for every $m \in M$ by $(\sum_{i \in I} S_i, m) = \sum_{i \in I_m} (S_i, m)$ where $I_m = \{i \in I \mid (S_i, m) \neq 0_A\}$. Then, e.g., every formal power series S is the sum of the monomials $(S, m)m$ for $m \in M$, i.e., the equation $S = \sum_{m \in M} (S, m)m$ holds, because the family $((S, m)m \mid m \in M)$ of monomials is locally finite. The formal sum $\sum_{m \in M} (S, m)m$ is also called the *expansion* of S.

In this paper we will instantiate M to Δ^* or Δ^+. An fps S over Δ^* and A is called *proper* if $(S, \varepsilon) = 0_A$. The *set of all fps (of all proper fps) over* Δ^* *and* A is denoted by $A\langle\langle\Delta^*\rangle\rangle$ (and $A_p\langle\langle\Delta^*\rangle\rangle$, respectively). The *set of all fps over* Δ^+ *and* A is denoted by $A\langle\langle\Delta^+\rangle\rangle$. It should be clear that the sets $A_p\langle\langle\Delta^*\rangle\rangle$ and $A\langle\langle\Delta^+\rangle\rangle$ are isomorphic.

Let $S \in A\langle\langle\Delta^*\rangle\rangle$ be a formal power series. If for every $w \in \Delta^*$, we have $(S, w) = 0_A$, then S is denoted by $\widetilde{0}_A$.

We identify every element $w \in \Delta^*$ with the monomial $1_A\, w$. Thus, $\Delta^* \subseteq A\langle\langle\Delta^*\rangle\rangle$.

3 Weighted automata with discounting

Let us first recall the concepts of weighted automaton (cf. Chapter VI, Section 6 of [Eil74]) and of φ-recognizability (cf. Definition 2.1 of [DK02, DK03]). A *weighted automaton* is a tuple $\mathcal{A} = (Q, \Delta, \delta, I, F, \underline{A})$ where

- Q is a finite set (of *states*),
- Δ is an alphabet (of *input symbols*),
- $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$ is a semiring,
- $\delta : Q \times \Delta \times Q \to A$ (*transition function*),
- $I : Q \to A$ is a mapping (*input mapping*), and
- $F : Q \to A$ is a mapping (*output mapping*).

We call a state $q \in Q$ *initial state* (*final state*) if $I(q) \neq 0_A$ (and $F(q) \neq 0_A$, respectively).

Let $q, q' \in Q$. A *run of $\mathcal{A}$ from q to q'* is a sequence

$$r = (q_0, a_1, q_1)(q_1, a_2, q_2) \ldots (q_{m-1}, a_m, q_m) \in (Q \times \Delta \times Q)^*$$

where $m \geq 0$, $q_0, \ldots, q_m \in Q$, $q = q_0$, $q' = q_m$, and $a_1, \ldots, a_m \in \Delta$. The *label of r* is the string $label(r) \in \Delta^*$ defined by

$$label(r) = a_1 \ldots a_m,$$

and the *weight of r* is the element $\delta(r) \in A$ defined by

$$\delta(r) = \delta(q_0, a_1, q_1) \odot_A \delta(q_1, a_2, q_2) \odot_A \ldots \odot_A \delta(q_{m-1}, a_m, q_m).$$

Note that, for $m = 0$, we have that $r = \varepsilon$, $label(r) = \varepsilon$, and $\delta(r) = 1_A$. Let $R_{\mathcal{A}}(q, q')$ denote the set of all runs of $\mathcal{A}$ from q to q', and for every $w \in \Delta^*$, let $R_{\mathcal{A}}(q, w, q')$ denote the subset of $R_{\mathcal{A}}(q, q')$ of all those runs r such that $label(r) = w$. Note that, for every $w \in \Delta^*$, $R_{\mathcal{A}}(q, w, q')$ is a finite set.

Let $q, q' \in Q$ and let $r = (q_0, a_1, q_1)(q_1, a_2, q_2) \ldots (q_{m-1}, a_m, q_m)$ be a run of $\mathcal{A}$ from q to q' where $q = q_0$ and $q' = q_m$. Moreover, let $\varphi : A \to A$ be an endomorphism on the semiring $\underline{A}$. The *φ-weight of* r is the element $\delta_\varphi(r) \in A$ defined by

$$\delta_\varphi(r) = \varphi^0(\delta(q_0, a_1, q_1)) \odot_A \varphi^1(\delta(q_1, a_2, q_2)) \odot_A \ldots \odot_A \varphi^{m-1}(\delta(q_{m-1}, a_m, q_m)),$$

where φ^{m-1} stands for the $(m-1)$-fold composition of φ (with φ^0 being the identity).

Definition 3.1 The *formal power series φ-accepted by* $\mathcal{A}$ is the formal power series $S_\varphi(\mathcal{A}) \in A\langle\langle\Delta^*\rangle\rangle$ defined for every $w \in \Delta^*$ by

$$(S_\varphi(\mathcal{A}), w) = \sum_{q, q' \in Q} I(q) \odot_A \delta_\varphi(q, w, q') \odot_A \varphi^{|w|}(F(q')),$$

where

$$\delta_\varphi(q, w, q') = \sum_{r \in R_{\mathcal{A}}(q, w, q')} \delta_\varphi(r).$$

□

Note that, taking the identity endomorphism $\mathrm{id} : A \to A$, the fps $S_{\mathrm{id}}(\mathcal{A})$ is exactly the formal power series accepted by $\mathcal{A}$ in the sense of [Eil74].

Definition 3.2 A formal power series $S \in A\langle\langle\Delta^*\rangle\rangle$ is *φ-recognizable* if there is a weighted automaton $\mathcal{A}$ such that $S = S_\varphi(\mathcal{A})$. The *class of φ-recognizable formal power series* is denoted by $A^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle$. □

Example 3.3 Let us consider the following weighted automaton $\mathcal{A} = (Q, \Delta, \delta, I, F, \underline{Trop_{\mathbb{Q}}})$ where

- $Q = \{q_0, q_1\}$,
- $\Delta = \{0, 1, .\}$ consisting of the three elements 0, 1, and the dot,

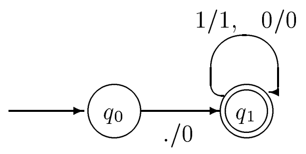

Figure 1: The visualization of the transitions of $\mathcal{A}$ in Example 3.3.

- $\underline{Trop_{\mathbb{Q}}} = (\mathbb{Q}_\infty, \min, +, \infty, 0)$ where $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ and the operations min and $+$ are extended from $\mathbb{Q}$ to $\mathbb{Q}_\infty$ in an obvious way,
- $\delta(q_0, ., q_1) = 0$, $\delta(q_1, 0, q_1) = 0$, $\delta(q_1, 1, q_1) = 1$, and for every other combination $(q, a, p) \in Q \times \Delta \times Q$, we let $\delta(q, a, p) = \infty$, and
- $I(q_0) = F(q_1) = 0$ and $I(q_1) = F(q_0) = \infty$.

The transitions of $\mathcal{A}$ including the marking of initial and final states can be visualized as seen in Figure 1.

Now consider the endomorphism $\varphi : \mathbb{Q}_\infty \to \mathbb{Q}_\infty$ defined for every $x \in \mathbb{Q}_\infty$ by $\varphi(x) = \frac{1}{2} \cdot x$. Also consider the word $w = .101$ as input. Then the set of runs from q_0 to q_1 with label .101 is the set

$$R_{\mathcal{A}}(q_0, w, q_1) = \{(q_0, ., q_1)(q_1, 1, q_1)(q_1, 0, q_1)(q_1, 1, q_1)\}$$

and

$$\delta_\varphi((q_0, ., q_1)(q_1, 1, q_1)(q_1, 0, q_1)(q_1, 1, q_1)) = (\frac{1}{2})^0 \cdot 0 \; + \; (\frac{1}{2})^1 \cdot 1 \; + \; (\frac{1}{2})^2 \cdot 0 \; + \; (\frac{1}{2})^3 \cdot 1 \; = \; 0.625$$

and hence, also

$$\delta_\varphi(q_0, .101, q_1) = \min_{r \in R_{\mathcal{A}}(q_0, .101, q_1)} \delta_\varphi(r) = 0.625.$$

Thus

$$\begin{array}{rl} & (S_\varphi(\mathcal{A}), w) \\ = & \sum_{q,q' \in Q} I(q) \odot_A \delta_\varphi(q, w, q') \odot_A \varphi^{|w|}(F(q')) \\ = & I(q_0) \odot_A \delta_\varphi(q_0, .101, q_1) \odot_A \varphi^{|.101|}(F(q_1)) \\ = & 0 + 0.625 + (\frac{1}{2})^4 \cdot 0 \\ = & 0.625 \end{array}$$

In general we can say that $\mathcal{A}$ computes for every input word $.w$ with $w \in \{0, 1\}^*$ its rational value. $\square$

4 Rational operations on formal power series

Here we define the rational operations on formal power series. These are the sum, the Cauchy-product, the Kleene-+ (and Kleene-$*$), and the multiplication with coefficients (cf. Chapter VII of [Eil74]; cf. also Definition 3.2 of [DK03]). In fact, we proceed as in [Eil74] and define the Kleene-+ and Kleene-$*$ on $A\langle\langle\Delta^+\rangle\rangle$.

Let $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$ be a semiring, $\varphi : A \to A$ be an endomorphism, and Δ an alphabet.

- The *sum* is the binary operation $\oplus : A\langle\langle\Delta^*\rangle\rangle \times A\langle\langle\Delta^*\rangle\rangle \to A\langle\langle\Delta^*\rangle\rangle$ defined, for every $S, T \in A\langle\langle\Delta^*\rangle\rangle$ and $w \in \Delta^*$ by $(S \oplus T, w) = (S, w) \oplus_A (T, w)$.
- The *φ-Cauchy-product* is the binary operation $\odot_\varphi : A\langle\langle\Delta^*\rangle\rangle \times A\langle\langle\Delta^*\rangle\rangle \to A\langle\langle\Delta^*\rangle\rangle$ defined, for every $S, T \in A\langle\langle\Delta^*\rangle\rangle$ and $w \in \Delta^*$, as follows:

$$(S \odot_\varphi T, w) = \sum_{\substack{w_1, w_2 \in \Delta^* \\ w = w_1 w_2}} (S, w_1) \odot_A \varphi^{|w_1|}((T, w_2)).$$

- The *φ-Kleene-+* is the unary operation $(.)^{+,\varphi} : A\langle\langle\Delta^+\rangle\rangle \to A\langle\langle\Delta^+\rangle\rangle$ defined, for every $S \in A\langle\langle\Delta^+\rangle\rangle$ and $w \in \Delta^+$, by

$$(S^{+,\varphi}, w) = \sum_{n \geq 1} (S^{n,\varphi}, w)$$

where

$$S^{1,\varphi} = S \text{ and } S^{n+1,\varphi} = S^{n,\varphi} \odot_\varphi S;$$

(note that $w \in \Delta^+$ admits only finitely many factorizations $w = w_1 \dots w_n$ with $w_1, \dots, w_n \in \Delta^+$ and thus, for every $w \in \Delta^+$ with length n and every $p > n$, we have that $(S^{p,\varphi}, w) = 0_A$).

- The *φ-Kleene-$*$* is the unary operation $(.)^{*,\varphi} : A\langle\langle\Delta^+\rangle\rangle \to A\langle\langle\Delta^*\rangle\rangle$ defined, for every $S \in A\langle\langle\Delta^+\rangle\rangle$, by $S^{*,\varphi} = S^{+,\varphi} \oplus 1_A\varepsilon$.
- The *multiplication* $\cdot : A \times A\langle\langle\Delta^*\rangle\rangle \to A\langle\langle\Delta^*\rangle\rangle$ *with coefficients* is the operation defined, for every $a \in A$, $S \in A\langle\langle\Delta^*\rangle\rangle$, and $w \in \Delta^*$, by $(a \cdot S, w) = a \odot_A (S, w)$.

Note that the φ-Kleene-$+$ and the φ-Kleene-$*$ are only defined on $A\langle\langle\Delta^+\rangle\rangle$. Also note that if φ is the identity id on A, then $\odot_\varphi$, $(.)^{+,\varphi}$, and $(.)^{*,\varphi}$ are the usual Cauchy-product, Kleene-$+$, and Kleene-$*$, respectively, as defined in Chapter VII of [Eil74].

If the operations sum, φ-Cauchy-product, and multiplication with coefficients are applied to fps which are elements of $A\langle\langle\Delta^+\rangle\rangle$, then the result is also an element in $A\langle\langle\Delta^+\rangle\rangle$. Thus we obtain the following operations on $A\langle\langle\Delta^+\rangle\rangle$:

- $\oplus' : A\langle\langle\Delta^+\rangle\rangle \times A\langle\langle\Delta^+\rangle\rangle \to A\langle\langle\Delta^+\rangle\rangle$,
- $\odot'_\varphi : A\langle\langle\Delta^+\rangle\rangle \times A\langle\langle\Delta^+\rangle\rangle \to A\langle\langle\Delta^+\rangle\rangle$, and
- $\cdot' : A \times A\langle\langle\Delta^+\rangle\rangle \to A\langle\langle\Delta^+\rangle\rangle$.

Similarly we can restrict the fps $\widetilde{0}_A$ to the fps

- $\widetilde{0}'_A \in A\langle\langle\Delta^+\rangle\rangle$ where for every $w \in \Delta^+$ we have $(\widetilde{0}'_A, w) = 0_A$.

However, in order to shorten notations, we drop the prime from the denotation of the latter three operations (it will be clear from the context whether they are applied to elements of $A\langle\langle\Delta^*\rangle\rangle$ or $A\langle\langle\Delta^+\rangle\rangle$).

Observation 4.1

1. The algebraic structure $\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle} = (A\langle\langle\Delta^*\rangle\rangle, \oplus, \odot_\varphi, \widetilde{0}_A, 1_A\varepsilon)$ is a semiring (cf. Lemma 3.1 of [DK03]); there this semiring is called the *semiring of skewed formal power series.*
2. For every $a, b \in A$ and $S, T \in A\langle\langle\Delta^+\rangle\rangle$, the following equations hold:

$$\begin{array}{llcll}
\text{(i)} & (a \odot_A b) \cdot S & = & a \cdot (b \cdot S) & \\
\text{(ii)} & (a \oplus_A b) \cdot S & = & (a \cdot S) \oplus (b \cdot S) & \\
\text{(iii)} & a \cdot (S \oplus T) & = & (a \cdot S) \oplus (a \cdot T) & \\
\text{(iv)} & 1_A \cdot S & = & S & \\
\text{(v)} & a \cdot \widetilde{0}_A & = & \widetilde{0}_A & = \quad 0_A \cdot S \\
\text{(vi)} & a \cdot (S \odot_\varphi T) & = & (a \cdot S) \odot_\varphi T &
\end{array}$$

Proof. Recall that $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$. To see that Equations (i)-(v) holds we first observe that $(A, \oplus_A)$ is a left-$\underline{A}$-semimodule with the scalar multiplication $\odot_A$ (for the notion of left-$\underline{A}$-semimodule cf., e.g., p.149 of [Gol99]). Then, e.g. by Example 14.4 of [Gol99], also $(A\langle\langle\Delta^*\rangle\rangle, \oplus)$ is a left-$\underline{A}$-semimodule with the scalar multiplication $\cdot$. Now Equations (i)-(v) are the axioms of the left-$\underline{A}$-semimodule $(A\langle\langle\Delta^*\rangle\rangle, \oplus)$.

Finally, we prove that Equation (vi) holds. Let $w \in \Delta^*$. Then, using the definitions of $\cdot$ and $\odot_\varphi$, the distributivity of $\odot_A$ over $\oplus$, and the associativuty of $\odot_A$, we obtain:

$$\begin{aligned}
& (a \cdot (S \odot_\varphi T), w) \\
= \ & a \odot_A (S \odot_\varphi T, w) \\
= \ & a \odot_A \textstyle\sum_{\substack{w_1, w_2 \in \Delta^* \\ w = w_1 w_2}} \big((S, w_1) \odot_A \varphi^{|w_1|}((T, w_2))\big) \\
= \ & \textstyle\sum_{\substack{w_1, w_2 \in \Delta^* \\ w = w_1 w_2}} \big(a \odot_A \big((S, w_1) \odot_A \varphi^{|w_1|}((T, w_2))\big)\big) \\
= \ & \textstyle\sum_{\substack{w_1, w_2 \in \Delta^* \\ w = w_1 w_2}} \big((a \odot_A (S, w_1)) \odot_A \varphi^{|w_1|}((T, w_2))\big) \\
= \ & \textstyle\sum_{\substack{w_1, w_2 \in \Delta^* \\ w = w_1 w_2}} \big((a \cdot S, w_1) \odot_A \varphi^{|w_1|}((T, w_2))\big) \\
= \ & ((a \cdot S) \odot_\varphi T, w)
\end{aligned}$$

$\square$

Definition 4.2

1. The $(A, +, \varphi)$-*algebra of formal power series over* Δ^+ is the algebra $(A\langle\langle\Delta^+\rangle\rangle, \oplus, \odot_\varphi, (.)^{+,\varphi}, \cdot, \widetilde{0}_A)$. The operations $\oplus$, $\odot_\varphi$, $(.)^{+,\varphi}$, and $\cdot$ are called the *rational operations*. (Note that the operations are defined on $A\langle\langle\Delta^+\rangle\rangle$ and the fps $\widetilde{0}_A \in A\langle\langle\Delta^+\rangle\rangle$, i.e., in view of the discussion above, we are dealing with the primed operations $\oplus'$, $\odot'_\varphi$, $\cdot'$, and $\widetilde{0}'_A$.)

2. The *class of* φ-*rational formal power series over* A *and* Δ^+, denoted by $A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$, is the carrier set of the subalgebra of the $(A, +, \varphi)$-algebra of formal power series over Δ^+, which is generated by the set $\{1_A w \mid w \in \Delta^+\}$.

3. The *class of* φ-*rational formal power series over* A *and* Δ^*, denoted by $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$, is the class of all formal power series $S \in A\langle\langle\Delta^*\rangle\rangle$ of the form
$$S = a \cdot \varepsilon \oplus T$$
where $a \in A$ and $T \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$.

We note that there is a slight inacuracy in the equation $S = a \cdot \varepsilon \oplus T$, because the fps $a \cdot \varepsilon \in A\langle\langle\Delta^*\rangle\rangle$ and the fps $T \in A\langle\langle\Delta^+\rangle\rangle$, and thus the type of $\oplus$ is not respected. However, we can use the fact that $A_p\langle\langle\Delta^*\rangle\rangle$ and $A\langle\langle\Delta^+\rangle\rangle$ are isomorphic and view T as an element of $A_p\langle\langle\Delta^*\rangle\rangle \subseteq A\langle\langle\Delta^*\rangle\rangle$. We will apply this point of view in the sequel whenever necessary.

Lemma 4.3 (cf. Proposition 4.1, p.171 of [Eil74]) The class $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ is the smallest class of formal power series in $A\langle\langle\Delta^*\rangle\rangle$ containing $\{1_A w \mid w \in \Delta^*\}$ and being closed under the rational operations with $(.)^{+,\varphi}$ restricted to elements of $A\langle\langle\Delta^+\rangle\rangle$.

Proof. First we prove that $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ contains $\{1w \mid w \in \Delta^*\}$ and is closed under the rational operations. Consider $1w$ with $w \in \Delta^*$. If $w = \varepsilon$, then define $S_w = 1_A \cdot \varepsilon \oplus \widetilde{0}_A$. If $w \neq \varepsilon$, then define $S_w = 0_A \cdot \varepsilon \oplus 1_A \cdot w$. In both cases $S_w = 1w \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$. Now let $S, S' \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ have the form $S = a \cdot \varepsilon \oplus T$ and $S' = a' \cdot \varepsilon \oplus T'$ with $T, T' \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$, respectively. Also let $b \in A$. Then we can calculate as follows:

$$\begin{aligned} & S \oplus S' \\ = \ & (a \cdot \varepsilon \oplus T) \oplus (a' \cdot \varepsilon \oplus T') \\ = \ & (a \cdot \varepsilon) \oplus (a' \cdot \varepsilon) \oplus (T \oplus T') && \text{(by associativity and commutativity of } \oplus) \\ = \ & (a \oplus_A a') \cdot \varepsilon \oplus (T \oplus T') && \text{(by Observation 4.1(2,ii))}. \end{aligned}$$

$$\begin{aligned} & S \odot_\varphi S' \\ = \ & (a \cdot \varepsilon \oplus T) \odot_\varphi (a' \cdot \varepsilon \oplus T') \\ = \ & (a \cdot \varepsilon \odot_\varphi a' \cdot \varepsilon) \oplus (a \cdot \varepsilon \odot_\varphi T') \oplus (T \odot_\varphi a' \cdot \varepsilon) \oplus (T \odot_\varphi T') \\ & \text{(because } \underline{A^\varphi\langle\langle\Delta^*\rangle\rangle} \text{ is a semiring by Observation 4.1(1), and hence } \odot_\varphi \text{ distributes over } \oplus) \\ = \ & (a \odot_A a') \cdot \varepsilon \oplus (a \cdot T' \oplus T \odot_\varphi a' \cdot \varepsilon \oplus T \odot_\varphi T'). \end{aligned}$$

Now observe that the three fps $a \cdot T'$, $T \odot_\varphi a' \cdot \varepsilon$, and $T \odot_\varphi T'$ are elements of $A\langle\langle\Delta^+\rangle\rangle$. (In particular, due to the definition of $\odot_\varphi$, we have that $supp(T \odot_\varphi a' \cdot \varepsilon) \subseteq \Delta^+$.) Hence, by Definition 4.2(3), we have $S \odot_\varphi S' \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$. (Note that we cannot reduce $T \odot_\varphi a' \cdot \varepsilon$ to $a' \cdot T$ because $\underline{A}$ need not be commutative in $\odot_A$.)

Let $S \in A\langle\langle\Delta^+\rangle\rangle$. Then $S = T$ and hence $S^{+,\varphi} \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle \subseteq A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$.

Finally, we prove that $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ is the smallest such class. Let $\mathcal{C}$ be another class which contains $\{1w \mid w \in \Delta^*\}$ and is closed under the rational operations with $(.)^{+,\varphi}$ restricted to elements of $A\langle\langle\Delta^+\rangle\rangle$. Then, by definition of $A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$, it follows that $A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle \subseteq \mathcal{C}$ and also $A \subseteq \mathcal{C}$ (where A is viewed as the set $A\langle\langle\{\varepsilon\}\rangle\rangle$ of all monomial formal power series $a \cdot \varepsilon$). Thus, every element $S = a \cdot \varepsilon \oplus T$ of $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ is also in $\mathcal{C}$, hence $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle \subseteq \mathcal{C}$. $\square$

5 Formalizing $S_\varphi(\mathcal{A})$ as algebraic path problem

In this section we formalize the computation of $S_\varphi(\mathcal{A})$ as an algebraic path problem. First let us fix some notations on graphs and paths.

A *finite directed graph* (for short: graph) is a tuple $G = (V, E)$ such that $V = \{1, \ldots, n\}$ for some $n \geq 1$ is a finite set and $E \subseteq V \times V$. The elements of V and E are called *vertices* and *edges*, respectively. Let $u, v \in V$ and $m \geq 1$. A *path p from u to v (of order m)* is a sequence $(v_0, v_1)(v_1, v_2) \ldots (v_{m-1}, v_m) \in E^+$ where $v_0, \ldots, v_m \in V$, $u = v_0$, and $v = v_m$. Moreover, if $u = v$, then ε is a path from u to u (of order 0). We also use $(v_0, v_1, v_2, \ldots, v_m)$ as an abbreviation of p. Let $p = (v_0, \ldots, v_m)$ be a path of order $m \geq 2$. Then the vertices $v_1, \ldots, v_{m-1}$ are the *inner vertices* of p. A path with order 1 or 0 does not have inner vertices. Let $m \geq 0$. The *set of paths from u to v (in G) of order m* is denoted by $P_{u,v}^{=m}$. Thus, in particular, $P_{u,v}^{=0} = \emptyset$ for $u \neq v$ and $P_{u,u}^{=0} = \{\varepsilon\}$. The *set of paths from u to v* is denoted by $P_{u,v}$; clearly, $P_{u,v} = \bigcup_{m \geq 0} P_{u,v}^{=m}$.

Since a path p is a word over E, i.e., $p \in E^*$, sets of paths can be considered as formal languages over E^*. Thus, the set theoretic operations $\cup$, $\cdot$, and $(.)^*$ on languages can also be applied to sets of paths. However, since not every $p \in E^*$ is a path, these operations yield sets of paths only if their arguments obey certain typing restrictions.

Let $G = (V, E)$ be a graph. Let $u, v \in V$ and $P_1, P_2 \subseteq P_{u,v}$. Then $P_1 \cup P_2 \subseteq P_{u,v}$. Let $u, v, w \in V$ and $P_1 \subseteq P_{u,v}$ and $P_2 \subseteq P_{v,w}$. Then $P_1 \cdot P_2 \subseteq P_{u,w}$. Let $u \in V$ and $P \subseteq P_{u,u}$. Then $P^* \subseteq P_{u,u}$.

Definition 5.1 Let A be a non-empty set and $G = (V, E)$ be a graph. Then (V, E, λ) is an *A-weighted graph* where $\lambda : E \to A$.

Definition 5.2 Let $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$ be a semiring. Moreover, let $G = (V, E, \lambda)$ be an A-weighted graph. The *$\underline{A}$-canonical extension of* λ is the path evaluation function

$$\widetilde{\lambda}_{\underline{A}} : \{U \mid U \subseteq P_{u,v}, u, v \in V\} \to A$$

which is defined for every path $p = (u_0, u_1, \ldots, u_m)$ with $m \geq 0$ and $u_0, \ldots, u_m \in V$ by

$$\widetilde{\lambda}_{\underline{A}}(\{p\}) = \lambda(u_0, u_1) \odot_A \lambda(u_1, u_2) \odot_A \ldots \odot_A \lambda(u_{m-1}, u_m)$$

(where $\widetilde{\lambda}_{\underline{A}}(\{\varepsilon\}) = 1_A$) and, for every $u, v \in V$ and $U \subseteq P_{u,v}$, by

$$\widetilde{\lambda}_{\underline{A}}(U) = \sum_{p \in U} \widetilde{\lambda}_{\underline{A}}(\{p\})$$

(where $\widetilde{\lambda}_{\underline{A}}(\emptyset) = 0_A$) if this sum is defined; if for some U the sum $\sum_{p \in U} \widetilde{\lambda}_{\underline{A}}(\{p\})$ is not defined, then there is no $\underline{A}$-canonical extension of λ. $\square$

We now show that weighted automata with discounting can be viewed as an algebraic path problem; the case of non-discounted weighted automata follows by taking φ to be the identity endomorphism.

We will consider a weighted automaton $\mathcal{A} = (Q, \Delta, \delta, I, F, \underline{A})$ over some semiring $\underline{A} = (A, \oplus_A, \odot_A, 0_A, 1_A)$. The weighted automaton $\mathcal{A}$ is split up into

- the $A\langle\langle\Delta^*\rangle\rangle$-weighted graph $G_{\mathcal{A}} = (Q, Q \times Q, \lambda)$ with $\lambda : Q \times Q \to A\langle\langle\Delta^*\rangle\rangle$ defined for every $q, q' \in Q$ by
$$\lambda(q, q') = \sum_{a \in \Delta} \delta(q, a, q')\ a.$$
and
- the $\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}$-canonical extension of λ where $\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle} = (A\langle\langle\Delta^*\rangle\rangle, \oplus, \odot_\varphi, \widetilde{0}_A, 1_A\varepsilon)$ is the semiring of skewed formal power series (cf. Observation 4.1(1)).

Thus, in particular, for every path $p = (q_0, q_1, \ldots, q_m)$ through $G_{\mathcal{A}}$, we have

$$\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\}) = \lambda(q_0, q_1) \odot_\varphi \lambda(q_1, q_2) \odot_\varphi \ldots \odot_\varphi \lambda(q_{m-1}, q_m) =$$

$$\widetilde{\varphi}^0(\lambda(q_0, q_1)) \odot_{\mathrm{id}} \widetilde{\varphi}^1(\lambda(q_1, q_2)) \odot_{\mathrm{id}} \ldots \odot_{\mathrm{id}} \widetilde{\varphi}^{m-1}(\lambda(q_{m-1}, q_m)),$$

where $\widetilde{\varphi} : A\langle\langle\Delta^*\rangle\rangle \to A\langle\langle\Delta^*\rangle\rangle$ is defined by $(\widetilde{\varphi}(S), w) = \varphi((S, w))$ for every $S \in A\langle\langle\Delta^*\rangle\rangle$ and $w \in \Delta^*$. We note that, for every $U \subseteq P_{q,q'}$, the sum

$$\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(U) = \sum_{p \in U} \widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\})$$

is defined, because the family $(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\}) \mid p \in P_{q,q'})$ of fps is locally finite. This holds because, for every $w \in \Delta^*$, we have that $(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\}), w) \neq 0_A$ iff $w \in supp(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\}))$ only if $p \in P^{=|w|}_{q,q'}$; and the set $P^{=|w|}_{q,q'}$ is finite.

First we show two easy properties of the canonical extension.

Lemma 5.3 Let $G_{\mathcal{A}} = (Q, Q \times Q, \lambda)$ be the $A\langle\langle\Delta^*\rangle\rangle$-weighted graph defined as above and $\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}$ be the $\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}$-canonical extension of λ. Then the following two statements hold.

1. For every path p in $G_{\mathcal{A}}$ of some order $m \geq 0$, we have $supp(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\})) \subseteq \Delta^m$.
2. For every two paths $p_1 = (q_0, \ldots, q_m)$ and $p_2 = (q_m, \ldots, q_l)$ and every word $w = a_1 \ldots a_m a_{m+1} \ldots a_l \in \Delta^*$ with $0 \leq m \leq l$, it holds that

$$(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_1p_2\}), w) = (\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_1\}), a_1 \ldots a_m) \odot_A \varphi^m \left((\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_2\}), a_{m+1} \ldots a_l) \right).$$

Proof. Proof of 1. For every edge (q, q'), we have $supp(\lambda(q, q')) \subseteq \Delta$, i.e., the support contains words of length one. By definition, for every path p of some order m, $\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\})$ is the m-fold Cauchy-product of some formal power series whose supports are subsets of Δ. Thus $supp(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p\})) \subseteq \Delta^m$.

Proof of 2.

$$\begin{aligned}
& \left(\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_1p_2\}), a_1 \ldots a_m a_{m+1} \ldots a_l\right) \\
= \; & \left(\widetilde{\varphi}^0(\lambda(q_0, q_1)) \odot_{\mathrm{id}} \ldots \odot_{\mathrm{id}} \widetilde{\varphi}^{l-1}(\lambda(q_{l-1}, q_l)), a_1 \ldots a_m a_{m+1} \ldots a_l\right) \\
= \; & \left(\widetilde{\varphi}^0(\lambda(q_0, q_1)), a_1\right) \odot_A \ldots \odot_A \left(\widetilde{\varphi}^{m-1}(\lambda(q_{m-1}, q_m)), a_m\right) \\
& \odot_A \left(\widetilde{\varphi}^m(\lambda(q_m, q_{m+1})), a_{m+1}\right) \odot_A \ldots \odot_A \left(\widetilde{\varphi}^{l-1}(\lambda(q_{l-1}, q_l)), a_l\right) \\
= \; & \varphi^0((\lambda(q_0, q_1), a_1)) \odot_A \ldots \odot_A \varphi^{m-1}((\lambda(q_{m-1}, q_m), a_m)) \\
& \odot_A \varphi^m((\lambda(q_m, q_{m+1}), a_{m+1})) \odot_A \ldots \odot_A \varphi^{l-1}((\lambda(q_{l-1}, q_l), a_l)) \\
= \; & (\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_1\}), a_1 \ldots a_m) \\
& \odot_A \varphi^m((\lambda(q_m, q_{m+1}), a_{m+1})) \odot_A \ldots \odot_A \varphi^{l-1}((\lambda(q_{l-1}, q_l), a_l)) \\
= \; & (\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_1\}), a_1 \ldots a_m) \\
& \odot_A \varphi^m \left(\varphi^0((\lambda(q_m, q_{m+1}), a_{m+1})) \odot_A \ldots \odot_A \varphi^{l-m-1}((\lambda(q_{l-1}, q_l), a_l))\right) \\
& \text{(because } \varphi \text{ is an endomorphism)} \\
= \; & (\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_1\}), a_1 \ldots a_m) \odot_A \varphi^m \left((\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(\{p_2\}), a_{m+1} \ldots a_l) \right).
\end{aligned}$$

□

Lemma 5.4 Let $\mathcal{A} = (Q, \Delta, \delta, I, F, \underline{A})$ be a weighted automaton and $\varphi : A \to A$ be an endomorphism. Then

$$S_\varphi(\mathcal{A}) = \sum_{q,q' \in Q} \widetilde{I}(q) \odot_\varphi \widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(P_{q,q'}) \odot_\varphi \widetilde{F}(q'),$$

where

- $\widetilde{I}(q) = I(q)\varepsilon \in A\langle\langle\{\varepsilon\}\rangle\rangle$ and $\widetilde{F}(q') = F(q')\varepsilon \in A\langle\langle\{\varepsilon\}\rangle\rangle$, and
- for every $q, q' \in Q$, the set $P_{q,q'}$ is the set of all paths from q to q' in the $A\langle\langle\Delta^*\rangle\rangle$-weighted graph $G_{\mathcal{A}}$.

Proof. In this proof we abbreviate $\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}$ by $\widetilde{\lambda}$. Then we can calculate, for every $w \in \Delta^*$, as follows

$$
\begin{array}{rl}
& (S_\varphi(\mathcal{A}), w) \\
= & \sum_{q,q' \in Q} I(q) \odot_A \delta_\varphi(q, w, q') \odot_A \varphi^{|w|}(F(q')) \\
= & \sum_{q,q' \in Q} I(q) \odot_A \left(\sum_{r \in R_\mathcal{A}(q,w,q')} \delta_\varphi(r)\right) \odot_A \varphi^{|w|}(F(q')) \\
=^{(*)} & \sum_{q,q' \in Q} I(q) \odot_A \left(\sum_{p \in P^{|w|}_{q,q'}} (\widetilde{\lambda}(\{p\}), w)\right) \odot_A \varphi^{|w|}(F(q')) \\
= & \sum_{q,q' \in Q} I(q) \odot_A \left(\sum_{p \in P_{q,q'}} (\widetilde{\lambda}(\{p\}), w)\right) \odot_A \varphi^{|w|}(F(q')) \\
& \text{(by Lemma 5.3(1))} \\
= & \sum_{q,q' \in Q} I(q) \odot_A \left(\sum_{p \in P_{q,q'}} \widetilde{\lambda}(\{p\}), w\right) \odot_A \varphi^{|w|}(F(q')) \\
= & \sum_{q,q' \in Q} I(q) \odot_A \left(\widetilde{\lambda}(P_{q,q'}), w\right) \odot_A \varphi^{|w|}(F(q')) \\
= & \sum_{q,q' \in Q} I(q) \odot_A \left(\widetilde{\lambda}(P_{q,q'}) \odot_\varphi \widetilde{F}(q'), w\right) \\
& \text{(because } \mathit{supp}(\widetilde{F}(q')) \subseteq \{\varepsilon\} \text{ and } (\widetilde{F}(q'), \varepsilon) = F(q')) \\
= & \left(\sum_{q,q' \in Q} \widetilde{I}(q) \odot_\varphi \widetilde{\lambda}(P_{q,q'}) \odot_\varphi \widetilde{F}(q'), w\right) \\
& \text{(because } \mathit{supp}(\widetilde{I}(q)) \subseteq \{\varepsilon\} \text{ and } (\widetilde{I}(q), \varepsilon) = I(q)).
\end{array}
$$

At equation $(*)$ we have used the fact that there is a one-to-one correspondence between the set $R_\mathcal{A}(q, w, q') = \{r \in R_\mathcal{A}(q, q') \mid \mathit{label}(r) = w\}$ of runs r of $\mathcal{A}$ from q to q' which are labeled by w, and the set $P^{=|w|}_{q,q'}$ of paths through $G_\mathcal{A}$ from q to q' of order $|w|$. In fact, for every $m \geq 0$ and $w = a_1 a_2 \ldots a_m$, the mapping $\theta_w : R_\mathcal{A}(q, w, q') \to P^{=|w|}_{q,q'}$ is a bijection where, for every run $r = (q_0, a_1, q_1)(q_1, a_2, q_2) \ldots (q_{m-1}, a_m, q_m)$, the path $\theta_w(r)$ is defined by

$$\theta_w(r) = (q_0, q_1)(q_1, q_2) \ldots, (q_{m-1}, q_m).$$

In particular, for the validity of (*) we have to prove that, for every run $r = (q_0, a_1, q_1)(q_1, a_2, q_2) \ldots (q_{m-1}, a_m, q_m)$, the following equality holds:

$$\delta_\varphi(r) = (\widetilde{\lambda}(\{\theta_w(r)\}), w),$$

where $w = a_1 a_2 \ldots a_m$. This can be seen as follows:

$$
\begin{array}{rl}
& \delta_\varphi(r) \\
= & \varphi^0(\delta(q_0, a_1, q_1)) \odot_A \varphi^1(\delta(q_1, a_2, q_2)) \odot_A \ldots \odot_A \varphi^{m-1}(\delta(q_{m-1}, a_m, q_m)) \\
= & \varphi^0((\lambda(q_0, q_1), a_1)) \odot_A \varphi^1((\lambda(q_1, q_2), a_2)) \odot_A \ldots \odot_A \varphi^{m-1}((\lambda(q_{m-1}, q_m), a_m)) \\
& \text{(because } \lambda(q_i, q_{i+1}) = \sum_{a \in \Delta} \delta(q_i, a, q_{i+1}) a) \\
= & (\widetilde{\varphi}^0(\lambda(q_0, q_1)), a_1) \odot_A (\widetilde{\varphi}^1(\lambda(q_1, q_2)), a_2) \odot_A \ldots \odot_A (\widetilde{\varphi}^{m-1}(\lambda(q_{m-1}, q_m)), a_m) \\
& \text{(by definition of } \widetilde{\varphi}) \\
= & (\widetilde{\varphi}^0(\lambda(q_0, q_1)) \odot_{\mathrm{id}} \widetilde{\varphi}^1(\lambda(q_1, q_2)) \odot_{\mathrm{id}} \ldots \odot_{\mathrm{id}} \widetilde{\varphi}^{m-1}(\lambda(q_{m-1}, q_m)), w) \\
= & (\lambda(q_0, q_1) \odot_\varphi \lambda(q_1, q_2) \odot_\varphi \ldots \odot_\varphi \lambda(q_{m-1}, q_m), w) \\
= & (\widetilde{\lambda}(\{(q_0, q_1)(q_1, q_2) \ldots (q_{m-1}, q_m)\}), w) \\
= & (\widetilde{\lambda}(\{\theta_w(r)\}), w).
\end{array}
$$

This verifies equation (*). Thus we obtain

$$S_\varphi(\mathcal{A}) = \sum_{q,q' \in Q} \widetilde{I}(q) \odot_\varphi \widetilde{\lambda}(P_{q,q'}) \odot_\varphi \widetilde{F}(q').$$

□

6 Unambiguous sets of paths

In Lemma 5.4 we have seen that we can express the formal power series $S_\varphi(\mathcal{A})$ accepted by some weighted automaton $\mathcal{A}$ in terms of the image of sets of paths under some appropriate path evaluation function. More precisely,

$$S_\varphi(\mathcal{A}) = \sum_{q,q' \in Q} \widetilde{I}(q) \odot_\varphi \widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(P_{q,q'}) \odot_\varphi \widetilde{F}(q').$$

Thus it is natural to deal with sets $P_{q,q'}$ of paths and their decompositions in order to obtain a kind of rational representation for them. More precisely, we will consider a particular subset $\widehat{P}^{(k)}_{q,q'}$ of $P_{q,q'}$ such that the inner vertices of every $p \in \widehat{P}^{(k)}_{q,q'}$ are elements of $\{1, \ldots, k\}$. Since the definition of $\widehat{P}^{(k)}_{q,q'}$ and the decomposition of its elements do not depend on the fact that the vertices represent states of some automaton, we again switch back to the system of notations and variables which we have used for arbitrary graphs.

Definition 6.1 Let $G = (V, E)$ be a directed graph such that $V = \{1, \ldots, n\}$. Moreover, let $u, v \in V$ and $k \in \{0, \ldots, n\}$. Define the set $\widehat{P}^{(k)}_{u,v}$ to be the set of all paths from u to v of some order $m \geq 1$ such that their inner vertices are contained in $\{1, \ldots, k\}$.

We note that these sets of paths are defined in a slightly different way than the sets $P^{(k)}_{u,v}$ in Definition 6.10 of Chapter IV of [HW98]; more precisely, in our sets $\widehat{P}^{(k)}_{u,v}$ the empty path ε is not involved. This reflects the way in which the rational fps in $A^{\mathrm{rat},\varphi}\langle\!\langle \Delta^* \rangle\!\rangle$ are defined, viz., such an fps S has the form $S = a \cdot \varepsilon \oplus T$ where $a \in A$ and $T \in A^{\mathrm{rat},\varphi}\langle\!\langle \Delta^+ \rangle\!\rangle$ (cf. Definition 4.2(3)). In fact, we will show that $\widetilde{\lambda}_{\underline{A^\varphi\langle\!\langle \Delta^* \rangle\!\rangle}}(\widehat{P}^{(n)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\!\langle \Delta^+ \rangle\!\rangle$ (cf. the proof of Theorem 7.4).

Observation 6.2

1. For every $u, v, k \in V$, we have $\varepsilon \notin \widehat{P}^{(k)}_{u,v}$.
2. If $(u, v) \in E$, then for every $0 \leq k \leq n$ we have that $(u, v) \in \widehat{P}^{(k)}_{u,v}$.
3. $\widehat{P}^{(0)}_{u,v} = \{(u, v)\}$ if $(u, v) \in E$, otherwise $\widehat{P}^{(0)}_{u,v} = \emptyset$.
4. For every $u, v \in V$, we have $\widehat{P}^{(0)}_{u,v} \subseteq \widehat{P}^{(1)}_{u,v} \subseteq \ldots \subseteq \widehat{P}^{(n)}_{u,v}$.
5. For every $u, v \in V$, we have $P_{u,v} = \widehat{P}^{(n)}_{u,v} \cup Z_{u,v}$ where $Z_{u,v} = \{\varepsilon\}$ if $u = v$, and $Z_{u,v} = \emptyset$ otherwise.

The sets of the form $\widehat{P}^{(k)}_{u,v}$ have some useful decomposition properties which we would like to give a name first.

Definition 6.3 Let $u, v \in V$.

1. Let $P_1, P_2 \subseteq P_{u,v}$. The set $P_1 \cup P_2$ is $\cup(P_1, P_2)$-*unambiguous* if $P_1 \cap P_2 = \emptyset$.
2. Let $k \in V$ and $P_1 \subseteq P_{u,k}$ and $P_2 \subseteq P_{k,v}$. The set $P_1 \cdot P_2$ is $\cdot(P_1, P_2)$-*unambiguous* if, for every path $p \in P_1 \cdot P_2$ there are uniquely determined paths $p_1 \in P_1$ and $p_2 \in P_2$ such that $p = p_1 \cdot p_2$.
3. Let $u = v$ and $P \subseteq P_{u,u}$. The set P^* is $*(P)$-*unambiguous* if, for every $p \in P^*$ either $p = \varepsilon$ or there is exactly one $l \geq 1$ and there are uniquely determined paths $p_1, \ldots, p_l \in P$, such that $p = p_1 \cdot \ldots \cdot p_l$.

Lemma 6.4 (cf. proof of Theorem 6.13 in Chapter IV of [HW98]) Let $G = (V, E)$ be a graph such that $V = \{1, \ldots, n\}$. Moreover, let $k, u, v \in V$. Then the following five statements hold.

1. The set $\widehat{P}^{(k)}_{u,v}$ of paths is $\cup(P_1, P_2)$-unambiguous where $P_1 = \widehat{P}^{(k-1)}_{u,v}$ and $P_2 = \widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$.
2. The set $\widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$ of paths is $\cdot(P_1, P_2)$-unambiguous where $P_1 = \widehat{P}^{(k-1)}_{u,k}$ and $P_2 = (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$.
3. The set $(\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$ of paths is $\cdot(P_1, P_2)$-unambiguous where $P_1 = (\widehat{P}^{(k-1)}_{k,k})^*$ and $P_2 = \widehat{P}^{(k-1)}_{k,v}$.
4. The set $(\widehat{P}^{(k-1)}_{k,k})^*$ is $*(P)$-unambiguous where $P = \widehat{P}^{(k-1)}_{k,k}$.
5. $\widehat{P}^{(k)}_{u,v} = \widehat{P}^{(k-1)}_{u,v} \cup \widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$.

Proof. We prove the statements one by one starting with Statement 1. Let $p \in \widehat{P}^{(k)}_{u,v}$. Note that $p \neq \varepsilon$. Then either k occurs in p as inner vertex or not. If it does not occur in p, then $p \in \widehat{P}^{(k-1)}_{u,v}$. Otherwise, $p \in \widehat{P}^{(k)}_{u,v} \setminus \widehat{P}^{(k-1)}_{u,v} = \widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$. This proves Statement 1.

Now let $p \in \widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$. Then p runs, say $l \geq 1$ times at inner vertices through k (indicated by superscripts), i.e., p has the form

$$p = (u, \underbrace{\ldots}_{\text{no k's}}, k^{(1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(2)}, \ldots, \underbrace{\ldots}_{\text{no k's}}, k^{(l)}, \underbrace{\ldots}_{\text{no k's}}, v).$$

Then we can partition p uniquely into

$p_1 = (u, \underbrace{\ldots}_{\text{no k's}}, k^{(1)}) \in \widehat{P}^{(k-1)}_{u,k}$ and

$p_2 = (k^{(1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(2)}, \ldots, \underbrace{\ldots}_{\text{no k's}}, k^{(l)}, \underbrace{\ldots}_{\text{no k's}}, v) \in (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$.

This proves Statement 2.

Now let $p \in (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$. Then, for some $l \geq 1$, p has the form

$$p = (k^{(1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(2)}, \ldots, \underbrace{\ldots}_{\text{no k's}}, k^{(l)}, \underbrace{\ldots}_{\text{no k's}}, v).$$

Then we can partition p uniquely into

$p_1 = (k^{(1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(2)}, \ldots, \underbrace{\ldots}_{\text{no k's}}, k^{(l)}) \in (\widehat{P}^{(k-1)}_{k,k})^*$, if $l > 1$ and

$p_1 = \varepsilon \in (\widehat{P}^{(k-1)}_{k,k})^*$, if $l = 1$ and

$p_2 = (k^{(l)}, \underbrace{\ldots}_{\text{no k's}}, v) \in \widehat{P}^{(k-1)}_{k,v}$.

This proves Statement 3.

Now let $p \in (\widehat{P}^{(k-1)}_{k,k})^*$. Then either $p = \varepsilon$ or there is an $l \geq 2$ such that p has the form

$$p = (k^{(1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(2)}, \ldots, \underbrace{\ldots}_{\text{no k's}}, k^{(l)}).$$

Then we can partition p uniquely into

$$p_1 = (k^{(1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(2)}) \in \widehat{P}^{(k-1)}_{k,k}, \quad \ldots, \quad p_{l-1} = (k^{(l-1)}, \underbrace{\ldots}_{\text{no k's}}, k^{(l)}) \in \widehat{P}^{(k-1)}_{k,k}.$$

This proves Statement 4 and the inclusion $\widehat{P}^{(k)}_{u,v} \subseteq \widehat{P}^{(k-1)}_{u,v} \cup \widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v}$ of Statement 5.

By Definition 6.1, it holds that $\widehat{P}^{(k-1)}_{u,v} \subseteq \widehat{P}^{(k)}_{u,v}$. Now given paths $p_1 \in \widehat{P}^{(k-1)}_{u,k}$, $p_2 \in (\widehat{P}^{(k-1)}_{k,k})^*$, and $p_3 \in \widehat{P}^{(k-1)}_{k,v}$, then $p_1 \cdot p_2 \cdot p_3 \in P_{u,v}$, because the target node of one path matches with the source node of the next path. In fact, we even have that $p_1 \cdot p_2 \cdot p_3 \in \widehat{P}^{(k)}_{u,v}$, because every subpath of $p_1 \cdot p_2 \cdot p_3$ only uses inner nodes from the set $\{1, \ldots, k\}$. This proves $\widehat{P}^{(k-1)}_{u,v} \cup \widehat{P}^{(k-1)}_{u,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,v} \subseteq \widehat{P}^{(k)}_{u,v}$ and thus also Statement 5. □

7 Recognizable Formal Power Series are Rational

In this section we will prove that the formal power series $S_\varphi(\mathcal{A})$ which is φ-recognizable by a weighted automaton $\mathcal{A}$, is rational. Since

- we have already represented $S_\varphi(\mathcal{A})$ in the form $S_\varphi(\mathcal{A}) = \sum_{q,q' \in Q} \widetilde{I}(q) \odot_\varphi \widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}(P_{q,q'}) \odot_\varphi \widetilde{F}(q')$ (cf. Lemma 5.4),
- and, for every $q, q' \in Q$, we have $P_{q,q'} = \widehat{P}^{(n)}_{q,q'} \cup Z_{q,q'}$ (cf. Observation 6.2), and

- the set $\widehat{P}^{(n)}_{q,q'}$ is decomposable by using rational operations on sets of paths (cf. Lemma 6.4),

it remains to show that the path evaluation function $\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}$ is compatible with the rational structure of $\widehat{P}^{(n)}_{q,q'}$. For this purpose we first define the concept of compatibility (cf. Definition 3.2 of [The95]). In this section we abbreviate $\widetilde{\lambda}_{\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}}$ by $\widetilde{\lambda}$.

Definition 7.1 Let $\underline{A}$ be a semiring, $\varphi : A \to A$ an endomorphism, Δ an alphabet, and $\oplus$, $\odot_\varphi$, and $(.)^{*,\varphi}$ be the operations on formal power series as defined in Section 4. Moreover, let $G = (Q, E, \lambda)$ be an $A\langle\langle\Delta^*\rangle\rangle$-weighted graph and $\theta : \{U \mid U \subseteq P_{q,q'}, q, q' \in Q\} \to A\langle\langle\Delta^*\rangle\rangle$ be a path evaluation function.

We say that θ is φ-*compatible* if the following three conditions hold for every $q, q', s \in Q$:

1. for every $P_1, P_2 \subseteq P_{q,q'}$, if $P_1 \cup P_2$ is $\cup(P_1, P_2)$-unambiguous, then $\theta(P_1 \cup P_2) = \theta(P_1) \oplus \theta(P_2)$,
2. for every $P_1 \subseteq P_{q,q'}$ and $P_2 \subseteq P_{q',q''}$, if $P_1 \cdot P_2$ is $\cdot(P_1, P_2)$-unambiguous, then $\theta(P_1 \cdot P_2) = \theta(P_1) \odot_\varphi \theta(P_2)$, and
3. for every $P \subseteq P_{q,q}$, if P^* is $*(P)$-unambiguous and $\varepsilon \notin \mathit{supp}(\theta(P))$, then $\theta(P^*) = (\theta(P))^{*,\varphi}$.

Example 7.2 In this example we would like to illustrate the effect of the (absence of the) condition in part 2 of Definition 7.1 on the $\cdot(P_1, P_2)$-unambiguity of $P_1 \cdot P_2$. For this purpose, consider the semiring $\underline{\mathbb{N}_\infty} = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ and the $\mathbb{N}\langle\langle\Delta^*\rangle\rangle$-weighted graph $G = (V, E, \lambda)$ with

- $V = \{1, 2, 3\}$ and $E = \{(1, 1), (1, 2), (2, 3)\}$, and
- $\Delta = \{a, b\}$ and $\lambda(1, 2) = \lambda(2, 3) = 1a$, $\lambda(2, 2) = 1b$.

Let $\widetilde{\lambda} : \{U \mid U \subseteq P_{q,q'}, q, q' \in V\} \to \mathbb{N}\langle\langle\Delta^*\rangle\rangle$ be the canonical extension of λ using φ to be the identity. Also consider the sets $P_1 = \{(1, 2), (1, 2, 2)\}$ and $P_2 = \{(2, 3), (2, 2, 3)\}$ of paths through G. Then $P_1 \cdot P_2 = \{(1, 2, 3), (1, 2, 2, 3), (1, 2, 2, 2, 3)\}$ and, clearly, $P_1 \cup P_2$ is <u>not</u> $\cdot(P_1, P_2)$-unambiguous. Then

$$\begin{aligned}
& \widetilde{\lambda}(P_1 \cdot P_2) \\
= \; & \widetilde{\lambda}(\{(1, 2, 3), (1, 2, 2, 3), (1, 2, 2, 2, 3)\}) \\
= \; & \widetilde{\lambda}(\{(1, 2, 3)\}) \oplus \widetilde{\lambda}(\{(1, 2, 2, 3)\}) \oplus \widetilde{\lambda}(\{(1, 2, 2, 2, 3)\}) \\
= \; & 1aa \oplus 1aba \oplus 1abba.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
& \widetilde{\lambda}(P_1) \odot_{\mathrm{id}} \widetilde{\lambda}(P_2) \\
= \; & \widetilde{\lambda}(\{(1, 2), (1, 2, 2)\}) \odot_{\mathrm{id}} \widetilde{\lambda}(\{(2, 3), (2, 2, 3)\}) \\
= \; & \Big(\widetilde{\lambda}(\{(1, 2)\}) \oplus \widetilde{\lambda}(\{(1, 2, 2)\})\Big) \odot_{\mathrm{id}} \Big(\widetilde{\lambda}(\{(2, 3)\}) \oplus \widetilde{\lambda}(\{(2, 2, 3)\})\Big) \\
= \; & (1a \oplus 1ab) \odot_{\mathrm{id}} (1a \oplus 1ba) \\
= \; & 1aa \oplus 1aba \oplus 1aba \oplus 1abba \\
= \; & 1aa \oplus 2aba \oplus 1abba
\end{aligned}$$

Thus $\widetilde{\lambda}(P_1 \cdot P_2) \neq \widetilde{\lambda}(P_1) \odot_{\mathrm{id}} \widetilde{\lambda}(P_2)$. □

Lemma 7.3 The $\underline{A^\varphi\langle\langle\Delta^*\rangle\rangle}$-canonical extension $\widetilde{\lambda}$ as defined in Definition 5.2 is φ-compatible.

Proof. Proof of Property 1.: Let $q, q' \in Q$, $P_1, P_2 \subseteq P_{q,q'}$ such that $P_1 \cup P_2$ is $\cup(P_1, P_2)$-unambiguous. Then $\widetilde{\lambda}(P_1 \cup P_2) = \sum_{p \in P_1 \cup P_2} \widetilde{\lambda}(\{p\}) =^* \sum_{p \in P_1} \widetilde{\lambda}(\{p\}) \oplus \sum_{p \in P_2} \widetilde{\lambda}(\{p\}) = \widetilde{\lambda}(P_1) \oplus \widetilde{\lambda}(P_2)$. Note that equation * holds because $P_1 \cap P_2 = \emptyset$.

Proof of Property 2.: Let $q, q', q'' \in Q$, $P_1 \subseteq P_{q,q'}$, $P_2 \subseteq P_{q',q''}$ such that $P_1 \cdot P_2$ is $\cdot(P_1, P_2)$-unambiguous. Moreover, let $w = a_1 \ldots a_m$ with the $a_i \in \Delta$. Then

$$
\begin{aligned}
& (\widetilde{\lambda}(P_1 \cdot P_2), w) \\
= \; & (\textstyle\sum_{p \in P_1 \cdot P_2} \widetilde{\lambda}(\{p\}), w) \\
= \; & \textstyle\sum_{p \in P_1 \cdot P_2} (\widetilde{\lambda}(\{p\}), w) \\
= \; & \textstyle\sum_{p_1 \in P_1, p_2 \in P_2} (\widetilde{\lambda}(\{p_1 p_2\}), w) \\
& \text{(because } P_1 \cdot P_2 \text{ is } \cdot(P_1, P_2)\text{-unambiguous)} \\
= \; & \textstyle\sum_{p_1 \in P_1, p_2 \in P_2} (\widetilde{\lambda}(\{p_1\}), a_1 \ldots a_{|p_1|}) \odot_A \varphi^{|p_1|}((\widetilde{\lambda}(\{p_2\}), a_{|p_1|+1} \ldots a_{|p_1 p_2|})) \\
& \text{(by Lemma 5.3(2))} \\
= \; & \textstyle\sum_{p_1 \in P_1, p_2 \in P_2} \sum_{\substack{u,v \in \Delta^*, w = uv \\ |u| = |p_1|, |v| = |p_2|}} (\widetilde{\lambda}(\{p_1\}), u) \odot_A \varphi^{|u|}((\widetilde{\lambda}(\{p_2\}), v)) \\
= \; & \textstyle\sum_{p_1 \in P_1, p_2 \in P_2} \sum_{u,v \in \Delta^*, w = uv} (\widetilde{\lambda}(\{p_1\}), u) \odot_A \varphi^{|u|}((\widetilde{\lambda}(\{p_2\}), v)) \\
& \text{(by Lemma 5.3(1))} \\
= \; & \textstyle\sum_{u,v \in \Delta^*, w = uv} \left(\sum_{p_1 \in P_1} (\widetilde{\lambda}(\{p_1\}), u) \right) \odot_A \left(\sum_{p_2 \in P_2} \varphi^{|u|}((\widetilde{\lambda}(\{p_2\}), v)) \right) \\
& \text{(by distributivity of } \odot_A \text{ over } \oplus_A \text{ and by commutativity of } \oplus_A) \\
= \; & \textstyle\sum_{u,v \in \Delta^*, w = uv} \left(\sum_{p_1 \in P_1} (\widetilde{\lambda}(\{p_1\}), u) \right) \odot_A \varphi^{|u|} \left(\sum_{p_2 \in P_2} (\widetilde{\lambda}(\{p_2\}), v) \right) \\
& \text{(because } \varphi \text{ is an endomorphism)} \\
= \; & \textstyle\sum_{u,v \in \Delta^*, w = uv} (\widetilde{\lambda}(P_1), u) \odot_A \varphi^{|u|}((\widetilde{\lambda}(P_2), v)) \\
= \; & (\widetilde{\lambda}(P_1) \odot_\varphi \widetilde{\lambda}(P_2), w)
\end{aligned}
$$

Proof of Property 3: Let $q \in Q$ and $P \subseteq P_{q,q}$ such that $\varepsilon \notin \mathit{supp}(\widetilde{\lambda}(P))$.

$$
\begin{aligned}
& (\widetilde{\lambda}(P^*), w) \\
= \; & (\textstyle\sum_{p \in P^*} \widetilde{\lambda}(\{p\}), w) \\
& \text{(note that the family } (\widetilde{\lambda}(\{p\}) \mid p \in P^*) \text{ of fps is locally finite)} \\
= \; & \textstyle\sum_{p \in P^*} (\widetilde{\lambda}(\{p\}), w) \\
= \; & \textstyle\sum_{m \geq 0} \sum_{p_1, \ldots, p_m \in P} (\widetilde{\lambda}(\{p_1 \ldots p_m\}), w) \\
& \text{(because } P^* \text{ is } *(P)\text{-unambiguous)} \\
= \; & \textstyle\sum_{m \geq 0} \sum_{p_1, \ldots, p_m \in P} \sum_{\substack{u_1, \ldots, u_m \in \Delta^+ \\ w = u_1 \ldots u_m}} \\
& \quad (\widetilde{\lambda}(\{p_1\}), u_1) \odot_A \varphi^{|u_1|} \left((\widetilde{\lambda}(\{p_2\}), u_2) \right) \odot_A \ldots \odot_A \varphi^{|u_1 \ldots u_{m-1}|} \left((\widetilde{\lambda}(\{p_m\}), u_m) \right) \\
& \text{(by repeated application of Lemma 5.3(2); note that we can choose the } u_i \text{ from } \Delta^+ \text{, because} \\
& \varepsilon \notin \mathit{supp}(\widetilde{\lambda}(P))) \\
= \; & \textstyle\sum_{m \geq 0} \sum_{\substack{u_1, \ldots, u_m \in \Delta^+ \\ w = u_1 \ldots u_m}} (\sum_{p_1 \in P} \widetilde{\lambda}(\{p_1\}), u_1) \odot_A \\
& \quad \varphi^{|u_1|} \left((\textstyle\sum_{p_2 \in P} \widetilde{\lambda}(\{p_2\}), u_2) \right) \odot_A \ldots \odot_A \varphi^{|u_1 \ldots u_{m-1}|} \left((\textstyle\sum_{p_m \in P} \widetilde{\lambda}(\{p_m\}), u_m) \right) \\
& \text{(by distributivity and the fact that } \varphi \text{ is an endomorphism)} \\
= \; & \textstyle\sum_{m \geq 0} \sum_{\substack{u_1, \ldots, u_m \in \Delta^+ \\ w = u_1 \ldots u_m}} (\widetilde{\lambda}(P), u_1) \odot_A \varphi^{|u_1|} \left((\widetilde{\lambda}(P), u_2) \right) \odot_A \ldots \odot_A \varphi^{|u_1 \ldots u_{m-1}|} \left((\widetilde{\lambda}(P), u_m) \right) \\
= \; & \textstyle\sum_{m \geq 0} (\widetilde{\lambda}(P)^{m,\varphi}, w) \\
= \; & (\textstyle\sum_{m \geq 0} \widetilde{\lambda}(P)^{m,\varphi}, w) \\
= \; & (\widetilde{\lambda}(P)^{*,\varphi}, w).
\end{aligned}
$$

□

Theorem 7.4 $A^{\mathrm{rec},\varphi}\langle\langle \Delta^* \rangle\rangle \subseteq A^{\mathrm{rat},\varphi}\langle\langle \Delta^* \rangle\rangle$.

Proof. Let $S \in A\langle\langle \Delta^* \rangle\rangle$ be a φ-recognizable formal power series over Δ^* and A. That means there is a weighted automaton $\mathcal{A} = (Q, \Delta, \delta, I, F, \underline{A})$ such that $S_\varphi(\mathcal{A}) = S$. By Lemma 5.4,

$$
S_\varphi(\mathcal{A}) = \sum_{q,q' \in Q} \widetilde{I}(q) \odot_\varphi \widetilde{\lambda}(P_{q,q'}) \odot_\varphi \widetilde{F}(q').
$$

Since, by Lemma 4.3, $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ is closed under $\oplus$ and $\odot_\varphi$, it remains to prove that the three formal power series $\widetilde{I}(q)$, $\widetilde{\lambda}(P_{q,q'})$, and $\widetilde{F}(q')$ are elements of $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$. For $\widetilde{I}(q)$ and $\widetilde{F}(q')$ this follows from the fact that $1_A\varepsilon \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ and the fact that $A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ is closed under multiplication with elements from A (cf. Lemma 4.3). It remains to prove that, for every $q, q' \in Q$, the membership $\widetilde{\lambda}(P_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$ holds.

Now we can calculate as follows:

$$
\begin{aligned}
& \widetilde{\lambda}(P_{q,q'}) \\
= \ & \widetilde{\lambda}(\widehat{P}^{(n)}_{q,q'} \cup Z_{q,q'}) \\
& \text{(by Observation 6.2(5))} \\
= \ & \widetilde{\lambda}(\widehat{P}^{(n)}_{q,q'}) \oplus \widetilde{\lambda}(Z_{q,q'}) \\
& \text{(because } \widehat{P}^{(n)}_{q,q'} \cup Z_{q,q'} \text{ is } \cup(\widehat{P}^{(n)}_{q,q'}, Z_{q,q'})\text{-unambiguous)} \\
= \ & \widetilde{\lambda}(Z_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(n)}_{q,q'}).
\end{aligned}
$$

If $q = q'$, then $\widetilde{\lambda}(Z_{q,q'}) = \widetilde{\lambda}(\{\varepsilon\}) = 1_A\varepsilon$, and if $q \neq q'$, then $\widetilde{\lambda}(Z_{q,q'}) = \widetilde{\lambda}(\emptyset) = \widetilde{0}_A$. Thus, $\widetilde{\lambda}(Z_{q,q'}) \in A\langle\langle\{\varepsilon\}\rangle\rangle$. Hence, by Definition 4.2(3),

$$\widetilde{\lambda}(Z_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(n)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle \quad \text{if} \quad \widetilde{\lambda}(\widehat{P}^{(n)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle.$$

We prove by induction that for every $0 \leq k \leq n$, the formal power series $\widetilde{\lambda}(\widehat{P}^{(k)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$.

<u>$k = 0$:</u> Let $q, q' \in Q$.

$$\widetilde{\lambda}(\widehat{P}^{(0)}_{q,q'}) = \begin{cases} \widetilde{\lambda}(\{(q,q')\}) & : \ (q,q') \in E \\ \widetilde{\lambda}(\emptyset) & : \ (q,q') \notin E \end{cases} = \begin{cases} \sum_{a\in\Delta} \delta(q,a,q')a & : \ (q,q') \in E \\ \widetilde{0}_A & : \ (q,q') \notin E \end{cases}$$

In both cases $\widetilde{\lambda}(\widehat{P}^{(0)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$. (Note that the fps $\widetilde{0}_A$ is viewed here as an element of $A\langle\langle\Delta^+\rangle\rangle$, i.e., in the sense of the discussion before Observation 4.1 we should have written $\widetilde{0}'_A$.)

<u>$k-1 \to k$:</u> Assume that, for every $u, v \in Q$, $\widetilde{\lambda}(\widehat{P}^{(k-1)}_{u,v}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$.

Now let $q, q' \in Q$ be two arbitrary states. We prove that $\widetilde{\lambda}(\widehat{P}^{(k)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$.

$$
\begin{aligned}
& \widetilde{\lambda}(\widehat{P}^{(k)}_{q,q'}) \\
= \ & \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,q'}) \oplus \widetilde{\lambda}\left(\widehat{P}^{(k-1)}_{q,k} \cdot (\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,q'}\right) \\
& \text{(by Lemma 6.4(1) and Lemma 7.3)} \\
= \ & \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,k}) \odot_\varphi \widetilde{\lambda}\left((\widehat{P}^{(k-1)}_{k,k})^* \cdot \widehat{P}^{(k-1)}_{k,q'}\right) \\
& \text{(by Lemma 6.4(2) and Lemma 7.3)} \\
= \ & \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,k}) \odot_\varphi \widetilde{\lambda}((\widehat{P}^{(k-1)}_{k,k})^*) \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,q'}) \\
& \text{(by Lemma 6.4(3) and Lemma 7.3)} \\
= \ & \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,k}) \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,k})^{*,\varphi} \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,q'}) \\
& \text{(by Lemma 6.4(4) and Lemma 7.3; note that } \varepsilon \notin \mathit{supp}(\widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,k})) \text{, because } \varepsilon \notin \widehat{P}^{(k-1)}_{k,k} \\
& \text{and hence, by Lemma 5.3(1), } \mathit{supp}(\widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,k})) \subseteq \textstyle\bigcup_{m\geq 1} \Delta^m) \\
= \ & \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,k}) \odot_\varphi (1_A\varepsilon \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,k})^{+,\varphi}) \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,q'}) \\
& \text{(by definition of Kleene-}*) \\
= \ & \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,k}) \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,q'}) \oplus \widetilde{\lambda}(\widehat{P}^{(k-1)}_{q,k}) \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,k})^{+,\varphi} \odot_\varphi \widetilde{\lambda}(\widehat{P}^{(k-1)}_{k,q'}) \\
& \text{(by Observation 4.1(1), in particular, } \odot_\varphi \text{ distributes over } \oplus)
\end{aligned}
$$

Using the induction hypothesis and the fact that $A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$ is closed under $\oplus$, $\odot_\varphi$, and $(.)^{+,\varphi}$, we have thus proved that $\widetilde{\lambda}(\widehat{P}^{(k)}_{q,q'}) \in A^{\mathrm{rat},\varphi}\langle\langle\Delta^+\rangle\rangle$ for every $0 \leq k \leq n$. □

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