Weighted Tree Automata over Strong Bimonoids

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Abstract: We consider weighted tree automata over strong bimonoids which are, roughly speaking, semirings without distributivity. We prove sufficient and necessary conditions under which the initial algebra semantics and the run semantics of these automata coincide. We prove closure properties of the class of recognizable tree series, a determinization result, and a characterization of recognizable step functions.

1 Introduction

In the past, weighted tree automata have been considered over different classes of algebras, viz. completely distributive lattices [10, 8], fields [2], commutative semirings [1], and continuous semirings [12, 7]. For a survey of results on recognizable tree series we refer the reader to [12, 7, 9].

In this paper we study weighted tree automata over strong bimonoids thereby following the line of research founded in [5] where weighted (string) automata over strong bimonoids have been considered. A strong bimonoid $(S, +, \cdot, 0, 1)$ consists of an additive monoid $(S, +, 0)$ and a multiplicative monoid $(S, \cdot, 1)$; moreover, the 0 is absorbing with respect to $\cdot$, i.e., $a \cdot 0 = 0 \cdot a = 0$ for every $a \in S$. In other words, a strong bimonoid is a semiring without distributivity laws. Each of the above mentioned algebras are particular strong bimonoids.

In the same way as for semiring-weighted tree automata [9], we define for a weighted tree automaton $\mathcal{A}$ over some strong bimonoid S two types of semantics: the initial algebra semantics and the run semantics, and we prove sufficient and necessary conditions under which these two semantics are equivalent. More precisely, let Σ be a ranked alphabet and S a strong bimonoid; then the following two statements are equivalent (cf. Theorem 4.1):

1. S is right distributive and, if Σ is non-monadic, then S is left distributive.
2. $r_{\mathcal{A}}^{run} = r_{\mathcal{A}}$ for every wta $\mathcal{A}$ over Σ and S.

We denote by $\mathrm{Rec}(\Sigma, S)$ (and bud-$\mathrm{Rec}(\Sigma, S)$) the class of tree series over Σ and S which are recognized by weighted tree automata (respectively, by bottom-up deterministic weighted tree automata) over the strong bimonoid S using the initial algebra semantics. We prove that $\mathrm{Rec}(\Sigma, S)$ and bud-$\mathrm{Rec}(\Sigma, S)$ are closed under sum (cf. Lemma 5.1) and, if S is commutative,

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then bud-Rec(Σ, S) is closed under Hadamard product (cf. Lemma 5.3). Moreover, bud-Rec(Σ, S) is closed under right multiplication with a coefficient a of S; Rec(Σ, S) is closed under right multiplication (respectively, left multiplication) with a, if S is right distributive (respectively, left distributive), cf. Theorem 5.4.

We prove that a recognizable tree series can be recognized by bottom-up deterministic weighted tree automata if the strong bimonoid is locally finite (cf. Theorem 6.4).

Finally, we prove that a tree series r is a recognizable step function if and only if r can be recognized by some crisp and bottom-up deterministic weighted tree automaton if and only if r has only finitely many images in S and each of the preimages is a recognizable tree language (cf. Theorem 7.3).

In most of the cases, the proof techniques that we employ are adapted from the proofs of corresponding results for weighted (string) automata over strong bimonoids [5] and for weighted tree automata over semirings [9].

2 Preliminaries

2.1 Sets, matrices, and functions

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. For a set A, we denote its set of subsets by $\mathcal{P}(A)$. The empty string is denoted by ε, and the length of a string w by $|w|$. We denote the cardinality of a finite set A by $|A|$.

Let S, I, and J be sets. An $I \times J-$*matrix over* S is a mapping $\mathcal{M} : I \times J \to S$; the set of all $I \times J-$matrices over S is denoted by $S^{I\times J}$. We write an entry $\mathcal{M}(i, j) \in S$ as $\mathcal{M}_{i,j}$. An I-*vector* v *over* S is defined analogously; the set of all I-vectors over S is denoted by S^I and an element $v(i) \in S$ is denoted by v_i. Let $\mathcal{M} \in S^{I\times J}$, $v_1 \in S^I$, and $v_2 \in S^J$. Then we define the *matrix-vector product* $v_1 \cdot \mathcal{M} \in S^J$ and $\mathcal{M} \cdot v_2 \in S^I$ as follows for every $i \in I$ and $j \in J$:

$$(v_1 \cdot \mathcal{M})_j = \sum_{i\in I}(v_1)_i \cdot \mathcal{M}_{i,j} \ ,$$
$$(\mathcal{M} \cdot v_2)_i = \sum_{j\in J} \mathcal{M}_{i,j} \cdot (v_2)_j .$$

For $u, v \in S^I$, we define the *scalar product* $u \cdot v \in S$ as $u \cdot v = \sum_{i\in I} u_i \cdot v_i$.

For two functions $f : A \to B$ and $g : B \to C$, we denote their composition by $g \circ f$ where $(g \circ f)(a) = g(f(a))$ for every $a \in A$.

Let A and B be two sets such that $A \subseteq B$. The *characteristic mapping of* A is the mapping $\chi_A : B \to \{0, 1\}$ such that $\chi_A(a) = 1$ if $a \in A$, and $\chi(a) = 0$ otherwise.

2.2 Trees

A *ranked alphabet* is a tuple (Σ, rk) where Σ is a finite set and $rk : \Sigma \to \mathbb{N}$ is a mapping called rank mapping. For every $k \geq 0$, we define $\Sigma^{(k)} = \{\sigma \in \Sigma \mid rk(\sigma) = k\}$. Sometimes we write $\sigma^{(k)}$ to emphasize that $\sigma \in \Sigma^{(k)}$. Σ is called *trivial* if $\Sigma^{(0)} = \emptyset$ or $\Sigma = \Sigma^{(0)}$, and Σ is *non-trivial* if Σ is not trivial. Thus Σ is non-trivial if Σ contains at least one nullary symbol and at least one non-nullary symbol. Σ is called *monadic* if $|\Sigma^{(0)}| = 1$, $|\Sigma^{(1)}| \geq 1$, and $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$. In fact, $T_\Sigma \cong \left(\Sigma^{(1)}\right)^*$ for monadic Σ. Σ is *non-monadic* if Σ is not monadic. A monadic ranked alphabet is non-trivial.

Let H be a set disjoint with Σ. The set of Σ-*terms over* H, denoted by $T_\Sigma(H)$, is the smallest set T such that (i) $\Sigma^{(0)} \cup H \subseteq T$ and (ii) if $k \geq 1$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in T$, then $\sigma(\xi_1, \ldots, \xi_k) \in T$. We denote $T_\Sigma(\emptyset)$ by T_Σ. Clearly, Σ is trivial iff T_Σ is finite. Since terms can be depicted in an illustrative way as trees, i.e., particular graphs, it has become a custom to call Σ-terms also *Σ-trees*. Every subset $L \subseteq T_\Sigma$ is called *Σ-tree language*.

We define $\mathrm{pos}(\xi) \subseteq \mathbb{N}^*$, the *set of positions* of tree $\xi \in T_\Sigma$ as follows: (i) for every $\alpha \in \Sigma^{(0)}$, $\mathrm{pos}(\alpha) = \{\varepsilon\}$, (ii) for every $\xi = \sigma(\xi_1, \ldots, \xi_k)$, where $k \geq 1$, $\mathrm{pos}(\xi) = \{\varepsilon\} \cup \{iv \mid 1 \leq i \leq k, v \in \mathrm{pos}(\xi_i)\}$.

Let $\xi \in T_\Sigma$ and $w \in \mathrm{pos}(\xi)$. The *subtree of ξ at w*, denoted by $\xi|_w$, is defined as follows: (i) for every $\alpha \in \Sigma^{(0)}$, $\alpha|_\varepsilon = \alpha$, (ii) for every $\xi = \sigma(\xi_1, \ldots, \xi_k)$ with $\sigma \in \Sigma^{(k)}$, $\xi|_\varepsilon = \xi$, and for every $1 \leq i \leq k$, $\xi|_{iv} = \xi_i|_v$.

> *In the rest of this paper, Σ will denote an arbitrary non-trivial ranked alphabet if not specified otherwise.*

2.3 Algebraic structures

A *bimonoid* $(S, +, \cdot, 0, 1)$ is an algebra which consists of a monoid $(S, +, 0)$, called additive monoid of S, and a monoid $(S, \cdot, 1)$, called multiplicative monoid of S. As usual, we identify the algebra $(S, +, \cdot, 0, 1)$ with its carrier set S. If the operation $+$ is commutative and 0 is absorbing with respect to $\cdot$, i.e., $a \cdot 0 = 0 \cdot a = 0$ for every $a \in S$, then S is called a *strong bimonoid* (for short: s-bimonoid). An s-bimonoid S is *commutative* if the operation $\cdot$ is commutative. We say that an s-bimonoid S is *right distributive* if it satisfies $(a+b) \cdot c = a \cdot c + b \cdot c$ for every $a, b, c \in S$. We call S *left distributive* if it satisfies $a \cdot (b+c) = a \cdot b + a \cdot c$ for every $a, b, c \in S$. An s-bimonoid which is left and right distributive is a *semiring*. The *Boolean semiring* is the semiring $(\mathbb{B}, \vee, \wedge, 0, 1)$ where $\mathbb{B} = \{0, 1\}$ and $\vee$ and $\wedge$ are the usual disjunction and conjunction, respectively. As another example, we recall from [13] the s-bimonoid $(\Sigma^* \cup \{\infty\}, \wedge, \cdot, \infty, \varepsilon)$, where $w_1 \wedge w_2$ is the longest common postfix of $w_1, w_2 \in \Sigma^* \cup \{\infty\}$ and $w_1 \cdot w_2$ is the concatenation of w_1 and w_2 (where $w_1 \cdot w_2 = \infty$ if $w_1 = \infty$ or $w_2 = \infty$). We note that this s-bimonoid is right distributive, but not left distributive.

An s-bimonoid S is *locally finite* if, for every finite $S' \subseteq S$, the sub-s-bimonoid of S generated by S', is finite.

> *In the rest of this paper, S will denote an arbitrary s-bimonoid $(S, +, \cdot, 0, 1)$ if not specified otherwise.*

A *Σ-algebra* (V, θ) consists of a non-empty set V and an arity preserving interpretation θ of symbols from Σ as operations over V, i.e., $\theta(\sigma) : V^k \to V$ for every $k \geq 0$ and $\sigma \in \Sigma^k$. The *Σ-term algebra* (T_Σ,top) is the Σ-algebra such that for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in T_\Sigma$, we have $\mathrm{top}(\sigma)(\xi_1, \ldots, \xi_k) = \sigma(\xi_1, \ldots, \xi_k)$. This Σ-algebra is *initial* in the class of all Σ-algebras, i.e., for every Σ-algebra (V, θ) there is a unique Σ-algebra homomorphism from T_Σ to V, which we denote by h_V.

2.4 Tree series

A *tree series over Σ and S* (or for short: tree series) is a mapping $r : T_\Sigma \to S$. For every $\xi \in T_\Sigma$, the element $r(\xi) \in S$ is called *coefficient* of ξ and it is denoted by (r, ξ). The set of all tree series over Σ and S is denoted by $S\langle\langle T_\Sigma\rangle\rangle$.

Let $r \in S\langle\langle T_\Sigma\rangle\rangle$. The *support of* r is defined as the set $\mathrm{supp}(r) = \{\xi \in T_\Sigma \mid (r,\xi) \neq 0\}$. For every $s \in S$ we define $r_{=s} = \{\xi \in T_\Sigma \mid (r,\xi) = s\}$. The *image of* r is the set $\mathrm{im}(r) = \{(r,\xi) \in S \mid \xi \in T_\Sigma\}$.

Let L be a tree language, i.e., $L \subseteq T_\Sigma$. We define the tree series $1_{(S,L)} : T_\Sigma \to S$ by $(1_{(S,L)},\xi) = \chi_L(\xi)$ for every $\xi \in T_\Sigma$. We call the tree series $1_{(S,L)}$ the *characteristic tree series of* L *with respect to* S. Obviously, $\mathrm{supp}(1_{(S,L)}) = L$.

Let $r_1, r_2 \in S\langle\langle T_\Sigma\rangle\rangle$. The *sum of* r_1 *and* r_2 and the *Hadamard product of* r_1 *and* r_2 are the tree series $r_1+r_2 \in S\langle\langle T_\Sigma\rangle\rangle$ and $r_1 \odot r_2 \in S\langle\langle T_\Sigma\rangle\rangle$, respectively, defined by $(r_1+r_2,\xi) = (r_1,\xi)+(r_2,\xi)$ and $(r_1 \odot r_2,\xi) = (r_1,\xi)\cdot(r_2,\xi)$ for every $\xi \in T_\Sigma$.

Let $a \in S$ and $r \in S\langle\langle T_\Sigma\rangle\rangle$. The *scalar left multiplication* of a and r is the tree series $a \cdot r \in S\langle\langle T_\Sigma\rangle\rangle$ defined by $(a \cdot r,\xi) = a \cdot (r,\xi)$ for every $\xi \in T_\Sigma$. The *scalar right multiplication* of a and r is the tree series $r \cdot a \in S\langle\langle T_\Sigma\rangle\rangle$ defined by $(r \cdot a,\xi) = (r,\xi) \cdot a$ for every $\xi \in T_\Sigma$.

3 Weighted tree automata

Now we define weighted tree automata over S. Actually, the definition is the same as that for semiring-weighted tree automata.

Definition 3.1. A *weighted tree automaton (over* Σ *and* S*)* (for short: wta) is a tuple $\mathcal{A} = (Q,\Sigma,S,\mu,\nu)$ where

- Q is a finite nonempty set, the *set of states*,
- Σ is a ranked alphabet, the *ranked input alphabet*,
- $\mu = (\mu_k \mid k \in \mathbb{N})$ is a family of mappings $\mu_k : \Sigma^{(k)} \to S^{Q^k \times Q}$, the *transition mappings*,
- $\nu \in S^Q$ is a Q-vector over S, the *root weight vector*.

For every transition $(w,q) \in Q^k \times Q$, the element $\mu_k(\sigma)_{w,q} \in S$ is called the *weight of* (w,q). We denote by $\mathrm{wts}(\mathcal{A})$ the set of all weights which occur in $\mathcal{A}$, i.e., $\mathrm{wts}(\mathcal{A}) = \{\mu_k(\sigma)_{w,q} \mid k \geq 0, \sigma \in \Sigma^{(k)}, w \in Q^k, q \in Q\} \cup \{\nu_q \mid q \in Q\}$. Note that $\mathrm{wts}(\mathcal{A}) \subseteq S$.

Initial algebra semantics
For every wta $\mathcal{A}$, we consider the Σ-algebra $(S^Q, \mu_\mathcal{A})$ where, for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$, the k-ary operation $\mu_\mathcal{A}(\sigma) : S^Q \times \ldots \times S^Q \to S^Q$ is defined by

$$\mu_\mathcal{A}(\sigma)(v_1,\ldots,v_k)_q = \sum_{q_1,\ldots,q_k \in Q} (v_1)_{q_1} \cdot \ldots \cdot (v_k)_{q_k} \cdot \mu_k(\sigma)_{q_1\ldots q_k,q}$$

for every $q \in Q$ and $v_1,\ldots,v_k \in S^Q$. The tree series $r_\mathcal{A} \in S\langle\langle T_\Sigma\rangle\rangle$ *recognized* by $\mathcal{A}$ is defined by:

$$(r_\mathcal{A},\xi) = h_\mu(\xi)\cdot \nu = \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q$$

for every $\xi \in T_\Sigma$, where h_μ denotes the unique Σ-algebra homomorphism from T_Σ to S^Q. A tree series $r \in S\langle\langle T_\Sigma\rangle\rangle$ is *recognizable* if there is a wta $\mathcal{A}$ such that $r = r_\mathcal{A}$. The class of all recognizable tree series over Σ and S is denoted by $\mathrm{Rec}(\Sigma, S)$.

Run semantics

A *run of* $\mathcal{A}$ *on* $\xi \in T_\Sigma$ is a mapping $\kappa : \mathrm{pos}(\xi) \to Q$. The set of all runs of $\mathcal{A}$ on ξ is denoted by $\mathcal{R}_\mathcal{A}(\xi)$. For every $\kappa \in \mathcal{R}_\mathcal{A}(\xi)$ and $w \in \mathrm{pos}(\xi)$, the *run induced by* κ *at position* w, denoted by $\kappa|_w \in \mathcal{R}_\mathcal{A}(\xi|_w)$, is the mapping $\kappa|_w : \mathrm{pos}(\xi|_w) \to Q$ defined by $\kappa|_w(w') = \kappa(ww')$ for every $w' \in \mathrm{pos}(\xi|_w)$. For every $\xi = \sigma(\xi_1, \ldots, \xi_k) \in T_\Sigma$, the *weight* $\mathrm{wt}(\kappa)$ of κ is

$$\mathrm{wt}(\kappa) = \mathrm{wt}(\kappa|_1) \cdot \ldots \cdot \mathrm{wt}(\kappa|_k) \cdot \mu_k(\sigma)_{\kappa(1)\ldots\kappa(k),\kappa(\varepsilon)}.$$

The *run semantics of* $\mathcal{A}$ is the tree series $r^{run}_\mathcal{A} \in S\langle\langle T_\Sigma\rangle\rangle$ defined by

$$(r^{run}_\mathcal{A}, \xi) = \sum_{\kappa \in \mathcal{R}_\mathcal{A}(\xi)} \mathrm{wt}(\kappa) \cdot \nu_{\kappa(\varepsilon)}$$

for every $\xi \in T_\Sigma$.

In fact, wta over monadic ranked alphabets correspond in a one-one relation to weighted finite automata over strong bimonoids as they are defined in [5]. Also the initial algebra semantics and run semantics pairwise correspond to each other.

Next we introduce bottom-up deterministic and crisp weighted tree automata.

Definition 3.2. Let $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ be a wta over Σ and S.

- We call $\mathcal{A}$ *bottom-up deterministic* (for short: bu-deterministic) if for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $w \in Q^k$ there is at most one $q \in Q$ such that $\mu_k(\sigma)_{w,q} \neq 0$.
- We call $\mathcal{A}$ *total* if for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $w \in Q^k$ there is at least one state q such that $\mu_k(\sigma)_{w,q} \neq 0$.
- We call $\mathcal{A}$ *crisp* if $\mu_k(\sigma)_{w,q} \in \{0, 1\}$ for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $(w, q) \in Q^k \times Q$.

Observation 3.3. Let $\mathcal{A}$ be a bu-deterministic wta over S. Then the following statements hold:

1. For every input tree $\xi \in T_\Sigma$, there is at most one $q \in Q$ such that $h_\mu(\xi)_q \neq 0$, and at most one run $\kappa \in \mathcal{R}_\mathcal{A}(\xi)$ such that $\mathrm{wt}(\kappa) \neq 0$. Moreover, there is such a state iff there is such a run. If there exists a state $q \in Q$ with $h_\mu(\xi)_q \neq 0$ and if there exists a run $\kappa \in \mathcal{R}_\mathcal{A}(\xi)$ with $\mathrm{wt}(\kappa) \neq 0$, then $\mathrm{wt}(\kappa) = h_\mu(\xi)_q$ and $\kappa(\varepsilon) = q$. In this case the operation $+$ of S is not used to compute $r_\mathcal{A}$ and $r^{run}_\mathcal{A}$.
2. If, additionally, the wta $\mathcal{A}$ is total and S is zero-divisor free, then for each input tree $\xi \in T_\Sigma$, there exists exactly one $q \in Q$ and exactly one $\kappa \in \mathcal{R}_\mathcal{A}(\xi)$, such that $h_\mu(\xi)_q = \mathrm{wt}(\kappa) \neq 0$ and $\kappa(\varepsilon) = q$.
3. If, additionaly, the wta $\mathcal{A}$ is crisp, then $\mathrm{im}(r_\mathcal{A}) \subseteq \{v_q \mid q \in Q\}$. Thus, in particular, $\mathrm{im}(r_\mathcal{A})$ is finite.

4. If, additionally, the wta $\mathcal{A}$ is total and crisp, then for every input tree $\xi \in T_\Sigma$, there exists exactly one $q \in Q$ and exactly one $\kappa \in \mathcal{R}_\mathcal{A}(\xi)$ such that $h_\mu(\xi)_q = \mathrm{wt}(\kappa) = 1$ and $\kappa(\varepsilon) = q$.

The tree series $r \in S\langle\langle T_\Sigma\rangle\rangle$ is *bu-deterministically recognizable* if there is a bu-deterministic wta $\mathcal{A}$ such that $r = r_\mathcal{A}$. The class of bu-deterministically recognizable tree series is denoted by bud-Rec(Σ, S). Clearly, bud-Rec$(\Sigma, S) \subseteq$ Rec(Σ, S).

Lemma 3.4. For every $r \in$ bud-Rec(Σ, S), there is a total bu-deterministic wta $\mathcal{A}$ such that $r = r_\mathcal{A}$.

Proof. Let $\mathcal{B} = (Q_\mathcal{B}, \Sigma, S, \mu^\mathcal{B}, \nu^\mathcal{B})$ be a bu-deterministic wta such that $r = r_\mathcal{B}$. Take $q_0 \notin Q_\mathcal{B}$ and let $Q = Q_\mathcal{B} \cup \{q_0\}$. We construct the wta $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ as follows. We define $\nu_{q_0} = 0$ and $\nu_q = \nu^\mathcal{B}_q$ for every $q \in Q_\mathcal{B}$. For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $w \in Q^k$:

- if $w \in Q^k_\mathcal{B}$ and there exists $q \in Q_\mathcal{B}$ such that $\mu^\mathcal{B}_k(\sigma)_{w,q} \neq 0$, then $\mu_k(\sigma)_{w,q} = \mu^\mathcal{B}_k(\sigma)_{w,q}$,
- if $w \in Q^k_\mathcal{B}$ and $\mu^\mathcal{B}_k(\sigma)_{w,q} = 0$ for every $q \in Q$, then $\mu_k(\sigma)_{w,q_0} = 1$,
- if $w \in Q^k \backslash Q^k_\mathcal{B}$, then $\mu_k(\sigma)_{w,q_0} = 1$,
- for every other combination $(v, p) \in Q^k \times Q$, we define $\mu_k(\sigma)_{v,p} = 0$.

Then, for every $\xi \in T_\Sigma$ and $q \in Q_\mathcal{B}$, we have that $h_{\mu^\mathcal{B}}(\xi)_q = h_\mu(\xi)_q$. Since $\nu_{q_0} = 0$, we have $r_\mathcal{A} = r$. ☐

Finally, we note that we obtain the classical concept of a finite state tree automaton as special case of our concept as follows. *A bottom-up finite state tree automaton* (for short: bu-fta) is a wta $\mathcal{A} = (Q, \Sigma, \mathbb{B}, \mu, \nu)$. In this case we write $\mathcal{A} = (Q, \Sigma, \mu, F)$ with $F = \nu^{-1}(1)$. The *tree language accepted by the bu-fta* $\mathcal{A}$ is the set $\mathcal{L}(\mathcal{A}) \subseteq T_\Sigma$, defined by $\mathcal{L}(\mathcal{A}) = \mathrm{supp}(r_\mathcal{A})$. The tree language $L \subseteq T_\Sigma$ is *recognizable* if there is a bu-fta $\mathcal{A}$ over Σ such that $L = \mathcal{L}(\mathcal{A})$. The class of all recognizable tree languages over Σ is denoted by Rec(Σ). A bu-fta $\mathcal{A} = (Q, \Sigma, \mu, F)$ is called *deterministic* (*total*) if the wta $(Q, \Sigma, \mathbb{B}, \mu, \nu)$ is bu-deterministic (total, respectively).

4 Initial algebra semantics versus run semantics

In the next theorem we will prove necessary and sufficient conditions under which the initial algebra semantics and the run semantics of a wta coincide. Actually, this theorem generalizes Lemma 6 of [5] from strings to trees, and it turns out that in the tree case the left distributivity of S has to be required additionally. Also, the theorem generalizes the fact that the initial algebra semantics and the run semantics of a wta over any semiring coincide (cf. p. 317 of [9]) by turning this fact into an equivalence.

Theorem 4.1. Let Σ be a ranked alphabet and S an s-bimonoid. Then the following statements are equivalent:

1. S is right distibutive and, if Σ is not monadic, then S is left distributive.
2. $r^{run}_\mathcal{A} = r_\mathcal{A}$ for every wta $\mathcal{A}$ over Σ and S.

Proof. 1. $\Rightarrow$ 2.: Let $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ be a wta. First, we will prove that the following statement holds:

$$(*) \qquad h_\mu(\xi)_q = \sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(\varepsilon) = q}} \mathrm{wt}(\kappa) \quad \text{for every } \xi \in T_\Sigma \text{ and } q \in Q.$$

Let $\xi = \sigma(\xi_1, \ldots, \xi_k)$ for some $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in T_\Sigma$. Then

$$\begin{aligned}
h_\mu(\xi)_q = {} & h_\mu(\sigma(\xi_1, \ldots, \xi_k))_q = \mu_\mathcal{A}(\sigma)(h(\xi_1), \ldots, h(\xi_k))_q \\
= {} & \sum_{q_1, \ldots, q_k \in Q} h_\mu(\xi_1)_{q_1} \cdot \ldots \cdot h_\mu(\xi_k)_{q_k} \cdot \mu_k(\sigma)_{q_1 \ldots q_k, q} \\
= {} & \sum_{q_1, \ldots, q_k \in Q} \Big(\sum_{\substack{\kappa_1 \in \mathcal{R}_\mathcal{A}(\xi_1) \\ \kappa_1(\varepsilon) = q_1}} \mathrm{wt}(\kappa_1) \Big) \cdot \ldots \cdot \Big(\sum_{\substack{\kappa_k \in \mathcal{R}_\mathcal{A}(\xi_k) \\ \kappa_k(\varepsilon) = q_k}} \mathrm{wt}(\kappa_k) \Big) \cdot \mu_k(\sigma)_{q_1 \ldots q_k, q} \\
& \hfill (\textit{Statement } (*)) \\
= {} & \sum_{q_1, \ldots, q_k \in Q} \sum_{\substack{\kappa_1 \in \mathcal{R}_\mathcal{A}(\xi_1) \\ \kappa_1(\varepsilon) = q_1}} \mathrm{wt}(\kappa_1) \cdot \Big(\ldots \cdot \Big(\sum_{\substack{\kappa_k \in \mathcal{R}_\mathcal{A}(\xi_k) \\ \kappa_k(\varepsilon) = q_k}} \mathrm{wt}(\kappa_k) \cdot \mu_k(\sigma)_{q_1 \ldots q_k, q} \Big) \ldots \Big) \\
& \hfill (\textit{S right distributive}) \\
= {} & \sum_{q_1, \ldots, q_k \in Q} \sum_{\substack{\kappa_1 \in \mathcal{R}_\mathcal{A}(\xi_1) \\ \kappa_1(\varepsilon) = q_1}} \cdots \sum_{\substack{\kappa_k \in \mathcal{R}_\mathcal{A}(\xi_k) \\ \kappa_k(\varepsilon) = q_k}} \mathrm{wt}(\kappa_1) \cdot \ldots \cdot \mathrm{wt}(\kappa_k) \cdot \mu_k(\sigma)_{q_1 \ldots q_k, q} \\
& \hfill (\Sigma \textit{ monadic or S left distributive}) \\
= {} & \sum_{q_1, \ldots, q_k \in Q} \sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(1) = q_1, \ldots, \kappa(k) = q_k \\ \kappa(\varepsilon) = q}} \mathrm{wt}(\kappa|_1) \cdot \ldots \cdot \mathrm{wt}(\kappa|_k) \cdot \mu_k(\sigma)_{q_1 \ldots q_k, q} \\
= {} & \sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(\varepsilon) = q}} \mathrm{wt}(\kappa|_1) \cdot \ldots \cdot \mathrm{wt}(\kappa|_k) \cdot \mu_k(\sigma)_{q_1 \ldots q_k, q} \\
= {} & \sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(\varepsilon) = q}} \mathrm{wt}(\kappa).
\end{aligned}$$

Hence,

$$\begin{aligned}
(r^{run}_\mathcal{A}, \xi) = {} & \sum_{\kappa \in \mathcal{R}_\mathcal{A}(\xi)} \big(\mathrm{wt}(\kappa) \cdot \nu_{\kappa(\varepsilon)} \big) = \sum_{q \in Q} \sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(\varepsilon) = q}} \big(\mathrm{wt}(\kappa) \cdot \nu_q \big) \\
= {} & \sum_{q \in Q} \Big(\sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(\varepsilon) = q}} \mathrm{wt}(\kappa) \Big) \cdot \nu_q \qquad (\textit{S right distributive}) \\
= {} & \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q = (r_\mathcal{A}, \xi).
\end{aligned}$$

2. $\Rightarrow$ 1.: Let $a, b, c \in S$. We should prove that

i) $(a+b)\cdot c = a\cdot c + b\cdot c$ and
ii) if Σ is non-monadic, then $a\cdot(b+c) = a\cdot b + a\cdot c$.

To prove the first equation, assume that $\alpha\in\Sigma^{(0)}$ and $\gamma\in\Sigma^{(k)}$ for some $k\geq 1$. We consider the wta $\mathcal{A}=(Q,\Sigma,S,\mu,\nu)$ defined as follows: $Q=\{p,q,1\}$, $\nu_p=\nu_1=0$, $\nu_q=c$. Moreover, we define the transition mappings as follows:

- $\mu_0(\alpha)_{\varepsilon,p}=a$, $\mu_0(\alpha)_{\varepsilon,q}=b$, $\mu_0(\alpha)_{\varepsilon,1}=1$,
- $\mu_k(\gamma)_{p1\ldots1,q}=\mu_k(\gamma)_{q1\ldots1,q}=1$,
- $\mu_k(\gamma)_{w,u}=0$ for every $(w,u)\notin\{(p1\ldots1,q),(q1\ldots1,q)\}$
- for every other input symbol $\sigma\in\Sigma^{(k)}$ with $k\geq 0$ and state behaviour $(w,r)\in Q^k\times Q$ we can define $\mu_k(\sigma)_{w,r}$ arbitrarily.

Take $\xi=\gamma(\underbrace{\alpha,\ldots,\alpha}_{k})\in T_\Sigma$. We will calculate $(r_{\mathcal{A}},\xi)$ and $(r^{run}_{\mathcal{A}},\xi)$.

$$\begin{aligned}
(r_{\mathcal{A}},\xi) &= \sum_{u\in Q} h_\mu(\xi)_u\cdot\nu_u = h_\mu(\xi)_q\cdot\nu_q\\
&= \Big(\sum_{u_1,\ldots,u_k\in Q} h_\mu(\alpha)_{u_1}\cdot\ldots\cdot h_\mu(\alpha)_{u_k}\cdot\mu_k(\sigma)_{u_1\ldots u_k,q}\Big)\cdot c\\
&= \big(h_\mu(\alpha)_p\cdot\underbrace{h_\mu(\alpha)_1\cdot\ldots\cdot h_\mu(\alpha)_1}_{k-1}\cdot\mu_k(\gamma)_{p1\ldots1,q}\,+\\
&\qquad\qquad + h_\mu(\alpha)_q\cdot\underbrace{h_\mu(\alpha)_1\cdot\ldots\cdot h_\mu(\alpha)_1}_{k-1}\cdot\mu_k(\gamma)_{q1\ldots1,q}\big)\cdot c\\
&= \big(a\cdot1\cdot\ldots\cdot1\cdot1 + b\cdot1\cdot\ldots\cdot1\cdot1\big)\cdot c\\
&= (a+b)\cdot c.
\end{aligned}$$

$$\begin{aligned}
(r^{run}_{\mathcal{A}},\xi) &= \sum_{\kappa\in\mathcal{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\kappa)\cdot\nu_{\kappa(\varepsilon)} = \sum_{\substack{\kappa\in\mathcal{R}_{\mathcal{A}}(\xi)\\ \kappa(\varepsilon)=q}}\big(\mathrm{wt}(\kappa)\cdot c\big)\\
&= \mathrm{wt}(\kappa_1)\cdot c + \mathrm{wt}(\kappa_2)\cdot c\\
&\qquad \text{where}\quad \kappa_1(\varepsilon)=q,\quad \kappa_1(1)=p,\quad \kappa_1(i)=*\quad\text{for every } i\in\{2,\ldots,k\}\\
&\qquad\qquad\quad\;\; \kappa_2(\varepsilon)=q,\quad \kappa_2(1)=q,\quad \kappa_2(i)=*\quad\text{for every } i\in\{2,\ldots,k\}\\
&= \mathrm{wt}(\kappa_1|_1)\cdot\mathrm{wt}(\kappa_1|_2)\cdot\ldots\cdot\mathrm{wt}(\kappa_1|_k)\cdot\mu_k(\gamma)_{p*\ldots*,q}\cdot c\,+\\
&\qquad + \mathrm{wt}(\kappa_2|_1)\cdot\mathrm{wt}(\kappa_2|_2)\cdot\ldots\cdot\mathrm{wt}(\kappa_2|_k)\cdot\mu_k(\gamma)_{q*\ldots*,q}\cdot c\\
&= \mu_0(\alpha)_{\varepsilon,p}\cdot\mu_0(\alpha)_{\varepsilon,*}\cdot\ldots\cdot\mu_0(\alpha)_{\varepsilon,*}\cdot1\cdot c + \mu_0(\alpha)_{\varepsilon,q}\cdot\mu_0(\alpha)_{\varepsilon,*}\cdot\ldots\cdot\mu_0(\alpha)_{\varepsilon,*}\cdot1\cdot c\\
&= a\cdot1\cdot\ldots\cdot1\cdot1\cdot c + b\cdot1\cdot\ldots\cdot1\cdot1\cdot c\\
&= a\cdot c + b\cdot c.
\end{aligned}$$

Since $r^{run}_{\mathcal{A}}=r_{\mathcal{A}}$, we obtain that Statement i) holds.

Now we prove Statement ii). Let Σ be non-monadic. Recall that Σ is non-trivial. Thus $\Sigma^{(0)}\neq\emptyset$ and there is $k\geq 2$ such that $\Sigma^{(k)}\neq\emptyset$. Let $\alpha\in\Sigma^{(0)}$ and $\sigma\in\Sigma^{(k)}$. Now consider the wta

$\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ with $Q = \{a, b, c, p, q, 1\}$, $\nu_q = 1$, $\nu_u = 0$ for every $u \neq q$, and the transition mappings:

- $\mu_0(\alpha)_{\varepsilon,x} = x$ for every $x \in \{a, b, c, 1\}$,
- $\mu_0(\alpha)_{\varepsilon,p} = \mu_0(\alpha)_{\varepsilon,q} = 0$,
- $\mu_k(\sigma)_{b1\ldots1,p} = \mu_k(\sigma)_{c1\ldots1,p} = \mu_k(\sigma)_{a1\ldots1p,q} = 1$,
- for every other combination $(w, r) \in Q^k \times Q$, we let $\mu_k(\sigma)_{w,r} = 0$,
- for every other input symbol $\delta \in \Sigma^{(k)}$, $k \geq 0$, and state behaviour $(w, r) \in Q^k \times Q$ we can define $\mu_k(\delta)_{w,r}$ arbitrarily.

Take $\xi = \sigma(\underbrace{\alpha, \ldots \alpha}_{k-1}, \sigma(\underbrace{\alpha, \ldots, \alpha}_{k})) \in T_\Sigma$. First we calculate

$$\begin{aligned}
h_\mu(\sigma(\underbrace{\alpha, \ldots, \alpha}_{k}))_p &= \sum_{u_1,\ldots,u_k \in Q} h_\mu(\alpha)_{u_1} \cdot h_\mu(\alpha)_{u_2} \cdot \ldots \cdot h_\mu(\alpha)_{u_{k-1}} \cdot h_\mu(\alpha)_{u_k} \cdot \mu_k(\sigma)_{u_1 u_2 \ldots u_{k-1} u_k, p} \\
&= h_\mu(\alpha)_b \cdot h_\mu(\alpha)_1 \cdot \ldots \cdot h_\mu(\alpha)_1 \cdot \mu_k(\sigma)_{b1\ldots1,p} + \\
&\qquad + h_\mu(\alpha)_c \cdot h_\mu(\alpha)_1 \cdot \ldots \cdot h_\mu(\alpha)_1 \cdot \mu_k(\sigma)_{c1\ldots1,p} \\
&= b + c.
\end{aligned}$$

Then,

$$\begin{aligned}
h_\mu\big(\sigma(\underbrace{\alpha, \ldots \alpha}_{k-1}, \sigma(\underbrace{\alpha, \ldots, \alpha}_{k}))\big)_q &= h_\mu(\alpha)_a \cdot h_\mu(\alpha)_1 \cdot \ldots \cdot h_\mu(\alpha)_1 \cdot h_\mu(\sigma(\alpha, \ldots, \alpha))_p \cdot \mu_k(\sigma)_{a1\ldots1p,q} \\
&= a \cdot (b + c).
\end{aligned}$$

Let $\kappa_b \in \mathcal{R}_\mathcal{A}(\xi)$ be such that $\kappa_b(\varepsilon) = q$, $\kappa_b(1) = a$, $\kappa_b(i) = 1$ for every $i \in \{2, \ldots, k-1\}$, $\kappa_b(k) = p$, $\kappa_b(k1) = b$, $\kappa_b(kj) = 1$ for every $j \in \{2, \ldots, k\}$, and $\kappa_c \in \mathcal{R}_\mathcal{A}(\xi)$ is the same as κ_b but $\kappa_c(k1) = c$. Then

$$(r^{run}_\mathcal{A}, \xi) = \sum_{\kappa \in \mathcal{R}_\mathcal{A}(\xi)} \mathrm{wt}(\kappa) \cdot \nu_{\kappa(\varepsilon)} = \sum_{\substack{\kappa \in \mathcal{R}_\mathcal{A}(\xi) \\ \kappa(\varepsilon) = q}} \mathrm{wt}(\kappa) = \mathrm{wt}(\kappa_b) + \mathrm{wt}(\kappa_c) = a \cdot b + a \cdot c.$$

Since $r^{run}_\mathcal{A} = r_\mathcal{A}$, we obtain $a \cdot (b + c) = a \cdot b + a \cdot c$. □

5 Closure properties

First we prove that $\mathrm{Rec}(\Sigma, S)$ and bud-$\mathrm{Rec}(\Sigma, S)$ are closed under sum. The construction is the straightforward "union" of the two given wta (cf., e.g., Lemma 6.4 of [4]).

Lemma 5.1. Let $r_1, r_2 \in S\langle\langle T_\Sigma \rangle\rangle$. Then the following statements hold:

1. If $r_1, r_2 \in \mathrm{Rec}(\Sigma, S)$, then $r_1 + r_2 \in \mathrm{Rec}(\Sigma, S)$.
2. If $r_1, r_2 \in$ bud-$\mathrm{Rec}(\Sigma, S)$, then $r_1 + r_2 \in$ bud-$\mathrm{Rec}(\Sigma, S)$.

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma, S, \mu^1, \nu^1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, S, \mu^2, \nu^2)$ be the wta such that $r_1 = r_{\mathcal{A}_1}$ and $r_2 = r_{\mathcal{A}_2}$. Clearly, we can choose Q_1 and Q_2 such that $Q_1 \cap Q_2 = \emptyset$. We construct the wta $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ with $Q = Q_1 \cup Q_2$. For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, q \in Q$, we define the mappings μ and ν as follows:

$$\mu_k(\sigma)_{q_1 \ldots q_k, q} = \begin{cases} \mu^1_k(\sigma)_{q_1 \ldots q_k, q}, & \text{if } q_1, \ldots q_k, q \in Q_1, \\ \mu^2_k(\sigma)_{q_1 \ldots q_k, q}, & \text{if } q_1, \ldots q_k, q \in Q_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_q = \begin{cases} \nu^1_q & \text{if } q \in Q_1, \\ \nu^2_q & \text{if } q \in Q_2. \end{cases}$$

Then, for every $\xi \in T_\Sigma$, we have $h_\mu(\xi)_q = h_{\mu^1}(\xi)_q$ if $q \in Q_1$, and $h_\mu(\xi)_q = h_{\mu^2}(\xi)_q$ if $q \in Q_2$. Thus,

$$\begin{aligned} (r_{\mathcal{A}}, \xi) = & \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q = \sum_{q \in Q_1} h_\mu(\xi)_q \cdot \nu_q + \sum_{q \in Q_2} h_\mu(\xi)_q \cdot \nu_q \\ = & \sum_{q \in Q_1} h_{\mu^1}(\xi)_q \cdot \nu^1_q + \sum_{q \in Q_2} h_{\mu^2}(\xi)_q \cdot \nu^2_q = (r_{\mathcal{A}'}, \xi) + (r_{\mathcal{A}''}, \xi) \\ = & (r_1, \xi) + (r_2, \xi) = (r_1 + r_2, \xi). \end{aligned}$$

Hence, $r_1 + r_2$ is recognizable.

If the wta $\mathcal{A}_1$ and $\mathcal{A}_2$ are bu-deterministic, then the wta $\mathcal{A}$ is bu-deterministic, which proves Statement 2. $\square$

In the following lemma we recall that $\mathrm{Rec}(\Sigma, S)$ and bud-$\mathrm{Rec}(\Sigma, S)$ are closed under Hadamard product if S is a commutative semiring, which has been proved in Corollary 3.9 of [3].

Lemma 5.2. (Corollary 3.9 of [3]) Let S be a commutative semiring, and $r_1, r_2 \in S\langle\langle T_\Sigma \rangle\rangle$. Then the following statements hold:

1. If $r_1, r_2 \in \mathrm{Rec}(\Sigma, S)$, then $r_1 \odot r_2 \in \mathrm{Rec}(\Sigma, S)$.
2. If $r_1, r_2 \in$ bud-$\mathrm{Rec}(\Sigma, S)$, then $r_1 \odot r_2 \in$ bud-$\mathrm{Rec}(\Sigma, S)$.

The next lemma generalizes Lemma 5.2(2) from commutative semirings to commutative s-bimonoids.

Lemma 5.3. Let S be commutative and $r_1, r_2 \in$ bud-$\mathrm{Rec}(\Sigma, S)$. Then $r_1 \odot r_2 \in$ bud-$\mathrm{Rec}(\Sigma, S)$.

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma, S, \mu^1, \nu^1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, S, \mu^2, \nu^2)$ be the bu-deterministic wta such that $r_1 = r_{\mathcal{A}_1}$ and $r_2 = r_{\mathcal{A}_2}$. We construct the wta $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ with $Q = Q_1 \times Q_2$. We define $\nu_{(q^1, q^2)} = \nu^1_{q^1} \cdot \nu^2_{q^2}$ and $\mu_k(\sigma)_{(q^1_1, q^2_1) \ldots (q^1_k, q^2_k), (q^1, q^2)} = \mu^1_k(\sigma)_{q^1_1 \ldots q^1_k, q^1} \cdot \mu^2_k(\sigma)_{q^2_1 \ldots q^2_k, q^2}$ for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $q^1_1, \ldots, q^1_k, q^1 \in Q_1$, and $q^2_1, \ldots, q^2_k, q^2 \in Q_2$.

Clearly, a wta $\mathcal{A}$ is bu-deterministic.

We show by structural induction that the following statement holds:

$$(**) \qquad h_\mu(\xi)_{(q^1, q^2)} = h_{\mu^1}(\sigma)_{q^1} \cdot h_{\mu^2}(\sigma)_{q^2} \quad \text{for every } \xi \in T_\Sigma,\ q^1 \in Q_1 \text{ and } q^2 \in Q_2.$$

Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $\xi_1, \ldots, \xi_k \in T_\Sigma$ and $\xi = \sigma(\xi_1, \ldots, \xi_k)$. Then

$$
\begin{aligned}
h_\mu(\xi)_{(q^1,q^2)} = & \sum_{\substack{q_1^1,\ldots,q_k^1\in Q_1\\ q_1^2,\ldots,q_k^2\in Q_2}} h_\mu(\xi_1)_{(q_1^1,q_1^2)}\cdot\ldots\cdot h_\mu(\xi_k)_{(q_k^1,q_k^2)}\cdot \mu_k(\sigma)_{(q_1^1,q_1^2)\ldots(q_k^1,q_k^2),(q^1,q^2)}\\
= & \sum_{\substack{q_1^1,\ldots,q_k^1\in Q_1\\ q_1^2,\ldots,q_k^2\in Q_2}} \big(h_{\mu^1}(\xi_1)_{q_1^1}\cdot h_{\mu^2}(\xi_1)_{q_1^2}\big)\cdot\ldots\cdot\big(h_{\mu^1}(\xi_k)_{q_k^1}\cdot h_{\mu^2}(\xi_k)_{q_k^2}\big)\cdot\\
& \qquad\cdot\big(\mu_k^1(\sigma)_{q_1^1\ldots q_k^1,q^1}\cdot \mu_k^2(\sigma)_{q_1^2\ldots q_k^2,q^2}\big)\\
& \qquad\qquad (\textit{induction hypothesis and definition of } \mu)\\
= & \sum_{q_1^1,\ldots,q_k^1\in Q_1}\sum_{q_1^2,\ldots,q_k^2\in Q_2}\big(h_{\mu^1}(\xi_1)_{q_1^1}\cdot\ldots\cdot h_{\mu^1}(\xi_k)_{q_k^1}\cdot\mu_k^1(\sigma)_{q_1^1\ldots q_k^1,q^1}\big)\cdot\\
& \qquad\cdot\big(h_{\mu^2}(\xi_1)_{q_1^2}\cdot\ldots\cdot h_{\mu^2}(\xi_k)_{q_k^2}\cdot\mu_k^2(\sigma)_{q_1^2\ldots q_k^2,q^2}\big)\\
& \qquad\qquad (\textit{commutativity})
\end{aligned}
$$

There are two possibilities:

1. there is an i with $1\le i\le k$ such that $h_\mu(\xi)_{q^1}=0$ for every $q^1\in Q_1$ or $h_\mu(\xi)_{q^2}=0$ for every $q^2\in Q_2$,
2. for every i with $1\le i\le k$ there are exactly one $p_i^1\in Q_1$ and exactly one $p_i^2\in Q_2$ such that $h_{\mu^1}(\xi_i)_{p_i^1}\neq 0$ and $h_{\mu^2}(\xi_i)_{p_i^2}\neq 0$.

Case 1. Let an $i\in\{1,\ldots,k\}$ be such that $h_\mu(\xi)_{q^1}=0$ for every $q^1\in Q_1$ (the proof is analogous if $h_\mu(\xi)_{q^2}=0$ for every $q^2\in Q_2$). Then we continue as follows:

$$
\begin{aligned}
= & \sum_{q_1^1,\ldots,q_k^1\in Q_1}\sum_{q_1^2,\ldots,q_k^2\in Q_2}\big(h_{\mu^1}(\xi_1)_{q_1^1}\cdot\ldots\cdot h_{\mu^1}(\xi_{i-1})_{q_{i-1}^1}\cdot 0\cdot\ldots\cdot h_{\mu^1}(\xi_k)_{q_k^1}\cdot\mu_k^1(\sigma)_{q_1^1\ldots q_k^1,q^1}\big)\cdot\\
& \qquad\cdot\big(h_{\mu^2}(\xi_1)_{q_1^2}\cdot\ldots\cdot h_{\mu^2}(\xi_k)_{q_k^2}\cdot\mu_k^2(\sigma)_{q_1^2\ldots q_k^2,q^2}\big)\\
= & \ 0 = \sum_{q_1^1,\ldots,q_k^1\in Q_1}\big(h_{\mu^1}(\xi_1)_{q_1^1}\cdot\ldots\cdot h_{\mu^1}(\xi_{i-1})_{q_{i-1}^1}\cdot 0\cdot\ldots\cdot h_{\mu^1}(\xi_k)_{q_k^1}\cdot\mu_k^1(\sigma)_{q_1^1\ldots q_k^1,q^1}\big)\cdot\\
& \qquad\cdot\sum_{q_1^2,\ldots,q_k^2\in Q_2}\big(h_{\mu^2}(\xi_1)_{q_1^2}\cdot\ldots\cdot h_{\mu^2}(\xi_k)_{q_k^2}\cdot\mu_k^2(\sigma)_{q_1^2\ldots q_k^2,q^2}\big)\\
= & \ h_{\mu^1}(\sigma)_{q^1}\cdot h_{\mu^2}(\sigma)_{q^2}.
\end{aligned}
$$

Case 2. We can continue as follows:

$$
\begin{aligned}
= & \ \big(h_{\mu^1}(\xi_1)_{p_1^1}\cdot\ldots\cdot h_{\mu^1}(\xi_k)_{p_k^1}\cdot\mu_k^1(\sigma)_{p_1^1\ldots p_k^1,p^1}\big)\cdot\\
& \qquad\cdot\big(h_{\mu^2}(\xi_1)_{p_1^2}\cdot\ldots\cdot h_{\mu^2}(\xi_k)_{p_k^2}\cdot\mu_k^2(\sigma)_{p_1^2\ldots p_k^2,p^2}\big)\\
= & \sum_{q_1^1,\ldots,q_k^1\in Q_1}\big(h_{\mu^1}(\xi_1)_{q_1^1}\cdot\ldots\cdot h_{\mu^1}(\xi_k)_{q_k^1}\cdot\mu_k^1(\sigma)_{q_1^1\ldots q_k^1,q^1}\big)\cdot\\
& \qquad\cdot\sum_{q_1^2,\ldots,q_k^2\in Q_2}\big(h_{\mu^2}(\xi_1)_{q_1^2}\cdot\ldots\cdot h_{\mu^2}(\xi_k)_{q_k^2}\cdot\mu_k^2(\sigma)_{q_1^2\ldots q_k^2,q^2}\big)\\
= & \ h_{\mu^1}(\sigma)_{q^1}\cdot h_{\mu^2}(\sigma)_{q^2}.
\end{aligned}
$$

This finishes the proof of Statement $(**)$.

Now take $\xi\in T_\Sigma$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are bu-deterministic, it is possible that $h_{\mu^1}(\xi)_p=0$ for every $p\in Q_1$ or $h_{\mu^2}(\xi)_q=0$ for every $q\in Q_2$, or, the second possibility is that there exists the unique $u\in Q_1$ such that $h_{\mu^1}(\xi)_u\neq 0$ and there exists the unique $v\in Q_2$ such that $h_{\mu^2}(\xi)_v\neq 0$.

In the first case, it follows from Statement (**) that $h_\mu(\xi)_{(q^1,q^2)} = 0$ for every $q^1 \in Q_1$ and $q^2 \in Q_2$. Then, one can easily check that $(r_{\mathcal{A}}, \xi) = 0 = (r_{\mathcal{A}_1}, \xi) \cdot (r_{\mathcal{A}_2}, \xi)$.

In the second case let $u \in Q_1$ (and $v \in Q_2$) be the unique state such that $h_{\mu^1}(\xi)_u \neq 0$ (respectively, $h_{\mu^2}(\xi)_v \neq 0$). Then we have that

$$\begin{aligned}
(r_{\mathcal{A}}, \xi) &= \sum_{\substack{q^1 \in Q_1 \\ q^2 \in Q_2}} h_\mu(\xi)_{(q^1,q^2)} \cdot \nu_{(q^1,q^2)} = \sum_{\substack{q^1 \in Q_1 \\ q^2 \in Q_2}} \big(h_{\mu^1}(\xi)_{q^1} \cdot h_{\mu^2}(\xi)_{q^2}\big) \cdot \big(\nu_{q^1} \cdot \nu_{q^2}\big) \\
&= h_{\mu^1}(\xi)_u \cdot \nu_u \cdot h_{\mu^2}(\xi)_v \cdot \nu_v = \sum_{q^1 \in Q_1} h_{\mu^1}(\xi)_{q^1} \cdot \nu_{q^1} \cdot \sum_{q^2 \in Q_2} h_{\mu^2}(\xi)_{q^2} \cdot \nu_{q^2} \\
&= (r_{\mathcal{A}_1}, \xi) \cdot (r_{\mathcal{A}_2}, \xi).
\end{aligned}$$

□

In Lemma 6.3 of [4] it was proved that the class of recognizable tree series over semirings is closed under left multiplication with a coefficient from the semiring. Here we deal with multiplication from left and right.

Theorem 5.4. Let $r \in \mathrm{Rec}(\Sigma, S)$ and $a \in S$. Then the following statements hold:

1. If $r \in \text{bud-Rec}(\Sigma, S)$, then $r \cdot a \in \text{bud-Rec}(\Sigma, S)$.
2. If S is right distributive, then $r \cdot a \in \mathrm{Rec}(\Sigma, S)$.
3. If S is left distributive or $r \in \text{bud-Rec}(\Sigma, S)$, then $a \cdot r \in \mathrm{Rec}(\Sigma, S)$.

Proof. Let $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ be some wta such that $r = r_{\mathcal{A}}$. We consider the wta $\mathcal{A}' = (Q, \Sigma, S, \mu, \nu')$ with $\nu'_q = \nu_q \cdot a$ for every $q \in Q$.

Proof of Statement 1: We assume that $\mathcal{A}$ is bu-deterministic. Then, clearly, also $\mathcal{A}'$ is bu-deterministic. Let $\xi \in T_\Sigma$. By Observation 3.3(1), there is at most one $q \in Q$ such that $h_\mu(\xi)_q \neq 0$. If $h_\mu(\xi)_q = 0$ for every $q \in Q$, then $(r_{\mathcal{A}'}, \xi) = \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu'_q = 0 = \big(\sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q\big) \cdot a = (r_{\mathcal{A}}, \xi) \cdot a$. Now, let $u \in Q$ be such that $h_\mu(\xi)_u \neq 0$. Then, $(r_{\mathcal{A}'}, \xi) = h_\mu(\xi)_u \cdot \nu'_u = h_\mu(\xi)_u \cdot \nu_u \cdot a = (r_{\mathcal{A}}, \xi) \cdot a = (r_{\mathcal{A}} \cdot a, \xi) = (r \cdot a, \xi)$. Thus, $r \cdot a$ is bu-deterministically recognizable.

Proof of Statement 2: We assume that S is right distributive. Thus, for every $\xi \in T_\Sigma$, we have $(r \cdot a, \xi) = \big(\sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q\big) \cdot a = \sum_{q \in Q} (h_\mu(\xi)_q \cdot \nu_q \cdot a) = \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu'_q = (r_{\mathcal{A}'}, \xi)$. Hence, $r \cdot a \in \mathrm{Rec}(\Sigma)$.

Proof of Statement 3: We define the wta $\tilde{\mathcal{A}} = (\tilde{Q}, \Sigma, S, \tilde{\mu}, \tilde{\nu})$ as follows:

- $\tilde{Q} = Q_0 \cup Q_1$ where $Q_0 = \{q^0 \mid q \in Q\}$ and $Q_1 = \{q^1 \mid q \in Q\}$,
- for every $q \in Q$ and $\alpha \in \Sigma^{(0)}$, we let $\tilde{\mu}(\alpha)_{\varepsilon, q^0} = \mu_0(\alpha)_{\varepsilon, q}$ and $\tilde{\mu}(\alpha)_{\varepsilon, q^1} = a \cdot \mu_0(\alpha)_{\varepsilon, q}$
- for every $k \geq 1$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, q \in Q$, we let $\tilde{\mu}_k(\sigma)_{q_1^1 q_2^0 \ldots q_k^0, q^1} = \tilde{\mu}_k(\sigma)_{q_1^0 q_2^0 \ldots q_k^0, q^0} = \mu_k(\sigma)_{q_1 q_2 \ldots q_k, q}$, and for every $s_1, \ldots, s_k, s \in \{0, 1\}$ such that $(s_1, \ldots, s_k, s) \notin \{(0, \ldots, 0, 0), (1, 0 \ldots, 0, 1)\}$, we let $\tilde{\mu}_k(\sigma)_{q_1^{s_1} \ldots q_k^{s_k}, q^s} = 0$,
- for every $q \in Q$, we let $\tilde{\nu}_{q^0} = 0$ and $\tilde{\nu}_{q^1} = \nu_q$.

First we prove by structural induction that the following statement holds:

$$(***) \qquad h_{\tilde{\mu}}(\xi)_{q^0} = h_\mu(\xi)_q \quad \text{for every } \xi \in T_\Sigma \text{ and } q \in Q.$$

Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $\xi_1, \ldots, \xi_k \in T_\Sigma$ and $\xi = \sigma(\xi_1, \ldots, \xi_k)$. Then

$$\begin{aligned}
h_\mu(\xi)_{q^0} &= \sum_{\tilde{q}_1,\ldots,\tilde{q}_k \in \tilde{Q}} h_{\tilde{\mu}}(\xi_1)_{\tilde{q}_1} \cdot \ldots \cdot h_{\tilde{\mu}}(\xi_k)_{\tilde{q}_k} \cdot \tilde{\mu}_k(\sigma)_{\tilde{q}_1\ldots\tilde{q}_k,q^0} \\
&= \sum_{\substack{q_1,\ldots,q_k \in Q \\ s_1,\ldots,s_k \in \{0,1\}}} h_{\tilde{\mu}}(\xi_1)_{q_1^{s_1}} \cdot \ldots \cdot h_{\tilde{\mu}}(\xi_k)_{q_k^{s_k}} \cdot \tilde{\mu}_k(\sigma)_{q_1^{s_1}\ldots q_k^{s_k},q^0} \\
&= \sum_{q_1,\ldots,q_k \in Q} h_{\tilde{\mu}}(\xi_1)_{q_1^0} \cdot \ldots \cdot h_{\tilde{\mu}}(\xi_k)_{q_k^0} \cdot \tilde{\mu}_k(\sigma)_{q_1^0\ldots q_k^0,q^0} \\
&\qquad\qquad (\textit{definition of } \tilde{\mu}\ ) \\
&= \sum_{q_1,\ldots,q_k \in Q} h_\mu(\xi_1)_{q_1} \cdot \ldots \cdot h_\mu(\xi_k)_{q_k} \cdot \mu_k(\sigma)_{q_1\ldots q_k,q} \\
&\qquad (\textit{induction hypothesis and definition of } \tilde{\mu}_k(\sigma)) \\
&= h_\mu(\xi)_q.
\end{aligned}$$

Now we show that

$$(*v) \qquad h_{\tilde{\mu}}(\xi)_{q^1} = a \cdot h_\mu(\xi)_q \quad \text{for every } \xi \in T_\Sigma \text{ and } q \in Q.$$

Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $\xi_1, \ldots, \xi_k \in T_\Sigma$ and $\xi = \sigma(\xi_1, \ldots, \xi_k)$. Then,

$$\begin{aligned}
h_{\tilde{\mu}}(\xi)_{q^1} &= \sum_{\tilde{q}_1,\ldots,\tilde{q}_k \in \tilde{Q}} h_{\tilde{\mu}}(\xi_1)_{\tilde{q}_1} \cdot \ldots \cdot h_{\tilde{\mu}}(\xi_k)_{\tilde{q}_k} \cdot \tilde{\mu}_k(\sigma)_{\tilde{q}_1\ldots\tilde{q}_k,q^1} \\
&= \sum_{\substack{q_1,\ldots,q_k \in Q \\ s_1,\ldots,s_k \in \{0,1\}}} h_{\tilde{\mu}}(\xi_1)_{q_1^{s_1}} \cdot \ldots \cdot h_{\tilde{\mu}}(\xi_k)_{q_k^{s_k}} \cdot \tilde{\mu}_k(\sigma)_{q_1^{s_1}\ldots q_k^{s_k},q^1} \\
&= \sum_{q_1,\ldots,q_k \in Q} h_{\tilde{\mu}}(\xi_1)_{q_1^1} \cdot h_{\tilde{\mu}}(\xi_2)_{q_2^0} \cdot \ldots \cdot h_{\tilde{\mu}}(\xi_k)_{q_k^0} \cdot \tilde{\mu}_k(\sigma)_{q_1^1 q_2^0\ldots q_k^0,q^1} \\
&= \sum_{q_1,\ldots,q_k \in Q} a \cdot h_\mu(\xi_1)_{q_1} \cdot h_\mu(\xi_2)_{q_2} \cdot \ldots \cdot h_\mu(\xi)_{q_k} \cdot \mu_k(\sigma)_{q_1\ldots q_k,q} \\
&\quad (\textit{induction hypothesis, property of } h \textit{ for } q_2^0, \ldots q_k^0, \textit{ and definition of } \tilde{\mu}_k(\sigma)).
\end{aligned}$$

If S is left distributive, then we can continue with

$$\begin{aligned}
&= a \cdot \sum_{q_1,\ldots,q_k \in Q} h_\mu(\xi_1)_{q_1} \cdot \ldots \cdot h_\mu(\xi_k)_{q_k} \cdot \mu_k(\sigma)_{q_1\ldots q_k,q} \\
&= a \cdot h_\mu(\xi)_q,
\end{aligned}$$

and we have proved Statement $(*v)$ if S is left distributive.

Now assume that $\mathcal{A}$ is bu-determinisic. Then there are two possibilities:

1. there is an i with $1 \leq i \leq k$ and $h_\mu(\xi_i)_q = 0$ for every $q \in Q$,
2. for every i with $1 \leq i \leq k$ there is exactly one $q_i \in Q$ with $h_\mu(\xi_i) \neq 0$.

Case 1. Then we can continue with

$$
\begin{aligned}
&= \sum_{q_1,\dots,q_k \in Q} a \cdot h_\mu(\xi_1)_{q_1} \cdot \ldots \cdot h_\mu(\xi_{i-1})_{q_{i-1}} \cdot 0 \cdot \ldots \cdot h_\mu(\xi_k)_{q_k} \cdot \mu_k(\sigma)_{q_1\dots q_k,q} \\
&= 0 = a \cdot \sum_{q_1,\dots,q_k \in Q} h_\mu(\xi_1)_{q_1} \cdot \ldots \cdot h_\mu(\xi_{i-1})_{q_{i-1}} \cdot 0 \cdot \ldots \cdot h_\mu(\xi_k)_{q_k} \cdot \mu_k(\sigma)_{q_1\dots q_k,q} \\
&= a \cdot h_\mu(\xi)_q.
\end{aligned}
$$

Case 2. Let $p_1, \dots, p_k \in Q$ be the unique states such that $h_\mu(\xi_i)_{p_i} \neq 0$. Then we can continue with

$$
\begin{aligned}
&= a \cdot h_\mu(\xi_1)_{p_1} \cdot h_\mu(\xi_2)_{p_2} \cdot \ldots \cdot h_\mu(\xi_k)_{p_k} \cdot \mu_k(\sigma)_{p_1\dots p_k,q} \\
&= a \cdot \sum_{q_1,\dots,q_k \in Q} h_\mu(\xi_1)_{q_1} \cdot \ldots \cdot h_\mu(\xi_k)_{q_k} \cdot \mu_k(\sigma)_{q_1\dots q_k,q} \\
&= a \cdot h_\mu(\xi)_q
\end{aligned}
$$

and we have proved Statement $(*v)$ if $\mathcal{A}$ is bu-deterministic. Hence Statement $(*v)$ holds.

Now we prove that $r_{\tilde{\mathcal{A}}} = a \cdot r_{\mathcal{A}}$. For this let $\xi \in T_\Sigma$. Then

$$
\begin{aligned}
(r_{\tilde{\mathcal{A}}}, \xi) &= \sum_{\tilde{q} \in \tilde{Q}} h_{\tilde{\mu}}(\xi)_{\tilde{q}} \cdot \tilde{\nu}_{\tilde{q}} = \sum_{q \in Q} h_{\tilde{\mu}}(\xi)_{q^1} \cdot \nu_q \\
&= \sum_{q \in Q} a \cdot h_\mu(\xi)_q \cdot \nu_q
\end{aligned}
$$

If S is left distributive, then we can continue as follows:

$$
= a \cdot \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q = a \cdot (r_{\mathcal{A}}, \xi) = (a \cdot r_{\mathcal{A}}, \xi).
$$

Now let $\mathcal{A}$ be bu-deterministic. Thus there are two cases:

1. $h_\mu(\xi)_q = 0$ for every $q \in Q$,
2. there is the unique $q \in Q$ such that $h_\mu(\xi)_q \neq 0$.

Case 1. Then we can continue as follows:

$$
= 0 = a \cdot 0 = a \cdot (r_{\mathcal{A}}, \xi) = (a \cdot r_{\mathcal{A}}, \xi).
$$

Case 2. Let $p \in Q$ be such that $h_\mu(\xi)_p \neq 0$. Then we can continue as follows:

$$
\begin{aligned}
&= a \cdot h_\mu(\xi)_p \cdot \nu_p = a \cdot \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q \\
&= a \cdot (r_{\mathcal{A}}, \xi) = (a \cdot r_{\mathcal{A}}, \xi).
\end{aligned}
$$

Hence, $a \cdot r$ is recognizable. □

6 Determinization

In this section we deal with the question under which conditions a recognizable tree series can be recognized by a bu-deterministic wta. We follow the approach of Section 3.4 of [9] (also cf. [5]). First, we prove that S is locally finite if and only if every wta over S and Σ can be represented by a finite Σ-algebra. The only-if direction is a generalization of Sec. 4 of [3] (also cf. Sec. 3.1 of [11] and Lemma 3.14 of [9]).

Lemma 6.1. The following statements are equivalent.

1. S is locally finite.

2. For every ranked alphabet Σ and every wta $\mathcal{A}$ over Σ and S there is a finite Σ-algebra (Q, θ) and a mapping $f : Q \to S$ such that $r_{\mathcal{A}} = f \circ h_Q$.

In particular, for every bu-fta $\mathcal{A}$ over Σ there is a finite Σ-algebra (Q, θ) and a subset $F \subseteq Q$ such that $\mathcal{L}(\mathcal{A}) = h_Q^{-1}(F)$.

Proof. 1. $\Rightarrow$ 2. Let Σ be a ranked alphabet and $\mathcal{A} = (P, \Sigma, S, \mu, \nu)$ be a wta over Σ and S. Moreover, let S' be the smallest sub-s-bimonoid containing $\mathrm{wts}(\mathcal{A})$. Then S' is finite, because S is locally finite. Clearly, $(S')^P \subseteq S^P$ and, by definition, $\mu_{\mathcal{A}}(\sigma) : (S^P)^k \to S^P$ for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$.

Now, let $v_1, \ldots, v_k \in (S')^P$. Then $\mu_{\mathcal{A}}(\sigma)(v_1, \ldots, v_k) \in (S')^P$, because S' is closed under addition and multiplication. Thus, we can define the operation $\mu'_{\mathcal{A}}(\sigma) : (S')^P \times \ldots \times (S')^P \to (S')^P$ by letting $\mu_{\mathcal{A}}(\sigma) = \mu'_{\mathcal{A}}(\sigma)$ for every $v_1, \ldots, v_k \in (S')^P$.

Let $Q = (S')^P$. Then, $(Q, \mu'_{\mathcal{A}})$ is a Σ-algebra and $h_\mu(\xi) = h_Q(\xi)$ for every $\xi \in T_\Sigma$. We define the mapping f by $f(v) = v \cdot \nu$ for $v \in Q$. Then $(r_{\mathcal{A}}, \xi) = h_\mu(\xi) \cdot \nu = h_Q(\xi) \cdot \nu = f(h_Q(\xi)) = (f \circ h_Q)(\xi)$. Thus, $r_{\mathcal{A}} = f \circ h_Q$.

2. $\Rightarrow$ 1. We prove this implication by contraposition: If S is not locally finite, then there is a ranked alphabet Σ and a wta $\mathcal{A}$ over Σ and S such that $r_{\mathcal{A}} \neq f \circ h_Q$ for every finite Σ-algebra (Q, θ) and every mapping $f : Q \to S$.

Since S is not locally finite, there is a set $A \subseteq S$ such that the set S', the smallest sub-s-bimonoid of S containing A, is not finite. We let $\tilde{\Sigma} = \{a^{(0)} \mid a \in A\} \cup \{\oplus^{(2)}, \odot^{(2)}\}$, and we define the mapping $\mathrm{val} : T_{\tilde{\Sigma}} \to S$ as follows: $\mathrm{val}(a^{(0)}) = a$ for every $a \in A$; $\mathrm{val}(\oplus(\xi_1, \xi_2)) = \mathrm{val}(\xi_1) + \mathrm{val}(\xi_2)$ and $\mathrm{val}(\odot(\xi_1, \xi_2)) = \mathrm{val}(\xi_1) \cdot \mathrm{val}(\xi_2)$ for every $\xi_1, \xi_2 \in T_{\tilde{\Sigma}}$.

Now we construct the wta $\mathcal{A} = (Q, \tilde{\Sigma}, S, \mu, \nu)$ with $Q = \{v, 1\}$ as follows: $\nu_v = 1$ and $\nu_1 = 0$, $\mu_0(a)_{\varepsilon,v} = a$ and $\mu_0(a)_{\varepsilon,1} = 1$ for every $a \in A$. Moreover, for every $p, q, r \in Q$ we let

$$\mu_2(\oplus)_{pq,r} = \begin{cases} 1, & \text{if } (pq, r) \in \{(11, 1), (v1, v)\}, (1v, v)\}; \\ 0, & \text{otherwise} \end{cases},$$

$$\mu_2(\odot)_{pq,r} = \begin{cases} 1, & \text{if } (pq, r) \in \{(11, 1), (vv, v)\}; \\ 0, & \text{otherwise} \end{cases}.$$

We show by structural induction that $h_\mu(\xi)_v = \mathrm{val}(\xi)$ and $h_\mu(\xi)_1 = 1$ for every $\xi \in T_{\tilde{\Sigma}}$. Take $a \in \tilde{\Sigma}^{(0)}$, we have $h_\mu(a)_v = \mu_0(a)_{\varepsilon,v} = a = \mathrm{val}(a)$ and $h_\mu(a)_1 = \mu_0(a)_{\varepsilon,1} = 1$. Moreover,

$$\begin{aligned} h_\mu(\oplus(\xi_1, \xi_2))_v &= \sum_{p,q \in Q} h_\mu(\xi_1)_p \cdot h_\mu(\xi_2)_q \cdot \mu_2(\oplus)_{pq,v} \\ &= h_\mu(\xi_1)_v \cdot h_\mu(\xi_2)_1 \cdot \mu_2(\oplus)_{v1,v} + h_\mu(\xi_1)_1 \cdot h_\mu(\xi_2)_v \cdot \mu_2(\oplus)_{1v,v} \\ &= \mathrm{val}(\xi_1) \cdot 1 \cdot 1 + 1 \cdot \mathrm{val}(\xi_2) \cdot 1 = \mathrm{val}(\xi_1) + \mathrm{val}(\xi_2) = \mathrm{val}(\oplus(\xi_1, \xi_2)), \\ h_\mu(\oplus(\xi_1, \xi_2))_1 &= \sum_{p,q \in Q} h_\mu(\xi_1)_p \cdot h_\mu(\xi_2)_q \cdot \mu_2(\oplus)_{pq,1} \\ &= h_\mu(\xi_1)_1 \cdot h_\mu(\xi_2)_1 \cdot \mu_2(\oplus)_{11,1} = 1, \end{aligned}$$

$$\begin{aligned}
h_\mu(\odot(\xi_1,\xi_2))_v = & \sum_{p,q\in Q} h_\mu(\xi_1)_p \cdot h_\mu(\xi_2)_q \cdot \mu_2(\odot)_{pq,v} \\
= & \ h_\mu(\xi_1)_v \cdot h_\mu(\xi_2)_v \cdot \mu_2(\odot)_{vv,v} = \mathrm{val}(\xi_1)\cdot \mathrm{val}(\xi_2) = \mathrm{val}(\odot(\xi_1,\xi_2)), \\
h_\mu(\odot(\xi_1,\xi_2))_1 = & \sum_{p,q\in Q} h_\mu(\xi_1)_p \cdot h_\mu(\xi_2)_q \cdot \mu_2(\odot)_{pq,1} \\
= & \ h_\mu(\xi_1)_1 \cdot h_\mu(\xi_2)_1 \cdot \mu_2(\odot)_{11,1} = 1\cdot 1 = 1.
\end{aligned}$$

Thus, $(r_{\mathcal{A}},\xi) = h_\mu(\xi)_v \cdot \nu_v = \mathrm{val}(\xi)$.

It is clear from the definition of the mapping val that for every $s \in S'$ there is some tree $\xi \in T_{\tilde{\Sigma}}$ such that $\mathrm{val}(\xi) = s$. (The elements of set A are the images of the elements of $\tilde{\Sigma}^{(0)}$, and for every $s_1, s_2 \in S'$ we have $s_1 + s_2 = \mathrm{val}(\oplus(\xi_1,\xi_2))$ and $s_1 \cdot s_2 = \mathrm{val}(\odot(\xi_1,\xi_2))$, where $\xi_1,\xi_2 \in T_{\tilde{\Sigma}}$ and $s_1 = \mathrm{val}(\xi_1)$, $s_2 = \mathrm{val}(\xi_2)$.) Hence, for every $s \in S'$, there is $\xi \in T_{\tilde{\Sigma}}$ such that $(r_{\mathcal{A}},\xi) = s$. Thus $S' \subseteq \mathrm{im}(r_{\mathcal{A}})$, and $\mathrm{im}(r_{\mathcal{A}})$ is not finite.

Now take any finite Σ-algebra (Q,θ) and any mapping $f : Q \to S$. Then the set $\mathrm{im}(f \circ h_Q)$ is finite, because Q is finite. Thus, $\mathrm{im}(r_{\mathcal{A}}) \neq \mathrm{im}(f \circ h_Q)$ for every finite Σ-algebra (Q,θ) and mapping $f : Q \to S$. This finishes the proof of 2. $\Rightarrow$ 1.

Now let $\mathcal{A}$ be a bu-fta over Σ. Since $\mathbb{B}$ is locally finite, there is a finite Σ-algebra (Q,θ) and a mapping $f : Q \to \{0,1\}$ such that $r_{\mathcal{A}} = f \circ h_Q$. Denote $f^{-1}(1)$ by F. Then, clearly, $F \subseteq Q$ and $\mathcal{L}(\mathcal{A}) = \mathrm{supp}(r_{\mathcal{A}}) = r_{\mathcal{A}}^{-1}(1) = h_Q^{-1}(f^{-1}(1)) = h_Q^{-1}(F)$. □

The next lemma shows how to implement a finite Σ-algebra by a crisp and bu-deterministic wta (cf. Lemma 3.10 of [9]).

Lemma 6.2. Let (Q,θ) be a finite Σ-algebra.

1. For every s-bimonoid S and mapping $f : Q \to S$ there is a crisp and bu-deterministic wta $\mathcal{A}$ over Σ and S such that $r_{\mathcal{A}} = f \circ h_Q$. Thus, in particular, $f \circ h_Q \in \mathrm{bud\text{-}Rec}(\Sigma, S)$.
2. For every $P \subseteq Q$ the language $h_Q^{-1}(P) \subseteq T_\Sigma$ is recognizable.

Proof. We construct the bu-deterministic wta $\mathcal{A} = (Q,\Sigma,S,\mu,\nu)$ by defining $\mu_k(\sigma)_{q_1\dots q_k,q} = 1$ if $\theta(\sigma)(q_1,\dots,q_k) = q$, and $\mu_k(\sigma)_{q_1\dots q_k,q} = 0$ otherwise, and $\nu_q = f(q)$ for every $\sigma \in \Sigma^{(k)}$, $q_1,\dots,q_k,q \in Q$. Then $h_\mu(\xi)$ is a vector such that $h_\mu(\xi)_p \in \{0,1\}$ for every $p \in Q$. It is easy to check that for every $\xi \in T_\Sigma$ and $q \in Q$ we have $h_\mu(\xi)_q = 1$ if $h_Q(\xi) = q$, and $h_\mu(\xi)_q = 0$ otherwise. (We note that no distributivity law is needed for the proof of this statement.)

Thus, for every $\xi \in T_\Sigma$ we have

$$\begin{aligned}
(r_{\mathcal{A}},\xi) = & \ h_\mu(\xi)\cdot\nu = \sum_{q\in Q} h_\mu(\xi)_q \cdot f(q) \\
= & \ h_\mu(\xi)_{h_Q(\xi)} \cdot f(h_Q(\xi)) = (f\circ h_Q)(\xi) = (f\circ h_Q,\xi).
\end{aligned}$$

Hence, $f \circ h_Q \in \mathrm{bud\text{-}Rec}(\Sigma,S)$.

Next we prove Statement 2. Let $f = 1_{(\mathbb{B},P)}$. Then, by Statement 1, there is a crisp bu-deterministic wta $\mathcal{A} = (Q,\Sigma,\mathbb{B},\mu,\nu)$ such that $r_{\mathcal{A}} = f \circ h_Q$, in particular, $\nu_q = f(q)$ for every $q \in Q$. In fact, $\mathcal{A}$ is the bu-fta (Q,Σ,μ,P) with $P = \{q \in Q \mid \nu_q = 1\}$ and $\mathcal{L}(\mathcal{A}) = \mathrm{supp}(r_{\mathcal{A}}) = \mathrm{supp}(f \circ h_Q) = \mathrm{supp}(1_{(\mathbb{B},P)} \circ h_Q) = \{\xi \in T_\Sigma \mid 1_{(\mathbb{B},P)}(h_Q(\xi)) = 1\} = \{\xi \in T_\Sigma \mid h_Q(\xi) \in P\} = h_Q^{-1}(P)$. Thus, $h_Q^{-1}(P) \subseteq T_\Sigma$ is recognizable. □

The next two theorems generalize Theorems 3.15 and 3.17 of [9] from semirings to s-bimonoids.

Theorem 6.3. Let S be locally finite, $E \subseteq S$, and $r \in \mathrm{Rec}(\Sigma, S)$. Then $r^{-1}(E) \in \mathrm{Rec}(\Sigma)$.

Proof. By Lemma 6.1, there is a finite Σ-algebra (Q, θ) and mapping $f : Q \to S$ such that $r = f \circ h_Q$. Then $r^{-1}(E) = (f \circ h_Q)^{-1}(E) = h_Q^{-1}(f^{-1}(E))$. Since $f^{-1}(E) \subseteq Q$ we obtain by Lemma 6.2 that $r^{-1}(E)$ is recognizable. □

Theorem 6.4. Let S be locally finite, $r \in \mathrm{Rec}(\Sigma, S)$, and $g : S \to S$. Then $g(r) \in$ bud-Rec(Σ, S). In particular, $\mathrm{Rec}(\Sigma, S) =$ bud-Rec(Σ, S).

Proof. By Lemma 6.1, there are a finite Σ-algebra Q and a mapping $f : Q \to S$ such that $r = f \circ h_Q$. Then $g(r) = g \circ (f \circ h_Q) = (g \circ f) \circ h_Q$. Since $g \circ f : Q \to S$ it follows, by Lemma 6.2, that $g(r) \in$ bud-Rec(Σ, S).
Take for g to be the identity mapping. Then for every $r \in \mathrm{Rec}(\Sigma, S)$, we obtain $r = g(r) \in$ bud-Rec(Σ, S), i.e., $\mathrm{Rec}(\Sigma, S) \subseteq$ bud-Rec(Σ, S). Thus, $\mathrm{Rec}(\Sigma, S)$bud-Rec(Σ, S). □

Since the Boolean semiring is locally finite, we obtain from Theorem 6.4 and Lemma 3.4 the well known result that every recognizable tree language can be recogized by a total deterministic bu-fta.

7 Recognizable step functions

Definition 7.1. A tree series $r \in S\langle\langle T_\Sigma \rangle\rangle$ is a *recognizable step function* if there are $n \geq 0$, recognizable tree languages $L_1, \ldots, L_n \subseteq T_\Sigma$, and $a_1, \ldots, a_n \in S$ such that $r = \sum_{i=1}^{n} a_i \cdot 1_{(S,L_i)}$.

It is easy to see that the characteristic tree series of L with respect to S is in bud-Rec(Σ, S) provided that L is a recognizable tree language (cf. Lemma 3.3 of [6] and Lemma 3.11 of [9]).

Lemma 7.2. If $L \subseteq T_\Sigma$ is a recognizable tree language, then $1_{(S,L)} \in$ bud-Rec(Σ, S).

Proof. Let L be a recognizable tree language. Then there is a bu-fta $\mathcal{A} = (Q, \Sigma, \mu, F)$ such that $L = \mathcal{L}(\mathcal{A})$. Clearly, $r_\mathcal{A} = 1_{(\mathbb{B},L)}$. Recall that (Q, Σ, μ, F) abbreviates $(Q, \Sigma, \mathbb{B}, \mu, \nu)$ with $\nu = \chi_F$. Since the Boolean semiring is locally finite, we obtain by Lemma 6.1 that there is a finite Σ-algebra (Q, θ) and a mapping $f : Q \to \mathbb{B}$ such that $r_\mathcal{A} = f \circ h_Q$. We define $g : \mathbb{B} \to S$ by $g(0) = 0$ and $g(1) = 1$. Clearly, $1_{(S,L)} = g \circ 1_{(\mathbb{B},L)}$. Thus, $1_{(S,L)} = g \circ 1_{(\mathbb{B},L)} = g \circ r_\mathcal{A} = (g \circ f) \circ h_Q$. Then, by Lemma 6.2(1) we obtain that $1_{(S,L)} \in$ bud-Rec(Σ, S). □

In the next theorem we will characterize recognizable step funtions (cf. Lemma 10 and Proposition 11 of [5]).

Theorem 7.3. Let $r \in S\langle\langle T_\Sigma \rangle\rangle$. Then the following three statements are equivalent:

1. r is a recognizable step function.

2. There exists a crisp and bu-deterministic wta $\mathcal{A}$ over Σ and S such that $r = r_\mathcal{A}$.

3. The set $\mathrm{im}(r)$ is finite and $r_{=a}$ is a recognizable tree language for every $a \in S$.

In particular, if r is a recognizable step function, then r is bu-deterministically recognizable.

Proof. 1. $\Rightarrow$ 2.: Let $n \in \mathbb{N}$, $L_1, \dots, L_n \subseteq T_\Sigma$, and $a_1, \dots, a_n \in S$ such that $L_1, \dots, L_n$ are recognizable and $r = \sum_{i=1}^n a_i \cdot 1_{(S,L_i)}$. For every $i \in \{1, \dots, n\}$, let $\mathcal{A}_i = (Q_i, \Sigma, \mu^i, F_i)$ be a total deterministic bu-fta over Σ such that $\mathcal{L}(\mathcal{A}_i) = L_i$. Let $Q = Q_1 \times \cdots \times Q_n$, then every $q \in Q$ is of the form $q = (q^1, \dots, q^n)$, where $q^i \in Q_i$ for every $i \in \{1, \dots, n\}$. We define the wta $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ as follows: for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q_1, \dots, q_k, q \in Q$:

$$\mu_k(\sigma)_{q_1 \dots q_k, q} = \begin{cases} 1, & \text{if } \mu_k(\sigma)_{q_1^i \dots q_k^i, q^i} = 1 \text{ for every } i \in \{1, \dots, n\}; \\ 0, & \text{otherwise} \end{cases},$$

$$\nu_q = \sum_{\substack{i \in \{1, \dots, n\} \\ q^i \in F_i}} a_i.$$

Clearly, $\mathcal{A}$ is total, bu-deterministic, and crisp.

One can easily check by structural induction that for every $q \in Q$ and $\xi \in T_\Sigma$ the following holds: $h_\mu(\xi)_q = 1$ iff $h_{\mu^i}(\xi)_{q^i} = 1$ for every $i \in \{1, \dots, n\}$.

Now, let $\xi \in T_\Sigma$. By Observation 3.3(4) there is a unique state, say $q_\xi \in Q$ such that $h_\mu(\xi)_{q_\xi} = 1$. Then $\xi \in L_i$ iff $q_\xi \in F_i$ for every $i \in \{1, \dots, n\}$. Let $I_\xi = \{i \in \{1, \dots, n\} \mid \xi \in L_i\}$. Then:

$$\begin{aligned} (r, \xi) &= \sum_{i \in \{1, \dots, n\}} a_i \cdot \left(1_{(S,L_i)}, \xi\right) = \sum_{i \in I_\xi} a_i = \sum_{\substack{i \in \{1, \dots, n\} \\ q_\xi^i \in F_i}} a_i \\ &= \nu_{q_\xi} = h_\mu(\xi)_{q_\xi} \cdot \nu_{q_\xi} = (r_\mathcal{A}, \xi). \end{aligned}$$

Thus r can be recognized by a total, bu-deterministic, and crisp wta.

2. $\Rightarrow$ 3.: Let $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ be a crisp and bu-deterministic wta over Σ and S such that $r_\mathcal{A} = r$. It follows, by Observation 3.3(3), that $\mathrm{im}(r)$ is finite. Let $a \in \mathrm{im}(r)$, we show that $r_{=a}$ is recognizable. Define the deterministic bu-fta $\mathcal{A}_a = (Q, \Sigma, \mu, F_a)$ by $F_a = \{q \in Q \mid \nu_q = a\}$. Then $r_{=a}$ is recognizable, because $r_{=a} = \mathcal{L}(\mathcal{A}_a)$.

3. $\Rightarrow$ 1.: Since $r = \sum_{a \in \mathrm{im}(r)} a \cdot 1_{(S, r_{=a})}$, it follows by assumption that r is recognizable step function. $\square$

The next lemma generalizes Theorem 13 of [5] from strings to trees.

Lemma 7.4. Let S be locally finite, and r recognizable tree series. Then r is recognizable step function.

Proof. Let $\mathcal{A} = (Q, \Sigma, S, \mu, \nu)$ be a wta such that $r = r_\mathcal{A}$. Let S' be the smallest sub-s-bimonoid containing $\mathrm{wts}(\mathcal{A})$. Since S is locally finite, S' is finite. Then $\mathrm{im}(r) = \mathrm{im}(r_\mathcal{A}) \subseteq S'$ is finite. By Theorem 6.3, for every $a \in S$, the tree language $r_{=a} = r^{-1}(a)$ is recognizable. Hence, by Theorem 7.3, r is recognizable step function. $\square$

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References

[1] A. Alexandrakis and S. Bozapalidis. Weighted grammars and Kleene's theorem. *Inform. Process. Lett.*, 24(1):1–4, 1987.

[2] J. Berstel and C. Reutenauer. Recognizable formal power series on trees. *Theoret. Comput. Sci.*, 18(2):115–148, 1982.

[3] B. Borchardt. A pumping lemma and decidability problems for recognizable tree series. *Acta Cybernet.*, 16(4):509–544, 2004.

[4] M. Droste, Chr. Pech, and H. Vogler. A Kleene theorem for weighted tree automata. *Theory Comput. Syst.*, 38:1–38, 2005.

[5] M. Droste, T. Stüber, and H. Vogler. Weighted automata over bimonoids. submitted for publication, 2008.

[6] M. Droste and H. Vogler. Weighted tree automata and weighted logics. *Theoret. Comput. Sci.*, 366:228–247, 2006.

[7] Z. Ésik and W. Kuich. Formal tree series. *J. Autom. Lang. Comb.*, 8(2):219–285, 2003.

[8] Z. Ésik and G. Liu. Fuzzy tree automata. *Fuzzy Sets and Systems*, 158:1450–1460, 2007.

[9] Z. Fülöp and H. Vogler. Weighted tree automata and tree transducers. In M. Droste, W. Kuich, and H. Vogler, editors, *Handbook of Weighted Automata*, chapter 9. Springer-Verlag, 2009. to appear.

[10] Y. Inagaki and T. Fukumura. On the description of fuzzy meaning of context-free languages. In *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, pages 301–328. Academic Press, New York, 1975.

[11] D. Kirsten and I. Mäurer. On the determinization of weighted automata. *J. Autom. Lang. Comb.*, 10:287–312, 2005.

[12] W. Kuich. Semirings and formal power series: Their relevance to formal languages and automata. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages, Vol. 1*, chapter 9, pages 609–677. Springer-Verlag, 1997.

[13] M. Mohri. Minimization algorithms for sequential transducers. *Theoretical Computer Science*, 234:177–201, 2000.