A Pumping Lemma and Decidability Problems for Recognizable Tree Series

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April 8, 2004

Abstract

In the present paper we show that given a tree series S, which is accepted by (a) a deterministic bottom-up finite state weighted tree automaton (for short: bu-w-fta) or (b) a non-deterministic bu-w-fta over a locally finite semiring, there exists for every input tree $t \in \mathrm{supp}(S)$ a decomposition $t = C'[C[s]]$ into contexts C, C' and an input tree s as well as there exist semiring elements a, a', b, b', c such that the equation $(S, C'[C^n[s]]) = a' \odot a^n \odot c \odot b^n \odot b'$ holds for every non-negative integer n. In order to prove this pumping lemma we extend the power-set construction of classical theories and show that for every non-deterministic bu-w-fta over a locally finite semiring there exists an equivalent deterministic one. By applying the pumping lemma we prove the decidability of a tree series S being constant on its support, S being constant, S being boolean, the support of S being the empty set, and the support of S being a finite set provided that S is accepted by (a) a deterministic bu-w-fta over a commutative semiring or (b) a non-deterministic bu-w-fta over a locally finite commutative semiring.

1 Introduction

Finite state automata (for short: fsa) can be generalized in several ways: in [Sch61] fsa were enriched by weights (or: costs, multiplicities), which are taken from a semiring. This leads to the model of finite state weighted automata (fwa). The idea is that every run on an input string has a weight, which is obtained by multiplying the weights of the applied transitions. Also leaving the system is reflected in weights, which depends on the state where the run ends. Non-determinism is finally handled by summing up the weights of all runs multiplied with the appropriate final weight. Thus an fwa accepts every input string with a weight, which is a semiring element. A survey paper on the theory of fwa is [Kui97b] (also cf. [Eil74, KS86, BR88]), while in [Moh97, BGW00, DK03] recent results are presented.

In [BR82] the generalization of adding weights to the transitions was applied to finite state tree automata (cf. [Eng75, GS84, GS97]), which yields the model of finite state weighted tree

*Research was financially supported by the German Research Council under grant (DFG, GRK 334/3).

automata (or: concept of recognizable tree series, also cf. [Boz91, Boz99]). In this paper we use the notion of bottom-up finite state weighted tree automata (for short: bu-w-fta), which are tuples $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, where Q is a finite set (of states), Σ is a ranked alphabet (of input symbols), $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, $\nu : Q \longrightarrow A$ is a (final weight) mapping, and $\mu = (\mu_k \mid k \in \mathbb{N})$ is a family of mappings $\mu_k : \Sigma^{(k)} \longrightarrow A^{Q^k \times Q}$, in which the transitions and their weights are encoded. Similar to fwa every run of a bu-w-fta on an input tree t causes a weight, which is obtained by multiplying the weights of the applied transitions and finally multiplying this product with the appropriate final weight $\nu(q)$ assuming that the considered run ends in state q. Non-determinism is handled by summing up the weights of all runs. Thus a bu-w-fta accepts every tree with a weight, which is taken from the underlying semiring and hence its semantics is a tree series. Note that every state of a bu-w-fta is potentially a final state and thus every run is successful. We observe that the concepts of weighted grammars (cf. [AB87]), representable tree series (cf. [Boz94, Boz97]), and K-Σ-algebras (cf. [BA89, Boz99]) are strongly related (and equally powerful) to the above two concepts. We also note that besides the aforementioned concepts there exist more weighted tree automata models, e.g., $\mathcal{A}$-cost automata of [Sei94], $\mathcal{A}'$-tree automata of [Kui97a], and finite state weighted tree automata with final states of [BV03]. In Section 3 of this paper we compare the power of these models. A survey on recognizable tree series can be found in [ÉK03], while further results are presented in e.g., [Boz91, FSW94, Boz01, Bor03, DPV03]. We note that weighted tree automata are instances of tree series transducers, which recently were deeply investigated (cf. e.g., [Kui99, EFV02, FV03]).

Let us now answer the question why we introduce bu-w-fta rather than using one of the existing notions of recognizable tree series. In classical automata theory it is a common strategy to prove theorems by additionally assuming that the given device is deterministic and thereby using that deterministic and non-deterministic devices are equally powerful. We would like to prove results on recognizable tree series in the same way. This requires a notion of determinism, which, to the best of our knowledge, only exists for weighted tree automata of [BV03]. Section 4 of the aforementioned paper provides a determinization construction, which extends the power-set construction of classical theories by associating weights to the states. Lemma 6.1 of [BV03] states that the extended power-set construction yields an equivalent deterministic device provided that the underlying semiring is a locally finite *semifield* (which is a semiring with multiplicative inverses). Hence results which are proven for deterministic devices also hold for non-deterministic automata over locally finite semifields. By equipping finite state weighted tree automata of [BV03] with final weights and thereby considering bu-w-fta we can prove stronger results: similar to [BV03] we extend the power-set constructions of classical theories. As done in the aforementioned paper we associate weights to the states. By considering bu-w-fta rather than bottom-up finite state weighted tree automata with final states the weights of the transitions of the constructed device can be defined such that all non-trivial computations of the automaton are shifted to the final weight mapping (cf. Definition 4.1). Thereby we obtain an equivalent deterministic bu-w-fta provided that the given bu-w-fta is defined over a locally finite *semiring* (cf. Theorem 4.8). Thus statements, which are proven for deterministic bu-w-fta, also hold for non-deterministic devices, if the underlying algebraic structure is a locally finite semiring.

We also prove a pumping lemma for recognizable tree series. In classical theories pumping lemmata state that, roughly speaking, parts of the input tree can be pumped such that recognizability is preserved. When considering bu-w-fta we would like to know how pumping is reflected in the weight the pumped tree is accepted with. Being more precise, in Theorem 5.6 we show that there exists a non-negative integer $m \in \mathbb{N}$ such that for every input tree $t \in \mathrm{supp}(S)$, which is contained in the support of S (i.e., t is mapped to a non-zero semiring element), and for every path of t of length $\geq m$ there exists a decomposition $t = C'[C[s]]$ along this path and semiring elements $a, a', b, b', c \in A$ such that $(S, C'[C^n[s]]) = a' \odot a^n \odot c \odot b^n \odot b'$ for every non-negative integer $n \in \mathbb{N}$. The pumping lemma assumes a deterministic bu-w-fta (or a non-deterministic device such that there exists an equivalent deterministic automaton). This is due to the pumping: in classical theories one can pump a context provided that there exists a run on this context, which starts and ends in the same state. There also might be additional runs, but they do not affect the accepting behavior. In weighted automata theory every run (with a non-zero weight) contributes to the weight an input tree is accepted with. Hence we restrict ourselves to deterministic devices and thereby apply the fact that in a deterministic device for every input tree there is at most one run (in our notion: there is at most one run with a non-zero weight). We note that in [BR82] a pumping lemma is proven for the concept of recognizable tree series. Theorem 9.2 of the aforementioned paper states that for every recognizable tree series S over a field there exists a constant m such that for every tree t of height $\geq m$, which is contained in the support of S, there exists a decomposition $t = C_1[C_2[C_3[\alpha]]]$ into contexts C_1, C_2, C_3 and a nullary input symbol α such that $C_1[C_2^*[C_3[\alpha]]] \cap \mathrm{supp}(S)$ is an infinite set. It is easily seen that Theorem 5.6 of the present paper generalizes the pumping lemma of [BR82] provided that the tree series is accepted by a deterministic device.

Similar to classical theories the pumping lemma can be applied for showing that a tree series is not accepted by a deterministic bu-w-fta. We prove that the particular tree series which maps every tree to its height is not recognized by a deterministic bu-w-fta over the arctic semiring. Since the set of all trees over some ranked alphabet is a recognizable tree language (i.e., a recognizable tree series over the Boolean semiring), we thereby show that recognizability is in general not preserved by associating weights to the transitions. The pumping lemma can also be used for deciding some common properties on tree series, e.g., is a given tree series constant on its support, constant, boolean, or is its support the empty or a finite set. We prove that all the aforementioned properties are decidable provided that the given tree series is accepted by (a) a deterministic bu-w-fta over a commutative semiring or (b) a non-deterministic bu-w-fta over a locally finite, commutative semiring. The decidability result of a tree series having finite support additionally assumes a zero-divisor free semiring. We note that in [Boz91] ([Boz97]) it is shown under the assumption that the underlying algebraic structure is a field (the semiring $\mathbb{N}$ of all non-negative integers or the semiring $\mathbb{R}_+$ of all non-negative reals, respectively) that the equivalence problem, i.e., are two recognizable tree series equal, and the minimization problem, i.e., is an automaton which accepts a given recognizable tree series minimal, are decidable.

This paper is organized as follows: In Section 2 we recall well-known notions on trees, semirings, and formal tree series. The concept of bu-w-fta is introduced in Section 3, where we also compare bu-w-fta with existing models of recognizable tree series. We investigate the determinization of bu-w-fta in Section 4. In Section 5 we prove pumping lemmata, which we apply in Section 6, where we present several decidability results.

2 Preliminaries

2.1 Notions on Trees

The sets of all non-negative and positive integers are denoted by $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$, respectively. The *star* $\mathbb{N}^*$ *of* $\mathbb{N}$ is defined to be the set $\mathbb{N}^* = \bigcup_{i \in \mathbb{N}} \mathbb{N}^i$, where $\mathbb{N}^0 = \{\varepsilon\}$ and $\mathbb{N}^{i+1} = \{n.w \mid n \in \mathbb{N}, w \in \mathbb{N}^i\}$ for every non-negative integer $i \in \mathbb{N}$. We note that $v.w$ denotes the concatenation of $v, w \in \mathbb{N}^*$. Moreover, for every two non-negative integers $m, n \in \mathbb{N}$ let $[m, n]$ be the interval $\{m, m+1, \ldots, n\}$ provided that $m \leq n$. Otherwise we set $[m, n] = \emptyset$. As usual we write $[n]$ rather than $[1, n]$. If S is a set, then the *cardinality* and the *power set* of S are denoted by $\mathrm{card}(S)$ and $\mathfrak{P}(S)$, respectively. Now let Σ be a non-empty finite set and $\mathrm{rk} : \Sigma \longrightarrow \mathbb{N}$ be a mapping. The tuple (Σ, rk) is called *ranked alphabet*. Throughout this paper we will be short in notation and write Σ rather than (Σ, rk). For every non-negative integer $k \in \mathbb{N}$ we define the set $\Sigma^{(k)} = \{\sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k\}$ of all symbols of Σ, which have rank k. An element $\sigma \in \Sigma^{(k)}$ is also written as $\sigma^{(k)}$.

Now let $n \in \mathbb{N}$ be a non-negative integer and $X_n = \{x_1, \ldots, x_n\}$ be a set of variables disjoint with Σ. The set $T_\Sigma(X_n)$ of *(finite, labeled, and ordered) trees over* Σ *(indexed by the set* X_n*)* is defined to be the smallest subset of $(\Sigma \cup X_n \cup \{(,)\} \cup \{,\})^*$ such that (i) $X_n \cup \Sigma^{(0)} \subseteq T_\Sigma(X_n)$ and (ii) $\sigma(t_1, \ldots, t_k) \in T_\Sigma(X_n)$ for every positive integer $k \in \mathbb{N}_+$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and input trees $t_1, \ldots, t_k \in T_\Sigma(X_n)$. The set $T_\Sigma(X_0)$ is denoted by T_Σ. The *substitution of* $x_1, \ldots, x_n$ *by* $s_1, \ldots, s_n \in T_\Sigma(X_n)$ *in* $t \in T_\Sigma(X_n)$ is the tree $t[s_1, \ldots, s_n] \in T_\Sigma(X_n)$ (as a shorthand for $t[x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n]$), where for every index $j \in [n]$ every occurrence of x_j in t is replaced by s_j. A tree $t \in T_\Sigma(X_n)$ is called Σ-n-*context* or *context*, if every variable $x \in X_n$ occurs precisely once in t. The set of all Σ-n-contexts is denoted by $C_\Sigma(X_n)$. The following observation shows that the set of Σ-1-contexts could also be defined by induction on its structure.

Observation 2.1
Let $C \in T_\Sigma(X_1)$. It holds that $C \in C_\Sigma(X_1)$, if and only if C is the trivial context x_1 or $C = \sigma(t_1, \ldots, t_{i-1}, C', t_{i+1}, \ldots, t_k)$ for some non-negative integers $k \in \mathbb{N}$ and $i \in [k]$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, context $C' \in C_\Sigma(X_1)$, and trees $t_j \in T_\Sigma$ for every index $j \in [k] \setminus \{i\}$. $\diamond$

Now let $t \in T_\Sigma(X_n)$ be a tree for some non-negative integer $n \in \mathbb{N}$. The *size* and *height of* t are inductively defined by $\mathrm{size}(x) = \mathrm{height}(x) = 1$ for every variable $x \in X_n$. Moreover, $\mathrm{size}(t) = 1 + \sum_{i \in [k]} \mathrm{size}(t_i)$ and $\mathrm{height}(t) = 1 + \max\{\mathrm{height}(t_i) \mid i \in [k]\}$ provided that $t = \sigma(t_1, \ldots, t_k) \in T_\Sigma(n)$ for some non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and trees $t_1, \ldots, t_k \in T_\Sigma(n)$. The set of *paths of* t is defined to be the image of the mapping $\mathrm{paths} : T_\Sigma(X_n) \longrightarrow \mathfrak{P}(\mathbb{N}^*)$, which is given by $\mathrm{paths}(t) = \{\varepsilon\} \cup \{i.w \mid i \in [k], w \in \mathrm{paths}(t_i)\}$. The *length of a path* $w = w_1 \ldots w_n \in \mathrm{paths}(t)$, where $w_i \in \mathbb{N}$ for every index $i \in [n]$, is defined to be $\mathrm{length}(w) = n$. We note that one could also look on the set $\mathrm{paths}(t)$ as the set of positions of t.

Observation 2.2
Let $t \in T_\Sigma$ be a tree. The length of a longest path of t is $\mathrm{height}(t) - 1$. $\diamond$

Let us finally define the subtrees of a tree $t \in T_\Sigma$ in terms of a function $\mathrm{paths}(t) \to T_\Sigma$: the *subtree* t/w *of* $t \in T_\Sigma$ *at the node* $w \in \mathrm{paths}(t)$ is defined inductively as follows: if $w = \varepsilon$ is the empty word, then $t/w = t$ and, if $w = i.w'$ for some integer $i \in [k]$ and word $w' \in \mathrm{paths}(t_i)$, then $t/w = t_i/w'$.

2.2 Semirings

In this section we briefly recall the concept of semirings, which is essential in weighted automata theory. For a more detailed presentation of semirings we refer the reader to [HW98]. Let A be a non-empty set, $\oplus$ and $\odot$ binary associative operations on A, and $\mathbf{0}, \mathbf{1}$ elements of A. As usual, $\odot$ is assumed to have a higher binding power than $\oplus$. The tuple $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is called *semiring*, if (i) $\mathbf{0}$ and $\mathbf{1}$ are the neutral elements of $\oplus$ and $\odot$, respectively ($a \oplus \mathbf{0} = a = \mathbf{0} \oplus a$ and $a \odot \mathbf{1} = a = \mathbf{1} \odot a$), (ii) $\oplus$ is commutative ($a \oplus b = b \oplus a$), (iii) $\odot$ is left- and right-distributive over $\oplus$ ($a \odot (b \oplus c) = a \odot b \oplus a \odot c$ and $(a \oplus b) \odot c = a \odot c \oplus b \odot c$), and (iv) $\mathbf{0}$ is absorbing ($a \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot a$).

For the rest of this paper let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring. As usual we lift the operations $\oplus$ and $\odot$ to sets $A_1, A_2 \subseteq A$ by defining $A_1 \oplus A_2 = \{a_1 \oplus a_2 \mid a_1 \in A_1, a_2 \in A_2\}$ and $A_1 \odot A_2 = \{a_1 \odot a_2 \mid a_1 \in A_1, a_2 \in A_2\}$. The semiring $\mathcal{A}$ is called *commutative*, if $\odot$ is commutative. We will shorten notation as follows: for every finite index set $I = \{i_1, \ldots, i_n\}$ for some non-negative integer $n \in \mathbb{N}$ and semiring elements $a_{i_1}, \ldots, a_{i_n} \in A$ let

$$\sum_{i \in I} a_i = \begin{cases} a_{i_1} \oplus \cdots \oplus a_{i_n} & \text{, if } I \neq \emptyset, \\ \mathbf{0} & \text{, otherwise.} \end{cases}$$

The semiring $\mathcal{A}$ is called *locally finite*, if, for every finite subset A' of A, the closure $\langle A' \rangle_{\{\oplus, \odot\}}$ of A' under the semiring operations $\oplus$ and $\odot$ is again a finite set. Clearly, every finite semiring is locally finite. Moreover, the min-max-semiring, which is defined below, is a locally finite semiring with an infinite carrier set.

Let us now present some well known semirings.

- The semiring of non-negative integers $Nat = (\mathbb{N}, +, \cdot, 0, 1)$ with the usual addition and multiplication. Nat can be used in automata theory for counting successful paths.
- The *Boolean* semiring $Bool = (\{0, 1\}, \vee, \wedge, 0, 1)$ with disjunction and conjunction. This semiring has a highly theoretical meaning, since there is a one-to-one correspondence between weighted (tree) automata over the Boolean semiring and unweighted (tree) automata.
- The *Tropical semiring* $Trop = (\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$, in which the semiring addition and multiplication are the natural extension of the minimum operation and addition of the non-negative integers to $\mathbb{N} \cup \{+\infty\}$, respectively. $Trop$ can be used for calculating shortest paths or minimal costs.
- The *Arctic semiring* $Arct = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, where, similar to the Tropical semiring, the semiring addition and multiplication are the extension of the maximum operation and addition of the non-negative integers to $\mathbb{N} \cup \{-\infty\}$, respectively. $Arct$ is used for calculating longest paths or critical costs.
- The min-max-*semiring* $MinMax = (\mathbb{N} \cup \{\pm\infty\}, \min, \max, +\infty, -\infty)$, in which the semiring addition and multiplication are the natural extension of the minimum and maximum operations of the non-negative integers to $\mathbb{N} \cup \{\pm\infty\}$, respectively. $MinMax$ can be used for solving capacity problems.

We note that many number structures, e.g., the integers $Int = (\mathbb{Z}, +, \cdot, 0, 1)$, the rational numbers $Rat = (\mathbb{Q}, +, \cdot, 0, 1)$, the real numbers $Real = (\mathbb{R}, +, \cdot, 0, 1)$, and the complex numbers $Comp = (\mathbb{C}, +, \cdot, 0, 1)$ with the usual addition and multiplication are semirings.

2.3 Formal Tree Series

Let us now recall the concept of formal tree series. A *(formal) tree series (over a ranked alphabet Σ and semiring $\mathcal{A}$)* is a mapping $S : T_\Sigma \longrightarrow A$. In what follows, we use another notation: the image $S(t) \in A$ of a tree $t \in T_\Sigma$ is called *coefficient of* t and, according to power series, which are known from analysis, the coefficient of t is denoted by (S, t). The tree series S now can be written as the sum $\sum_{t \in T_\Sigma} (S, t)\, t$. The set of all tree series over Σ and $\mathcal{A}$ is denoted by $\mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle$. The *support of* S is the set $\mathrm{supp}(S) = \{t \in T_\Sigma \mid (S, t) \neq \mathbf{0}\}$. A tree series S is called *boolean*, if $(S, t) \in \{\mathbf{0}, \mathbf{1}\}$ for every tree $t \in T_\Sigma$. Moreover, S is called *constant on its support*, if there exists a semiring element $a \in A$ such that $(S, t) = a$ for every tree $t \in \mathrm{supp}(S)$. A tree series S, which is constant on its support, is called *constant tree series*, denoted by $S = \widetilde{a}$, if there exists a semiring element $a \in A$ such that $(S, t) = a$ for every tree $t \in T_\Sigma$.

We conclude this section by defining two operations on tree series. Let $\mathcal{A}$ be a semiring and $S, T \in \mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle$ tree series. The *sum* $S \oplus T$, and the *Hadamard product* $S \odot T$ are defined for every tree $t \in T_\Sigma$ by $(S \oplus T) = (S, t) \oplus (T, t)$ and $(S \odot T, t) = (S, t) \odot (T, t)$, respectively.

For more details on formal tree series we refer the reader to [Kui99].

3 Bottom-Up Finite State Weighted Tree Automata

In this section we introduce bottom-up finite state weighted tree automata with final weights. There is a tight relationship between bottom-up finite state weighted tree automata of [BV03] which have final states rather than final weights, and the devices, which we define below. This relationship will be discussed in the course of this section as well as the relationships to further weighted tree automata models. We also present an application, namely tree pattern matching. We conclude this section by proving that the cross product of two bottom-up finite state weighted tree automata M_1 and M_2 accepts the Hadamard product of the tree series, which are accepted by M_1 and M_2.

Let us start this section by defining tree representations. Tree representations encode the transitions and their weights. We note that for technical reasons at this time we do *not* assume a finite set of states.

Definition 3.1 (Tree representation, cf. [BV03], Definition 3.1)
Let Q be a not necessarily finite set (of states), Σ a ranked alphabet (of input symbols), and $\mathcal{A}$ a semiring. A (bottom-up) tree representation (over Q, Σ, and $\mathcal{A}$) *is a family $\mu = (\mu_k \mid k \in \mathbb{N})$ of mappings $\mu_k : \Sigma^{(k)} \longrightarrow A^{Q^k \times Q}$. A tree representation is called* finite*, if the underlying set Q of states is finite. Moreover, a tree representation μ is called* deterministic*, if for every non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and k-tuple of states $(q_1, \ldots, q_k) \in Q^k$ there is at most one state $q \in Q$ such that $\mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq \mathbf{0}$.* $\diamond$

Every finite tree representation μ induces a family of mappings $(\overline{\mu_k(\sigma)} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)})$ in the following way:

$$\overline{\mu_k(\sigma)} : A^Q \times \ldots \times A^Q \longrightarrow A^Q :$$
$$\overline{\mu_k(\sigma)}(V_1, \ldots, V_k)_q = \sum_{(q_1,\ldots,q_k) \in Q^k} (V_1)_{q_1} \odot \ldots \odot (V_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q}$$

for every state $q \in Q$ and vectors $V_1, \ldots, V_k \in A^Q$. We observe that $\big(A^Q, \big(\overline{\mu_k(\sigma)} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}\big)\big)$ is a Σ-algebra. Its unique homomorphism $h_\mu : T_\Sigma \to A^Q$ is given for every non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and trees $t_1, \ldots t_k \in T_\Sigma$ by

$$h_\mu(\sigma(t_1, \ldots t_k)) = \overline{\mu_k(\sigma)}(h_\mu(t_1), \ldots, h_\mu(t_k)).$$

We call $h_\mu(t)$ the *characteristic vector of the tree $t \in T_\Sigma$ (with respect to the tree representation μ)*. Let us now define bottom-up weighted tree automata. For technical reasons we define automata with an infinite set of states as well as automata with a finite set of states.

Definition 3.2 (Bottom-up (finite state) tree automata)
Let Q be a set (of states), Σ a ranked alphabet (of input symbols), $\mathcal{A}$ a semiring, $\nu : Q \longrightarrow A$ a mapping (final weight mapping), and μ a tree representation. The tuple $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ is called bottom-up weighted tree automaton (with final weight mapping, for short bu-w-ta)*. A bu-w-ta is called* deterministic*, if its tree representation is deterministic. A bu-w-ta is called* bottom-up finite state weighted tree automaton (for short bu-w-fta)*, if its tree representation is finite. The tree series S_M, which is* accepted or recognized *by a (finite) bu-w-fta M, is defined for every tree $t \in T_\Sigma$ by $(S_M, t) = \sum_{q \in Q} h_\mu(t)_q \odot \nu(q)$. We denote by $\mathcal{A}^{n,bu}\langle\langle T_\Sigma \rangle\rangle$ and $\mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$ the classes of all tree series, which are accepted by bu-w-fta and deterministic bu-w-fta, respectively.* $\diamond$

In Example 3.3 we present a bu-w-fta over the Arctic semiring, which accepts every input tree with its height. Note that in Example 5.9 we prove that this tree series is not accepted by any deterministic bu-w-fta over the Arctic semiring, which shows that deterministic and non-deterministic bu-w-fta are in general not equally powerful.

Example 3.3 ($\sum_{t \in T_\Sigma}$ height(t) t accepted by some non-deterministic bu-w-fta)
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ the bu-w-fta, which is defined by $Q = \{q, q_0\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, $\nu(q) = 0$, $\nu(q_o) = -\infty$, $\mathcal{A} = Arct$,

$$\begin{gathered} \mu_0(\alpha)_{(),q} = 1, \quad \mu_0(\alpha)_{(),q_0} = 0, \\ \mu_2(\sigma)_{(q,q_0),q} = 1, \quad \mu_2(\sigma)_{(q_0,q),q} = 1, \quad \mu_2(\sigma)_{(q_0,q_0),q_0} = 0, \end{gathered}$$

and, for every three states $q_1, q_2, q_3 \in Q$, for which $\mu_2(\sigma)_{(q_1,q_2),q_3}$ is not yet defined, let $\mu_2(\sigma)_{(q_1,q_2),q_3} = -\infty$. The following straightforward inductive proof shows that $h_\mu(t)_q =$ height(t) and $h_\mu(t)_{q_0} = 0$ for every tree $t \in T_\Sigma$: if $t = \alpha$, then

$$h_\mu(t)_q = \overline{\mu_0(\alpha)}()_q = \mu_0(\alpha)_{(),q} = 1 = \mathrm{height}(t)$$

and similarly $h_\mu(t)_{q_0} = 0$. If $t = \sigma(t_1, t_2)$ for some trees $t_1, t_2 \in T_\Sigma$, then

$$\begin{aligned} & h_\mu(\sigma(t_1, t_2))_q \\ & = \overline{\mu_2(\sigma)}(h_\mu(t_1), h_\mu(t_2))_q \\ & = \max\{h_\mu(t_1)_{p_1} + h_\mu(t_2)_{p_2} + \mu_2(\sigma)_{(p_1,p_2),q} \mid (p_1, p_2) \in Q^2\} \\ & = \max\{h_\mu(t_1)_q + h_\mu(t_2)_{q_0} + \mu_2(\sigma)_{(q,q_0),q}, h_\mu(t_1)_{q_0} + h_\mu(t_2)_q + \mu_2(\sigma)_{(q_0,q),q}, -\infty\} \\ & = \max\{\mathrm{height}(t_1) + 0 + 1, 0 + \mathrm{height}(t_2) + 1, -\infty\} \\ & = 1 + \max\{\mathrm{height}(t_1), \mathrm{height}(t_2)\} \\ & = \mathrm{height}(t). \end{aligned}$$

By a similar calculation one easily proves that $h_\mu(\sigma(t_1,t_2))_{q_0} = 0$. Hence M accepts every input tree $t \in T_\Sigma$ with $(S_M, t) = \mathrm{height}(t)$. $\diamond$

Let us now discuss the relationship between (finite) bu-w-fta and other weighted tree automata models. Obviously representable tree series, which are considered in e.g. [Boz94] are precisely the tree series, which are accepted by non-deterministic bu-w-fta. Let us now compare bu-w-fta with bottom-up finite state weighted tree automata (with final states), which were introduced in [BV03]. The latter devices are defined to be tuples $M = (Q, \Sigma, Q_d, \mathcal{A}, \mu)$, where Q, Σ, $\mathcal{A}$, and μ are as in Definition 3.2 and Q_d is a subset of Q (of final states). M accepts every input tree $t \in T_\Sigma$ with the weight $(S_M, t) = \sum_{q \in Q_d} h_\mu(t)_q$. A bottom-up finite state weighted tree automaton M with final states can be modeled by a bu-w-fta by taking the same set of states, ranked alphabet, semiring, and tree representation. The final weight mapping maps every final state to $\mathbf{1}$ and every non-final state to $\mathbf{0}$. The equivalence of both devices is easily seen. Conversely, a bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ with final weight mapping can be modeled by a bottom-up finite state weighted tree automaton $M' = (Q', \Sigma, Q'_d, \mathcal{A}, \mu')$ with final states by introducing a new state $* \notin Q$, which is the unique final state: we set $Q' = Q \cup \{*\}$, $Q'_d = \{*\}$, and for every non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and states $q'_1, \ldots, q'_k, q' \in Q'$,

$$\mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'} = \begin{cases} \mu_k(\sigma)_{(q'_1,\ldots,q'_k),q'} & \text{, if } q'_1,\ldots,q'_k,q' \in Q, \\ \sum_{q \in Q} \mu_k(\sigma)_{(q'_1,\ldots,q'_k),q} \odot \nu(q) & \text{, if } q'_1,\ldots,q'_k \in Q \text{ and } q' = *, \\ \mathbf{0} & \text{, otherwise.} \end{cases}$$

The (inductive) proof of equivalence is very straightforward. We therefore leave it to the reader. By the above two constructions it is shown that the two non-deterministic models of bottom-up finite state weighted tree automata are equally powerful. Unfortunately, the latter construction does not preserve determinism. Being more precise, deterministic bu-w-fta are in general more powerful than deterministic bottom-up finite state weighted tree automata with final states.

In Section 3 of [BV03] it is shown that bottom-up finite state weighted tree automata with final states and hence bu-w-fta of the present paper are particular $\mathcal{A}$-cost automata of [Sei94] and $\mathcal{A}'$-tree automata of [Kui97a] (by considering the equally powerful top-down devices).

Let us now compare bu-w-fta with the concept of recognizable tree series, which was introduced in [BR82] (also cf. [Boz91, Boz99, ÉK03]). For the algebraic notions we refer the reader to any good algebra textbook. A recognizable tree series is defined in terms of a Σ-algebra $\mathcal{V} = (V, a)$, where V is a vector space and $a = (a_\sigma : V^k \longrightarrow V \mid \sigma \in \Sigma^{(k)})$ is a family of multi-linear mappings. As usual, the family a of multi-linear mappings is extended to a mapping $\mu_\mathcal{V} : T_\Sigma \longrightarrow V$, which is inductively defined for every non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and trees $t_1, \ldots, t_k \in T_\Sigma$ by $\mu_\mathcal{V}(\sigma(t_1,\ldots,t_k)) = a_\sigma(\mu_\mathcal{V}(t_1), \ldots, \mu_\mathcal{V}(t_k))$. A tree series $S \in \mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle$ is recognizable, if there exists a realization $(\mathcal{V}, \varphi)$ of S, which is a pair consisting of a Σ-algebra $\mathcal{V}$ over a finite dimensional vector space (as introduced above) and a linear form $\varphi : V \longrightarrow A$ such that $S = \varphi(\mu_\mathcal{V})$. We now briefly show that a tree series is recognizable in the sense of [BR82], if and only if it is accepted by a bu-w-fta provided that the underlying semiring is commutative. First let a bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be given. A realization $(\mathcal{V}, \varphi)$ of the tree series S_M, which is accepted by M, can be defined as follows: the underlying Σ-algebra $\mathcal{V} = (V, a)$ is given by the vector space $V = A^Q$ and the sequence $a = (a_\sigma : V^k \longrightarrow V \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)})$ of multi-linear mappings $a_\sigma = \overline{\mu_k(\sigma)}$ for every k-ary input symbol $\sigma \in \Sigma^{(k)}$. We observe that $\overline{\mu_k(\sigma)}$ is a multi-linear mapping provided that the underlying semiring is commutative. We define the linear form φ for every vector $v \in V$ by $\varphi(v) = \sum_{q \in Q} v_q \odot \nu(q)$. The (inductive) proof of correctness is very straightforward and hence left to the reader. Conversely, let S be a recognizable tree series in the sense of [BR82], i.e., there exists a finite dimensional realization $(\mathcal{V}, \varphi)$ with $\mathcal{V} = (V, a)$ and $a = (a_\sigma : V^k \longrightarrow V \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)})$ of S. We define a bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, which accepts the tree series S, as follows: Q is a basis of the vector space V. Moreover, for all states $q_1, \ldots, q_k, q \in Q$ we define the final weight mapping ν and the tree representation μ by $\nu(q) = \varphi(q)$ and $\mu_k(\sigma)_{(q_1,\ldots,q_k),q} = a_\sigma(q_1, \ldots, q_k)_q$, respectively. One easily proves by induction on the structure of the input tree $t \in T_\Sigma$ that the bu-w-fta M accepts the recognizable tree series (in the sense of [BR82]). Summing up, we have shown that our notion of recognizable tree series coincides with the classical notion of [BR82] provided that the underlying semiring is commutative.

Let us now present an application of weighted tree automata.

Example 3.4 (Tree pattern matching, also cf. [FSW94])
Consider a tree $t \in T_\Sigma$ and a pattern C. We would like to find all occurrences of C in t and, roughly speaking, give references to the root of the occurrences of C. This can be formalized as follows: let $t \in T_\Sigma$ and $C \in C_\Sigma(X_m)$ for some non-negative integer $m \in \mathbb{N}$. We call C *pattern of t at $w \in$ paths(t)*, if $t/w = C[t_1, \ldots, t_m]$ for some trees $t_1, \ldots, t_m \in T_\Sigma$. Before we define the tree series S_C, which maps every tree t to the set of all $w \in \text{paths}(t)$ such that C is a pattern of t at w, we introduce the semiring, which we will work with. Consider the tuple $(\mathfrak{P}(\mathbb{N}^*), \cup, \circ, \emptyset, \{\varepsilon\})$, where $\circ$ is the binary operation on $\mathfrak{P}(\mathbb{N}^*)$, which is defined for every two subsets $A, B \in \mathfrak{P}(\mathbb{N}^*)$ by $A \circ B = \{b.a \in \mathbb{N}^* \mid a \in A, b \in B\}$. Recall that $b.a$ denotes the concatenation of the words b and a. One easily proves that $(\mathfrak{P}(\mathbb{N}^*), \cup, \circ, \emptyset, \{\varepsilon\})$ is a semiring.

Let us now define the tree series S_C over the semiring $(\mathfrak{P}(\mathbb{N}^*), \cup, \circ, \emptyset, \{\varepsilon\})$ as follows: for every tree $t \in T_\Sigma$ let

$$(S_C, t) = \{w \in \mathbb{N}^* \mid (\exists t_1, \ldots, t_m \in T_\Sigma) : t/w = C[t_1, \ldots, t_m]\}.$$

We claim that the tree series S_C can be computed by a bu-w-fta over the semiring $\mathfrak{P}(\mathbb{N}^*)$. Since the general case is very technical and the intention of this paper is to prove a pumping lemma rather than discussing tree pattern matching, we now restrict ourselves to the particular ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and pattern $C = \sigma(\sigma(\alpha, \alpha), x_1)$, and note that the general case is very similar. Let us now define a bu-w-fta $M_C = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, which accepts the tree series S_C. M_C is given by $Q = \{q_\perp, q_\alpha, q_{\sigma(\alpha,\alpha)}, q_C\}$, $\nu(q_C) = 1$ and $\nu(q) = 0$ for every state $q \in Q \setminus \{q_C\}$, $\mathcal{A} = (\mathfrak{P}(\mathbb{N}^*), \cup, \circ, \emptyset, \{\varepsilon\})$, and

$$\begin{aligned}
\mu_0(\alpha)_{(),q_\perp} &= \{\varepsilon\}, & \mu_0(\alpha)_{(),q_\alpha} &= \{\varepsilon\}, \\
\mu_2(\sigma)_{(q_\perp,q_\perp),q_\perp} &= \{\varepsilon\}, & \mu_2(\sigma)_{(q_\alpha,q_\alpha),q_{\sigma(\alpha,\alpha)}} &= \{\varepsilon\}, \\
\mu_2(\sigma)_{(q_{\sigma(\alpha,\alpha)},q_\perp),q_C} &= \{\varepsilon\}.
\end{aligned}$$

Moreover, for every state $q \in Q \setminus \{q_C\}$,

$$\mu_2(\sigma)_{(q_C,q),q_C} \quad = \quad \{1\}, \qquad\qquad \mu_2(\sigma)_{(q,q_C),q_C} \quad = \quad \{2\}.$$

Otherwise we set $\mu_2(\sigma)_{(q_1,q_2),q} = \emptyset$ for every three states $q_1, q_2, q \in Q$. Let us now briefly discuss the intended meaning of the states. This requires us to consider "runs" on an input tree t. If a "run" ends up in the state q_α, ($q_{\sigma(\alpha,\alpha)}$, $q_\perp$, respectively), then it has either weight $\emptyset$ or $\{\varepsilon\}$ and we have just met an α-tree (a $\sigma(\alpha, \alpha)$-tree, an arbitrary tree, respectively). If a "run" ends up in the state q_C, then again, either it has weight $\emptyset$ or we have met the pattern C while traversing the input tree and the weight of the "run" is $\{w\}$, where $w \in \mathrm{paths}(t)$ and $t/w = C[t']$ for some tree $t' \in T_\Sigma$. The inductive proof of correctness is very straightforward. We leave it to the reader. $\diamond$

Later on, for a given input tree $t \in T_\Sigma$ and bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, we will work with the set $\widetilde{\mu}(t)$ of all those states $q \in Q$ such that, roughly speaking, there exists a "run" of the automaton M on t ending in state q such that every "transition associated to this run" has a weight different from zero. Formally, the mapping $\widetilde{\mu} : T_\Sigma \longrightarrow \mathfrak{P}(Q)$ is inductively defined for every input tree $t = \sigma(t_1, \ldots, t_k)$, where $k \in \mathbb{N}$ is a non-negative integer, $\sigma \in \Sigma^{(k)}$ is a k-ary input symbol, and $t_1, \ldots, t_k \in T_\Sigma$ are trees, by

$$\widetilde{\mu}(t) = \{q \in Q \mid (\forall i \in [k]), (\exists q_i \in \widetilde{\mu}(t_i)), \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq \mathbf{0}\}.$$

Observation 3.5
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a bu-w-fta, $s, t \in T_\Sigma$ trees, and $C = \sigma(t_1, \ldots, t_{i-1}, x_1, t_{i+1}, \ldots, t_k)$ a context for some positive integers $k \in \mathbb{N}_+$ and $i \in [k]$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and trees $t_j \in T_\Sigma$ for every index $j \in [k] \setminus \{i\}$.

(i) If $q \in Q \setminus \widetilde{\mu}(s)$, then $h_\mu(s)_q = \mathbf{0}$.

(ii) If $s \in \mathrm{supp}(S_M)$, then $\widetilde{\mu}(s) \neq \emptyset$.

(iii) If $\widetilde{\mu}(s) = \emptyset$, then $\widetilde{\mu}(C[s]) = \emptyset$.

(iv) If $\widetilde{\mu}(s) = \widetilde{\mu}(t)$, then $\widetilde{\mu}(C[s]) = \widetilde{\mu}(C[t])$.

(v) If M is a deterministic bu-w-fta, then $\widetilde{\mu}(s)$ is either the empty set or a singleton. In the latter case we identify $\widetilde{\mu}(s)$ with the state contained in $\widetilde{\mu}(s)$.

(vi) If M is a deterministic bu-w-fta and $\widetilde{\mu}(s) \in Q$, then $(S_M, s) = h_\mu(s)_{\widetilde{\mu}(s)} \odot \nu(\widetilde{\mu}(s))$. $\diamond$

By a repeated application of Observation 3.5 (iii) and (iv) we obtain the following statement:

Corollary 3.6
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a bu-w-fta, $s, t \in T_\Sigma$ trees, and $C \in C_\Sigma(X_1)$ a context.

(i) If $\widetilde{\mu}(s) = \emptyset$, then $\widetilde{\mu}(C[s]) = \emptyset$.

(ii) If $\widetilde{\mu}(s) = \widetilde{\mu}(t)$, then $\widetilde{\mu}(C[s]) = \widetilde{\mu}(C[t])$. $\diamond$

In classical automata theory the cross product $\mathfrak{A}_1 \times \mathfrak{A}_2$ of two automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is defined by setting the set of states (initial states, final states, respectively) of $\mathfrak{A}_1 \times \mathfrak{A}_2$ to the cross product of the sets of states (initial states, final states, respectively) of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. The transitions are defined in the obvious way. It is well known that $\mathfrak{A}_1 \times \mathfrak{A}_2$ accepts the intersection of the languages, which are accepted by $\mathfrak{A}_1$ and $\mathfrak{A}_2$. We now define the cross product $M_1 \times M_2$ of bu-w-fta M_1 and M_2 and prove that, if the underlying semiring is commutative, then $M_1 \times M_2$ accepts the Hadamard product of the tree series, which are recognized by M_1 and M_2.

Definition 3.7 (Cross product)
Let $M_1 = (Q_1, \Sigma, \nu_1, \mathcal{A}, \mu_1)$ and $M_2 = (Q_2, \Sigma, \nu_2, \mathcal{A}, \mu_2)$ be bu-w-fta. The cross product *of* M_1 *and* M_2 *is defined to be the bu-w-fta $M_1 \times M_2 = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, where $Q = Q_1 \times Q_2$, $\nu((p,q)) = \nu_1(p) \odot \nu_2(q)$ for every two states $p \in Q_1$ and $q \in Q_2$, and μ is defined for every non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and states $p_1, \ldots, p_k, p \in Q_1$, $q_1, \ldots, q_k, q \in Q_2$ by*

$$\mu_k(\sigma)_{((p_1,q_1),\ldots,(p_k,q_k)),(p,q)} = (\mu_1)_k(\sigma)_{(p_1,\ldots,p_k),p} \odot (\mu_2)_k(\sigma)_{(q_1,\ldots,q_k),q}. \qquad \diamond$$

Lemma 3.8
Let $\mathcal{A}$ be a commutative semiring and M_1, M_2, and $M_1 \times M_2$ bu-w-fta as required/defined in Definition 3.7. It holds that $(S_{M_1 \times M_2}, t) = (S_{M_1}, t) \odot (S_{M_2}, t)$ for every input tree $t \in T_\Sigma$.

Proof:
Let us first show that the equation $h_\mu(t)_{(p,q)} = h_{\mu_1}(t)_p \odot h_{\mu_2}(t)_q$ holds for every two states $p \in Q_1$ and $q \in Q_2$, which we prove by induction on the structure of the input tree $t \in T_\Sigma$. Note that the induction base is covered by the induction step. Let $t = \sigma(t_1, \ldots, t_k)$ for some non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and trees $t_1, \ldots, t_k \in T_\Sigma$. For every two states $p \in Q_1$ and $q \in Q_2$,

$$\begin{aligned}
& h_\mu(\sigma(t_1, \ldots, t_k))_{(p,q)} \\
&= \sum_{\substack{((p_1,q_1),\ldots,(p_k,q_k)) \\ \in (Q_1 \times Q_2)^k}} h_\mu(t_1)_{(p_1,q_1)} \odot \cdots \odot h_\mu(t_k)_{(p_k,q_k)} \odot \mu_k(\sigma)_{((p_1,q_1),\ldots,(p_k,q_k)),(p,q)} \\
&= \sum_{\substack{((p_1,q_1),\ldots,(p_k,q_k)) \\ \in (Q_1 \times Q_2)^k}} (h_{\mu_1}(t_1)_{p_1} \odot h_{\mu_2}(t_1)_{q_1}) \odot \cdots \odot (h_{\mu_1}(t_k)_{p_k} \odot h_{\mu_2}(t_k)_{q_k}) \odot \\
&\qquad \odot \mu_k(\sigma)_{((p_1,q_1),\ldots,(p_k,q_k)),(p,q)} \\
&\quad \text{(by induction hypothesis)} \\
&= \sum_{\substack{((p_1,q_1),\ldots,(p_k,q_k)) \\ \in (Q_1 \times Q_2)^k}} (h_{\mu_1}(t_1)_{p_1} \odot h_{\mu_2}(t_1)_{q_1}) \odot \cdots \odot (h_{\mu_1}(t_k)_{p_k} \odot h_{\mu_2}(t_k)_{q_k}) \odot \\
&\qquad \odot (\mu_1)_k(\sigma)_{(p_1,\ldots,p_k),p} \odot (\mu_2)_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&= \sum_{\substack{(p_1,\ldots,p_k) \in Q_1^k \\ (q_1,\ldots,q_k) \in Q_2^k}} h_{\mu_1}(t_1)_{p_1} \odot \cdots \odot h_{\mu_1}(t_k)_{p_k} \odot (\mu_1)_k(\sigma)_{(p_1,\ldots,p_k),p} \odot \\
&\qquad \odot h_{\mu_2}(t_1)_{q_1} \odot \cdots \odot h_{\mu_2}(t_k)_{q_k} \odot (\mu_2)_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&= \left(\sum_{(p_1,\ldots,p_k) \in Q_1^k} h_{\mu_1}(t_1)_{p_1} \odot \cdots \odot h_{\mu_1}(t_k)_{p_k} \odot (\mu_1)_k(\sigma)_{(p_1,\ldots,p_k),p} \right) \odot \\
&\quad \odot \left(\sum_{(q_1,\ldots,q_k) \in Q_2^k} h_{\mu_2}(t_1)_{q_1} \odot \cdots \odot h_{\mu_2}(t_k)_{q_k} \odot (\mu_2)_k(\sigma)_{(q_1,\ldots,q_k),q} \right) \\
&= h_{\mu_1}(\sigma(t_1, \ldots, t_k))_p \odot h_{\mu_2}(\sigma(t_1, \ldots, t_k))_q.
\end{aligned}$$

This proves the claim. Let us now show that the lemma holds. We have

$$\begin{aligned}
(S_{M_1 \times M_2}) &= \sum_{(p,q)\in Q_1 \times Q_2} h_\mu(t)_{(p,q)} \odot \nu((p,q)) \\
&= \sum_{(p,q)\in Q_1 \times Q_2} h_{\mu_1}(t)_p \odot h_{\mu_2}(t)_q \odot \nu_1(p) \odot \nu_2(q) \\
&= \left(\sum_{p\in Q_1} h_{\mu_1}(t)_p \odot \nu_1(p) \right) \odot \left(\sum_{q\in Q_2} h_{\mu_2}(t)_q \odot \nu_2(q) \right) \\
&= (S_{M_1}, t) \odot (S_{M_2}, t),
\end{aligned}$$

which proves the lemma. ∎

Corollary 3.9 (cf. Proposition 5.1 of [BR82])
Let $\mathcal{A}$ be a commutative semiring.

(i) If $S_1, S_2 \in \mathcal{A}^{n,bu}\langle\langle T_\Sigma \rangle\rangle$, then $S_1 \odot S_2 \in \mathcal{A}^{n,bu}\langle\langle T_\Sigma \rangle\rangle$.

(ii) If $S_1, S_2 \in \mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$, then $S_1 \odot S_2 \in \mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$.

Proof:
The claims follow from Lemma 3.8 and the observation that the cross product $M_1 \times M_2$ of two deterministic bu-w-fta M_1 and M_2 again is a deterministic bu-w-fta. ∎

We note that Definition 3.7, Lemma 3.8, and Corollary 3.9 cover the corresponding theory of fwa (cf. [KS86]).

4 Determinization

In this section we construct for a given bu-w-fta M an equivalent deterministic device. To do so we introduce a natural extension of the power set construction known from the theory of bottom-up finite tree automata (cf. [GS84]). We encode the characteristic vector of a tree into the states of the generated automaton $det^p(M)$. It turns out that the extended power set construction is partial in the sense that there exist a bu-w-fta M and an equivalent deterministic bu-w-fta M' such that $det^p(M)$ is an infinite device.

For the rest of this section let

- $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a bu-w-fta,
- $q'_t = \bigcup_{q\in Q}\{(q, h_\mu(t)_q)\} \in \mathfrak{P}(Q \times A)$ be the state associated with the tree $t \in T_\Sigma$, and
- $Q' = \{q'_t \mid t \in T_\Sigma\}$ the set of all states, which are associated with a tree over Σ.

We observe that two states q'_t and $q'_{t'}$ are equal, if and only if the characteristic vectors $h_\mu(t)$ and $h_\mu(t')$ are equal. Later on we need to express the weight of a state $q \in Q$ of the given device M in a state $q' \in Q'$ of the constructed device $det^p(M)$. Therefore we define for every state $q' \in Q'$ of the constructed device the mapping $a_{q'} : Q \longrightarrow A$, which is defined for every state $q \in Q$ of the given device by

$$a_{q'}(q) = h_\mu(t)_q \qquad \text{assuming that} \qquad q' = q'_t \text{ for some } t \in T_\Sigma.$$

Hence, $q'_t = \bigcup_{q \in Q} \{(q, a_{q'_t}(q))\}$. Now let us introduce the extended power set construction. We note that the constructed automaton might have an infinite set of states, i.e., we only generate a bu-w-ta. Later on we show that, if the underlying semiring is locally finite, then M is a finite automaton (cf. Lemma 4.7).

Definition 4.1 (Extended power set construction)
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a bu-w-fta. The bu-w-ta $det^p(M) = (Q', \Sigma, \nu', \mathcal{A}, \mu')$ is defined for every non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and states $q'_1, \dots, q'_k, q' \in Q'$ by $\nu'(q') = \sum_{q \in Q} a_{q'}(q) \odot \nu(q)$ and

$$\mu'_k(\sigma)_{(q'_1,\dots,q'_k),q'} = \begin{cases} \mathbf{1} & \text{, if } a_{q'}(q) = \sum_{(q_1,\dots,q_k) \in Q^k} a_{q'_1}(q_1) \odot \cdots \odot a_{q'_k}(q_k) \odot \mu_k(\sigma)_{(q_1,\dots,q_k),q} \\ & \quad \text{for every state } q \in Q, \\ \mathbf{0} & \text{, otherwise.} \end{cases} \qquad \diamond$$

In order to shorten notation, let $det^p(M) = (Q', \Sigma, \nu', \mathcal{A}, \mu')$ for the rest of this section. In the following we show that the extended power set construction is partial. By this we mean that there exists a bu-w-fta such that the extended power set construction might generate an infinite device (cf. Example 4.6). If the extended power set construction outputs a (finite) bu-w-fta as in Example 4.5, then by Observation 4.2 and Lemma 4.4 the generated automaton is a deterministic device being equivalent to the given automaton.

Observation 4.2
The tuple $det^p(M)$ is a deterministic bu-w-ta. $\diamond$

Now we prove that, if $det^p(M)$ is a finite automaton, then M and $det^p(M)$ are equivalent bu-w-fta, which is stated in Lemma 4.4. This proof requires the following preparing lemma.

Lemma 4.3
Let $k \in \mathbb{N}$ be a non-negative integer, $\sigma \in \Sigma^{(k)}$ a k-ary input symbol, and $q'_1, \dots, q'_k, q' \in Q'$ states. Then $\mu'_k(\sigma)_{(q'_1,\dots,q'_k),q'} = \mathbf{1}$, if and only if there exist trees $t_1, \dots, t_k \in T_\Sigma$ such that $q' = q'_{\sigma(t_1,\dots,t_k)}$ and $q'_i = q'_{t_i}$ for every index $i \in [k]$.

Proof:
Since $q'_1, \dots, q'_k, q' \in Q'$ are states of the constructed device, there exist trees $t_1, \dots, t_k \in T_\Sigma$ such that $q'_i = q'_{t_i}$ for every index $i \in [k]$. Hence

$$\begin{aligned} & \mu'_k(\sigma)_{(q'_1,\dots,q'_k),q'} = \mathbf{1} \\ \iff & (\forall q \in Q) : \; a_{q'}(q) = \sum_{(q_1,\dots,q_k) \in Q^k} a_{q'_1}(q_1) \odot \cdots \odot a_{q'_k}(q_k) \odot \mu_k(\sigma)_{(q_1,\dots,q_k),q} \\ \iff & (\forall q \in Q) : \; a_{q'}(q) = \sum_{(q_1,\dots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\dots,q_k),q} \\ \iff & (\forall q \in Q) : \; a_{q'}(q) = h_\mu(\sigma(t_1,\dots,t_k))_q \\ \iff & q' = q'_{\sigma(t_1,\dots,t_k)}, \end{aligned}$$

which proves the lemma. ∎

Lemma 4.4
If the set Q' of states of $det^p(M)$ is finite, then M and $det^p(M)$ are equivalent bu-w-fta .

Proof:
The proof uses the following statement, which we denote by (*): if Q' is a finite set, then for every tree $t \in T_\Sigma$ and state $q' \in Q'$ it holds that $h_{\mu'}(t)_{q'} = \mathbf{1}$, if $q' = q'_t$, and $h_{\mu'}(t)_{q'} = \mathbf{0}$ otherwise. Let us show by induction on the structure of t that (*) holds. Note that the induction base is covered by the induction step. Let $t = \sigma(t_1, \ldots, t_k)$ for some non-negative integer $k \in \mathbb{N}$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, and trees $t_1, \ldots, t_k \in T_\Sigma$. Then

$$\begin{aligned}
h_{\mu'}(t)_{q'_t} &= \sum_{(q'_1,\ldots,q'_k)\in(Q')^k} h_{\mu'}(t_1)_{q'_1} \odot \cdots \odot h_{\mu'}(t_k)_{q'_k} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'_t} \\
&= h_{\mu'}(t_1)_{q'_{t_1}} \odot \cdots \odot h_{\mu'}(t_k)_{q'_{t_k}} \odot \mu'_k(\sigma)_{(q'_{t_1},\ldots,q'_{t_k}),q'_t} \\
&\qquad \text{(by ind. hyp., } h_{\mu'}(t_i)_{q'_i} = \mathbf{0} \text{ whenever } q'_i \neq q'_{t_i} \text{ for every } i \in [k]) \\
&= \mathbf{1}. \\
&\qquad \text{(by induction hypothesis and Lemma 4.3)}
\end{aligned}$$

Hence $q'_t \in \widetilde{\mu}'(t)$. Since $det^p(M)$ is a deterministic bu-w-fta by assumption (finite) and Observation 4.2 (deterministic), we deduce from Observation 3.5(v) that $\widetilde{\mu}'(t) = q'_t$. Thus by Observation 3.5(i) $h_{\mu'}(t)_{q'} = \mathbf{0}$ for every state $q' \in Q \setminus \{q'_t\}$.

Let us now prove the equivalence of the devices M and $det^p(M)$ provided that the set Q' of states of the constructed device is finite. For every tree $t \in T_\Sigma$ the following holds:

$$\begin{aligned}
(S_{det^p(M)}, t) &= \sum_{q'\in Q'} h_{\mu'}(t)_{q'} \odot \nu'(q') =_{(\text{by } (*))} h_{\mu'}(t)_{q'_t} \odot \nu'(q'_t) \\
&=_{(\text{by } (*))} \mathbf{1} \odot \sum_{q\in Q} h_\mu(t)_q \odot \nu(q) = (S_M, t).
\end{aligned}$$

Thus M and $det^p(M)$ are equivalent bu-w-fta provided the latter device is a finite automaton. ■

Let us now present an example of the extended power set construction.

Example 4.5 ($det^p(M)$ is a finite bu-w-fta)
Consider the bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, which is given by $Q = \{q_\alpha, q_\perp\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$, $\nu(q_\alpha) = 1$, $\nu(q_\perp) = 0$, $\mathcal{A} = Bool$, and

$$\mu_0(\alpha) = \begin{array}{c} q_\alpha \;\; q_\perp \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}, \quad \mu_0(\beta) = \begin{array}{c} q_\alpha \;\; q_\perp \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array}, \quad \mu_2(\sigma) = \begin{array}{cc} q_\alpha \;\; q_\perp & \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{array}{c} (q_\alpha, q_\alpha) \\ (q_\alpha, q_\perp) \\ (q_\perp, q_\alpha) \\ (q_\perp, q_\perp) \end{array} \end{array}.$$

It is easily seen that $h_\mu(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and thus $(S_M, t) = 1$, if the input tree t contains a node labeled with α and $h_\mu(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and hence $(S_M, t) = 0$, if it does not. It follows that the set of states of $det^p(M)$ is given by $Q' = \{q'_\alpha, q'_\beta\}$. Moreover, $\nu'(q'_\alpha) = 1$, $\nu'(q'_\beta) = 0$, and

$$\mu'_0(\alpha) = \begin{array}{c} \begin{array}{cc} q'_\alpha & q'_\beta \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}, \qquad \mu'_0(\beta) = \begin{array}{c} \begin{array}{cc} q'_\alpha & q'_\beta \end{array} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array}, \qquad \mu'_0(\sigma) = \begin{array}{cc} \begin{array}{cc} q'_\alpha & q'_\beta \end{array} & \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{c} (q'_\alpha, q'_\alpha) \\ (q'_\alpha, q'_\beta) \\ (q'_\beta, q'_\alpha) \\ (q'_\beta, q'_\beta) \end{array} \end{array}.$$

Clearly, $det^p(M)$ is a deterministic bu-w-fta. Moreover, a straightforward calculation shows that $h_{\mu'}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and thus $(S_{M'}, t) = 1$, if the input tree t contains a node labeled with α and $h_{\mu'}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and hence $(S_{M'}, t) = 0$, if it does not. Thereby we have shown that the automata M and $det^p(M)$ are equivalent. $\diamond$

Let us now investigate, under which conditions $det^p(M)$ is a finite automaton. We observe that $det^p(M)$ is a (finite) bu-w-fta, if and only if the set $\{h_\mu(t) \mid t \in T_\Sigma\}$ of characteristic vectors is of finite cardinality. In Example 4.5 we have presented a bu-w-fta such that the extended power set construction generates a finite device. Unfortunately there even exists a deterministic bu-w-fta such that the extended power set construction produces an infinite bu-w-ta.

Example 4.6 ($\boldsymbol{det^p(M)}$ is an infinite bu-w-fta)
Consider the deterministic bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, which is given by $Q = \{q\}$, $\Sigma = \{\alpha^{(0)}, \beta^{(0)}, \sigma^{(2)}\}$, $\nu(q) = 0$, $\mathcal{A} = Trop$, and $\mu_0(\alpha) = (1)$, $\mu_0(\beta) = (0)$, and $\mu_2(\sigma) = (0)$. A straightforward inductive proof shows that for every input tree t, $h_\mu(t)_q$ equals the number of nodes of t, which are labeled with α. We observe that for every non-negative integer $n \in \mathbb{N}$ the tree t_n, which is inductively defined by $t_0 = \alpha$ and $t_{i+1} = \sigma(\alpha, t_i)$ for every non-negative integer $i \in \mathbb{N}$, contains precisely $n + 1$ nodes labeled with α. Hence, $\{h_\mu(t) \mid t \in T_\Sigma\}$ is an infinite set and thus $det^p(M)$ is an infinite device. $\diamond$

Let us now consider a bu-w-fta M over a finite semiring. Since A^Q is a finite set and $\{h_\mu(t) \mid t \in T_\Sigma\}$ is a subset of A^Q, $\{h_\mu(t) \mid t \in T_\Sigma\}$ is also a finite set. Hence $det^p(M)$ is a (finite) bu-w-fta. In Lemma 4.7 we extend this result by showing that the determinization of a bu-w-fta over a locally finite semiring generates a finite device.

Lemma 4.7
If $\mathcal{A}$ is a locally finite semiring, then $det^p(M)$ is a (finite) bu-w-fta.

Proof:
Let $A' = \{\mu_k(\sigma)_{(q_1,\ldots,q_k),q} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q_1, \ldots, q_k, q \in Q\}$, which is a finite set. Clearly, $h_\mu(t)_q \in \langle A' \rangle_{\{\oplus,\odot\}}$ for every input tree $t \in T_\Sigma$ and state q. Thus, $\{h_\mu(t) \mid t \in T_\Sigma\}$ is a subset of $(\langle A' \rangle_{\{\oplus,\odot\}})^Q$, which by local finiteness is finite. Hence $\{h_\mu(t) \mid t \in T_\Sigma\}$ is a finite set and thus $det^p(M)$ is a (finite) bu-w-fta. ∎

The following theorem is a consequence of Observation 4.2, Lemma 4.4, and Lemma 4.7. It provides a sufficient condition, under which deterministic and non-deterministic bu-w-fta are equally powerful.

Theorem 4.8 (Sufficient condition for $\mathcal{A}^{d,bu}\langle\langle T_\Sigma\rangle\rangle = \mathcal{A}^{n,bu}\langle\langle T_\Sigma\rangle\rangle$)
If $\mathcal{A}$ is a locally finite semiring, then $\mathcal{A}^{d,bu}\langle\langle T_\Sigma\rangle\rangle = \mathcal{A}^{n,bu}\langle\langle T_\Sigma\rangle\rangle$.

Proof:
The claim follows from Lemma 4.7. ∎

5 Pumping Lemma

In this section we prove a pumping lemma for recognizable tree series. Since there is a one-to-one correspondence between recognizable tree languages and recognizable tree series over the Boolean semiring, we thereby generalize the pumping lemma for classical tree languages, which is (cf. [GS84], Lemma 10.1 of Chapter 2): for every recognizable tree language $L \subseteq T_\Sigma$ there exists an integer $m \in \mathbb{N}$ such that for every tree $t \in L$ with $\text{height}(t) \geq m$ there exist contexts $C, C' \in C_\Sigma(X_1)$ and an input tree $s \in T_\Sigma$ such that

(i) $t = C'[C[s]]$,

(ii) $\text{height}(C) > 1$ (i.e., $C \neq x_1$), and

(iii) $C'[C^n[s]] \in L$ for every non-negative integer $n \in \mathbb{N}$.

Similar to classical theories we first show a pumping lemma for tree series, which are accepted by deterministic bu-w-fta. By applying Theorem 4.8 we obtain a result on recognizable tree series. Recall that in classical theories for every device there exists an equivalent deterministic one, while for bu-w-fta a similar result only holds under the assumption that the underlying semiring is locally finite. Hence, the general version of the pumping lemma, which we prove in this section, assumes a locally finite semiring.

Let us start this section by proving a relationship between the characteristic vectors with respect to a deterministic bu-w-fta of the two trees $s, C[s] \in T_\Sigma$, where the latter tree is obtained from the first one by plugging it into a context $C \in C_\Sigma(X_1)$. Since a "run" on the tree $C[s]$ can be decomposed into a "run" on the tree s and a "run" on the context C, one might assume that $h_\mu(C[s])_q$ is the product of the "run" (if it exists) on the input tree s, which has a weight different from $\mathbf{0}$ and ends up in some state p, and the "run" on the context C, which starts at the variable x_1 in state p and which ends in state q. Indeed, such a result was proven in [Bor03], Theorem 1, for commutative semirings. We prove a similar result without the assumption of $\mathcal{A}$ being commutative. For this purpose let us define the two mappings $a_M, b_M : Q \times Q \times C_\Sigma(X_1) \longrightarrow A$, which are inductively defined for every two states $p, q \in Q$ by

$$a_M(p,q,C) = b_M(p,q,C) = \begin{cases} \mathbf{1} & \text{, if } p = q, \\ \mathbf{0} & \text{, otherwise,} \end{cases}$$

provided that C is the trivial context x_1, and

$$a_M(p,q,C) = \begin{cases} h_\mu(t_1)_{\widetilde{\mu}(t_1)} \odot \cdots \odot h_\mu(t_{i-1})_{\widetilde{\mu}(t_{i-1})} \odot a_M(p,r,C') \\ \quad \text{, if } \widetilde{\mu}(t_j) \in Q \text{ for every } j \in [k] \setminus \{i\} \text{ and exists a } r \in Q \text{ such} \\ \quad \text{ that } r = \widetilde{\mu}(C'[t]) \text{ for every } t \in T_\Sigma \text{ with } p = \widetilde{\mu}(t), \\ \mathbf{0} \quad \text{, otherwise,} \end{cases}$$

$$b_M(p,q,C) = \begin{cases} b_M(p,r,C') \odot h_\mu(t_{i+1})_{\widetilde{\mu}(t_{i+1})} \odot \cdots \odot h_\mu(t_k)_{\widetilde{\mu}(t_k)} \odot \mu_k(\sigma)_{\vec{q},q} \\ \quad \text{, if } \widetilde{\mu}(t_j) \in Q \text{ for every } j \in [k] \setminus \{i\}\text{, exists } r \in Q \text{ such} \\ \quad \text{that } r = \widetilde{\mu}(C'[t]) \text{ for every } t \in T_\Sigma \text{ with } p = \widetilde{\mu}(t) \text{ and} \\ \quad \vec{q} = (\widetilde{\mu}(t_1), \ldots, \widetilde{\mu}(t_{i-1}), r, \widetilde{\mu}(t_{i+1}), \ldots, \widetilde{\mu}(t_k)), \\ \mathbf{0} \quad \text{, otherwise,} \end{cases}$$

if $C = \sigma(t_1, \ldots, t_{i-1}, C', t_{i+1}, \ldots, t_k)$ for some positive integers $k \in \mathbb{N}_+$ and $i \in [k]$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, context $C' \in C_\Sigma(X_1)$, and input trees $t_j \in T_\Sigma$ for every index $j \in [k] \setminus \{i\}$.

Lemma 5.1
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a deterministic bu-w-fta. For every tree $s \in T_\Sigma$ with $\widetilde{\mu}(s) \neq \emptyset$, state $q \in Q$, and context $C \in C_\Sigma(X_1)$ the equation $h_\mu(C[s])_q = a_M(\widetilde{\mu}(s), q, C) \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), q, C)$ holds.

Proof:
The proof is by induction on the length $n \in \mathbb{N}$ of the path from the root of C to the node labeled with the variable x_1 in the context C.

Induction base: Let $n = 0$, i.e., C is the trivial context x_1. Clearly, $h_\mu(C[s])_q = h_\mu(s)_q = a_M(\widetilde{\mu}(s), q, C) \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), q, C)$.

Induction step: Now let $n \geq 1$, i.e., $C = \sigma(t_1, \ldots, t_{i-1}, C', t_{i+1}, \ldots, t_k)$ for some positive integers $k \in \mathbb{N}_+$ and $i \in [k]$, k-ary input symbol $\sigma \in \Sigma^{(k)}$, context $C' \in C_\Sigma(X_1)$, and input trees $t_j \in T_\Sigma$ for every index $j \in [k] \setminus \{i\}$. Let us prove the claimed statement by a case analysis.

Case 1: There exists an index $j \in [k] \setminus \{i\}$ such that $\widetilde{\mu}(t_j) = \emptyset$. Clearly, $\mathbf{0} = a_M(p,q,C) = b_M(p,q,C)$ for every states $p, q \in Q$. By Observation 3.5(iii) also $\widetilde{\mu}(C[s]) = \emptyset$. Moreover, by Observation 3.5(i) $h_\mu(C[s])_q = \mathbf{0} = a_M(\widetilde{\mu}(s), q, C) \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), q, C)$ for every $q \in Q$.

Case 2: We have $\widetilde{\mu}(C'[s]) = \emptyset$. By Observation 3.5(iii) it holds that $\widetilde{\mu}(C[s]) = \emptyset$ and by Observation 3.5(i) we have $h_\mu(C[s])_q = \mathbf{0}$ for every state $q \in Q$. On the other hand, $a_M(\widetilde{\mu}(s), q, C') = \mathbf{0} = b_M(\widetilde{\mu}(s), q, C')$ and thus $h_\mu(C[s])_q = \mathbf{0} = a_M(\widetilde{\mu}(s), q, C) \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), q, C)$ for every state $q \in Q$.

Case 3: The sets $\widetilde{\mu}(C'[s])$ and $\widetilde{\mu}(t_j)$ are singletons for every index $j \in [k] \setminus \{i\}$. In order to shorten notation let $\vec{q} = (\widetilde{\mu}(t_1), \ldots, \widetilde{\mu}(t_{i-1}), \widetilde{\mu}(C'[s]), \widetilde{\mu}(t_{i+1}), \ldots, \widetilde{\mu}(t_k))$. From Observation 3.5(iv) we deduce that $\widetilde{\mu}(C'[s]) = \widetilde{\mu}(C'[t])$ for every tree $t \in T_\Sigma$ with $\widetilde{\mu}(s) = \widetilde{\mu}(t)$, i.e., $a_M(\widetilde{\mu}(s), q, C) = h_\mu(t_1)_{\widetilde{\mu}(t_1)} \odot \cdots \odot h_\mu(t_{i-1})_{\widetilde{\mu}(t_{i-1})} \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[s]), C')$ and $b_M(\widetilde{\mu}(s), q, C) = b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[s]), C') \odot h_\mu(t_{i+1})_{\widetilde{\mu}(t_{i+1})} \odot \cdots \odot h_\mu(t_k)_{\widetilde{\mu}(t_k)} \odot \mu_k(\sigma)_{\vec{q},q}$. Hence, for every state $q \in Q$,

$$\begin{aligned} & h_\mu(C[s])_q \\ = & \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_{i-1})_{q_{i-1}} \odot h_\mu(C'[s])_{q_i} \odot h_\mu(t_{i+1})_{q_{i+1}} \odot \cdots \odot \\ & \qquad \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \end{aligned}$$

$$= h_\mu(t_1)_{\widetilde{\mu}(t_1)} \odot \cdots \odot h_\mu(t_{i-1})_{\widetilde{\mu}(t_{i-1})} \odot h_\mu(C'[s])_{\widetilde{\mu}(C'[s])} \odot h_\mu(t_{i+1})_{\widetilde{\mu}(t_{i+1})} \odot \cdots \odot$$
$$\odot\, h_\mu(t_k)_{\widetilde{\mu}(t_k)} \odot \mu_k(\sigma)_{\vec{q},q}$$
(by Observation 3.5(i) and (v))
$$= h_\mu(t_1)_{\widetilde{\mu}(t_1)} \odot \cdots \odot h_\mu(t_{i-1})_{\widetilde{\mu}(t_{i-1})} \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[s]), C') \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot$$
$$\odot\, b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[s]), C') \odot h_\mu(t_{i+1})_{\widetilde{\mu}(t_{i+1})} \odot \cdots \odot h_\mu(t_k)_{\widetilde{\mu}(t_k)} \odot \mu_k(\sigma)_{\vec{q},q}$$
(by induction hypothesis)
$$= a_M(\widetilde{\mu}(s), q, C) \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), q, C).$$

We note that the above case analysis is complete by Observation 3.5(v). ■

Now we show, how pumping a context is reflected in the weights of the runs.

Lemma 5.2
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a deterministic bu-w-fta, $C \in C_\Sigma(X_1)$ a context, and $s \in T_\Sigma$ a tree such that $\widetilde{\mu}(s) = \widetilde{\mu}(C[s]) \in Q$ is a state. For every non-negative integer $n \in \mathbb{N}$ the equations

(i) $\widetilde{\mu}(s) = \widetilde{\mu}(C^n[s]) \in Q$ *and*

(ii) $h_\mu(C^n[s])_{\widetilde{\mu}(s)} = a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n.$

hold.

Proof:
Claim (i) follows by a repeated application of Corollary 3.6(ii). Let us now prove claim (ii) by induction on $n \in \mathbb{N}$. By Lemma 5.1 the equation $h_\mu(C[t])_q = a_M(\widetilde{\mu}(t), q, C) \odot h_\mu(t)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(t), q, C)$ holds for every state $q \in Q$ and tree $t \in T_\Sigma$ with $\widetilde{\mu}(t) \in Q$.

Induction base: If $n = 0$, then $h_\mu(C^0[s])_{\widetilde{\mu}(s)} = h_\mu(s)_{\widetilde{\mu}(s)} = a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^0 \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^0$.

Induction step: Now assume $n \geq 1$. The following computation proves the claim.

$$h_\mu(C^n[s])_{\widetilde{\mu}(s)}$$
$$= h_\mu(C[C^{n-1}[s]])_{\widetilde{\mu}(s)}$$
$$= a_M(\widetilde{\mu}(C^{n-1}[s]), \widetilde{\mu}(s), C) \odot h_\mu(C^{n-1}[s])_{\widetilde{\mu}(C^{n-1}[s])} \odot b_M(\widetilde{\mu}(C^{n-1}[s]), \widetilde{\mu}(s), C)$$
(by Lemma 5.1)
$$= a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot h_\mu(C^{n-1}[s])_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)$$
(by Claim (i))
$$= a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot \big(a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^{n-1} \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^{n-1}\big) \odot$$
$$\odot\, b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)$$
(by induction hypothesis)
$$= a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n$$
■

The following lemma assumes a decomposition of a given tree $t \in T_\Sigma$ into contexts $C, C' \in C_\Sigma(X_1)$ and a tree $s \in T_\Sigma$ such that there are “runs” on the trees s and $C[s]$ with a non-zero weight, which end up in the same state. In Lemma 5.4 we prove that such a decomposition exists provided that the input tree satisfies an assumption, which is related the height of the input tree.

Lemma 5.3
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a deterministic bu-w-fta, $C, C' \in C_\Sigma(X_1)$ contexts, and $s \in T_\Sigma$ an input tree such that $\widetilde{\mu}(s) = \widetilde{\mu}(C[s]) \in Q$ and $\widetilde{\mu}(C'[C[s]]) \in Q$ are states. For every non-negative integer $n \in \mathbb{N}$ it holds that

$$\begin{aligned} h_\mu(C'[C^n[s]])_{\widetilde{\mu}(C'[C^n[s]])} \quad = \quad & a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot \\ & \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C'). \end{aligned}$$

Proof:
First we observe that by a repeated application of Corollary 3.6(i) it holds that $\widetilde{\mu}(C^n[s]) = \widetilde{\mu}(C[s]) \in Q$ and thus $\widetilde{\mu}(C'[C^n[s]]) = \widetilde{\mu}(C'[C[s]]) \in Q$. Let us denote this last equation, which shows that the claimed equations are well-defined, by (*). Moreover,

$$\begin{aligned} & h_\mu(C'[C^n[s]])_{\widetilde{\mu}(C'[C^n[s]])} \\ = \quad & a_M(\widetilde{\mu}(C^n[s]), \widetilde{\mu}(C'[C^n[s]]), C') \odot h_\mu(C^n[s])_{\widetilde{\mu}(C^n[s])} \odot b_M(\widetilde{\mu}(C^n[s]), \widetilde{\mu}(C'[C^n[s]]), C') \\ & \quad \text{(by Lemma 5.1)} \\ = \quad & a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot h_\mu(C^n[s])_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \\ & \quad \text{(by (*))} \\ = \quad & a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot \\ & \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C'). \\ & \quad \text{(by Lemma 5.2(ii))} \end{aligned}$$

■

Next we show that a decomposition of an input tree $t = C'[C[s]]$ into contexts $C, C' \in C_\Sigma(X_1)$ and a tree $s \in T_\Sigma$ such that $\widetilde{\mu}(s) = \widetilde{\mu}(C[s])$ exists provided there exists a path w in t with more nodes than the given bu-w-fta has states. This can be ensured by requiring $\text{length}(w) \geq \text{card}(Q)$.

Lemma 5.4
Let M be a deterministic bu-w-fta and $t \in T_\Sigma$ an input tree such that $\widetilde{\mu}(t) \in Q$. For every path $w = w_1 \ldots w_{\text{length}(w)} \in \text{paths}(t)$ of t with $w_1, \ldots, w_{\text{length}(w)} \in \mathbb{N}$ and $\text{length}(w) \geq \text{card}(Q)$ there exist indices $i, j \in [0, \text{length}(w)]$ such that $i < j$ and $\widetilde{\mu}(t/(w_1 \ldots w_i)) = \widetilde{\mu}(t/(w_1 \ldots w_j)) \in Q$.

Proof:
Let us start with two observations. For every index $i \in [0, \text{length}(w)]$ there exists a context $C \in C_\Sigma(X_1)$ such that $t = C[t/(w_1 \ldots w_i)]$. By Corollary 3.6(ii) and since $\widetilde{\mu}(t) \neq \emptyset$, also $\widetilde{\mu}(t/(w_1 \ldots w_i)) \neq \emptyset$. Moreover, by Observation 3.5(v) it holds that $\widetilde{\mu}(t/(w_1 \ldots w_i)) \in Q$. The claim now follows from the pigeon hole principle. ■

Before we state a smooth version of the pumping lemma for recognizable tree series, let us present a more powerful version of the pumping lemma. A last intermediate result is proven in Lemma 5.5.

Lemma 5.5
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a deterministic bu-w-fta. For every input tree $t \in T_\Sigma$ such that $\widetilde{\mu}(t) \in Q$ is a state and for every path $w = w_1 \dots w_l \in \mathrm{paths}(t)$ of t with $w_1, \dots, w_{\mathrm{length}(w)} \in \mathbb{N}$ and $\mathrm{length}(w) = l \geq \mathrm{card}(Q)$ there exist contexts $C, C' \in C_\Sigma(X_1)$ and an input tree $s \in T_\Sigma$ such that

(i) $t = C'[C[s]]$,

(ii) there exist indices $i, j \in [0, l]$ with $i < j$ and $l - i \leq \mathrm{card}(Q)$ such that $s = t/(w_1 \dots w_j)$ and $C[s] = t/w_1 \dots w_i$,

(iii) $\widetilde{\mu}(C^n[s]) = \widetilde{\mu}(s) \in Q$ and $\widetilde{\mu}(C'[C^n[s]]) = \widetilde{\mu}(t) \in Q$ for every non-negative integer $n \in \mathbb{N}$, and

(iv) for every non-negative integer $n \in \mathbb{N}$ it holds that

$$
\begin{aligned}
&h_\mu(C'[C^n[s]])_{\widetilde{\mu}(C'[C^n[s]])}\\
&= a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot\\
&\quad \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C').
\end{aligned}
$$

Proof:
Clearly, $w_{l-\mathrm{card}(Q)+1} \dots w_l$ is a path of length $\mathrm{card}(Q)$ of the tree $t/(w_1 \dots w_{l-\mathrm{card}(Q)})$. Since $\widetilde{\mu}(t) \in Q$, we deduce from Corollary 3.6(i) that $\widetilde{\mu}(t/w_1 \dots w_{l-\mathrm{card}(Q)}) \neq \emptyset$ and hence by Lemma 5.4 there exist indices $i', j' \in [0, \mathrm{card}(Q)]$ with $i' < j'$ such that

$$
\begin{aligned}
\widetilde{\mu}(t/(w_1 \dots w_{l-\mathrm{card}(Q)+i'})) &= \widetilde{\mu}((t/(w_1 \dots w_{l-\mathrm{card}(Q)}))/(w_{l-\mathrm{card}(Q)+1} \dots w_{l-\mathrm{card}(Q)+i'}))\\
&= \widetilde{\mu}((t/(w_1 \dots w_{l-\mathrm{card}(Q)}))/(w_{l-\mathrm{card}(Q)+1} \dots w_{l-\mathrm{card}(Q)+j'}))\\
&= \widetilde{\mu}(t/(w_1 \dots w_{l-\mathrm{card}(Q)+j'})).
\end{aligned}
$$

Let us denote this statement by (*). We set $i = l - \mathrm{card}(Q) + i'$, $j = l - \mathrm{card}(Q) + j'$, and $s = t/(w_1 \dots w_j)$. Moreover, we choose the contexts $C, C' \in C_\Sigma(X_1)$ such that $C[s] = t/w_1 \dots w_i$ and $C'[C[s]] = t$. Then (i) is satisfied. Furthermore, $i, j \in [0, l]$ and $i < j$ by the choice of the indices i' and j' and since $l \geq \mathrm{card}(Q)$. Hence (ii) holds. From Corollary 3.6(ii), the assumption $\widetilde{\mu}(t) \in Q$, and Condition (i) we deduce that $\widetilde{\mu}(s) \in Q$. Thus, also applying (*),

$$\widetilde{\mu}(s) = \widetilde{\mu}(t/(w_1 \dots w_j)) = \widetilde{\mu}(t/(w_1 \dots w_i)) = \widetilde{\mu}(C[s]) \in Q$$

by (*). Hence, by Lemma 5.2(ii) and Corollary 3.6(ii) it holds that $\widetilde{\mu}(C^n[s]) = \widetilde{\mu}(s) \in Q$ and $\widetilde{\mu}(C'[C^n[s]]) = \widetilde{\mu}(t) \in Q$ for every non-negative integer $n \in \mathbb{N}$, which shows Property (iii). Statement (iv) follows from Property (iii) and Lemma 5.3. ∎

Theorem 5.6 (Pumping Lemma (i))
Let (a) $\mathcal{A}$ be a semiring and $S \in \mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$ or (b) $\mathcal{A}$ be a locally finite semiring and $S \in \mathcal{A}^{n,bu}\langle\langle T_\Sigma \rangle\rangle$. There exists a non-negative integer $m \in \mathbb{N}$ such that for every tree $t \in \mathrm{supp}(S)$ and path $w = w_1 \dots w_l \in path(t)$ of t with $w_1, \dots, w_{\mathrm{length}(w)} \in \mathbb{N}$ and $\mathrm{length}(w) = l \geq m$, there exist contexts $C, C' \in C_\Sigma(X_1)$, a tree $s \in T_\Sigma$, and semiring elements $a, a', b, b', c \in A$ such that

(i) $t = C'[C[s]]$,

(ii) there exist indices $i, j \in [0, l]$ with $i < j$ and $l - i \leq m$ such that $s = t/w_1 \dots w_j$ and $C[s] = t/w_1 \dots w_i$, and

(iii) $(S, C'[C^n[s]]) = a' \odot a^n \odot c \odot b^n \odot b'$ for every non-negative integer $n \in \mathbb{N}$.

Moreover, if $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ is a deterministic bu-w-fta, which accepts S, then

(iv) m can be chosen to be the number of states of M,

(v) $\widetilde{\mu}(C^n[s]) = \widetilde{\mu}(s) \in Q$ and $\widetilde{\mu}(C'[C^n[s]]) = \widetilde{\mu}(t) \in Q$ for every non-negative integer $n \in \mathbb{N}$, and

(vi) the semiring elements $a, a', b, b', c \in A$ can be set to

$$\begin{array}{rclrcl} a &=& a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C), & a' &=& a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C'), \\ b &=& b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C), & b' &=& b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot \nu(C'[C[s]]), \\ c &=& h_\mu(s)_{\widetilde{\mu}(s)}. \end{array}$$

Proof:
By Theorem 4.8 we have $\mathcal{A}^{n,bu}\langle\langle T_\Sigma \rangle\rangle = \mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$, if the underlying semiring is locally finite. Hence it suffices to consider a tree series $S \in \mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$, which is accepted by a deterministic device. Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ such a bu-w-fta. We set $m = \mathrm{card}(Q)$. Furthermore, let $t \in \mathrm{supp}(S)$ be an input tree and $w = w_1 \dots w_l \in \mathrm{paths}(t)$ a path of t of length $l \geq m$. In particular, $\widetilde{\mu}(t) \in Q$ by Observation 3.5(i), (v), and (vi). By Lemma 5.5 there exist contexts $C, C' \in C_\Sigma(X_1)$ and a tree $s \in T_\Sigma$ such that Properties (i), (ii), (iv), and (v) are satisfied. Moreover, by Observation 3.5(vi) it holds for every non-negative integer $n \in \mathbb{N}$ that

$$\begin{array}{rcl} (S, C'[C^n[s]]) &=& h_\mu(C'[C^n[s]])_{\widetilde{\mu}(C'[C^n[s]])} \odot \nu(\widetilde{\mu}(C'[C^n[s]])) \\ &=& h_\mu(C'[C^n[s]])_{\widetilde{\mu}(C'[C^n[s]])} \odot \nu(\widetilde{\mu}(C'[C[s]])) \\ &=& a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot \\ && \odot\, b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot \nu(\widetilde{\mu}(C'[C[s]])), \end{array}$$

which proves the remaining Statements (iii) and (vi). ∎

Let us now present a smoother, but less powerful version of the pumping lemma:

Lemma 5.7
Let $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ be a deterministic bu-w-fta. For every input tree $t \in T_\Sigma$ with $\mathrm{height}(t) \geq \mathrm{card}(Q) + 1$ and $\widetilde{\mu}(t) \in Q$ there exist contexts $C, C' \in C_\Sigma(X_1)$, a tree $s \in T_\Sigma$, and semiring elements $a, a', b, b' \in A$ such that

(i) $t = C'[C[s]]$,

(ii) $\mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1$ and $C \neq x_1$, and

(iii) $\widetilde{\mu}(C^n[s]) = \widetilde{\mu}(s) \in Q$ and $\widetilde{\mu}(C'[C^n[s]]) = \widetilde{\mu}(t) \in Q$ for every non-negative integer $n \in \mathbb{N}$,

(iv) for every non-negative integer $n \in \mathbb{N}$ it holds that

$$\begin{array}{l} h_\mu(C'[C^n[s]])_{\widetilde{\mu}(C'[C^n[s]])} \\ = a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)^n \odot \\ \quad \odot\, b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C'). \end{array}$$

Proof:
Let $t \in T_\Sigma$ be an input tree of height $\geq \mathrm{card}(Q)+1$ such that $\widetilde{\mu}(t) \in Q$ and $w = w_1 \ldots w_l \in \mathrm{paths}(t)$ be a longest path of t, where $w_1, \ldots, w_l \in \mathbb{N}$. By Observation 2.2 it holds that $l \geq \mathrm{card}(Q)$. Hence by Lemma 5.5 there exist contexts $C, C' \in C_\Sigma(X_1)$, a tree $s \in T_\Sigma$ and semiring elements $a, a', b, b', c \in A$ such that (i), (iii), and (iv) are satisfied. Moreover, there exist indices $i, j \in [0, l]$ with $i < j$ and $l - i \leq \mathrm{card}(Q)$ such that $s = t/(w_1 \ldots w_j)$ and $C[s] = t/(w_1 \ldots w_i)$. We observe that $w_{i+1} \ldots w_l$ is a longest path of the tree $C[s]$ of length $l - i$ and thus $\mathrm{height}(C[s]) = l - i + 1 \leq \mathrm{card}(Q) + 1$ by Observation 2.2. Furthermore, C is not the trivial context x_1 by $i < j$, $s = t/(w_1 \ldots w_j)$, and $C[s] = t/(w_1 \ldots w_i)$. Thus also claim (ii) holds. ■

Corollary 5.8 (Pumping Lemma (ii))
Let (a) $\mathcal{A}$ be a semiring and $S \in \mathcal{A}^{d,bu}\langle\langle T_\Sigma \rangle\rangle$ or (b) $\mathcal{A}$ be a locally finite semiring and $S \in \mathcal{A}^{n,bu}\langle\langle T_\Sigma \rangle\rangle$. There exists a non-negative integer $m \in \mathbb{N}$ such that for every tree $t \in \mathrm{supp}(S)$ with $\mathrm{height}(t) \geq m+1$ there exist contexts $C, C' \in C_\Sigma(X_1)$, a tree $s \in T_\Sigma$, and semiring elements $a, a', b, b', c \in A$ such that

(i) $t = C'[C[s]]$,

(ii) $\mathrm{height}(C[s]) \leq m+1$ *and* $C \neq x_1$, *and*

(iii) $(S, C'[C^n[s]]) = a' \odot a^n \odot c \odot b^n \odot b'$ *for every non-negative integer* $n \in \mathbb{N}$.

Furthermore, if $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$ is a deterministic bu-w-fta, which accepts S, then

(iv) m can be chosen to be the number of states of M.

(v) $\widetilde{\mu}(C^n[s]) = \widetilde{\mu}(s) \in Q$ and $\widetilde{\mu}(C'[C^n[s]]) = \widetilde{\mu}(t) \in Q$ for every non-negative integer $n \in \mathbb{N}$,

(vi) the semiring elements $a, a', b, b', c \in A$ can be set to

$$\begin{array}{rclcrcl} a & = & a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C), & & a' & = & a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C'), \\ b & = & b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C), & & b' & = & b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[C[s]]), C') \odot \nu(C'[C[s]]) \\ c & = & h_\mu(s)_{\widetilde{\mu}(s)}. & & & & \end{array}$$

Proof:
The proof is very similar to the proof of Theorem 5.6. We note that it uses Lemma 5.7 rather than Lemma 5.5 and leave it to the reader. ■

We note that the two versions of the pumping lemma do not say anything about the support of the considdered tree series unless the underlying semiring is zero-divisor free.

Similar to classical theories the pumping lemma can be applied for proving, that a tree series is not recognizable. Let us give an example for this application. We prove that the tree series, which maps every input tree to its height, is not recognized by a deterministic device over the Arctic semiring. This result is surprising by two reasons: first, the height of a tree is defined in terms of max and $+$, which are the operations of the Arctic semiring. Second, the support of this tree series is T_Σ, which is recognizable by a classical one-state (and hence deterministic) bottom-up finite tree automata. Thus, associating weights to a recognizable tree language, might destroy recognizability.

Example 5.9 ($\sum_{t \in T_\Sigma}$ height(t) t not acceptable by deterministic bu-w-fta)
Let $(S,t) = \mathrm{height}(t)$ for every input tree $t \in T_\Sigma$, where the underlying ranked alphabet is given by $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and the underlying algebraic structure is the Arctic semiring. We show by contradiction that S is not recognizable by a deterministic bu-w-fta. Assume the converse. By Theorem 5.6 there exists a non-negative integer $m \in \mathbb{N}$ such that for every tree $t \in \mathrm{supp}(S)$ and path $w = w_1 \dots w_l \in \mathrm{paths}(t)$ of t with $w_1, \dots, w_l \in \mathbb{N}$ and $\mathrm{length}(w) = l \geq m$ there exist contexts $C, C' \in C_\Sigma(X_1)$, a tree $s \in T_\Sigma$, and semiring elements $a, a', b, b', c \in A$ such that (i) $t = C'[C[s]]$, (ii) there exist indices $i, j \in [0, l]$ with $i < j$ and $l - i \leq m$ such that $s = t/w_1 \dots w_j$ and $C[s] = t/w_1 \dots w_i$, and (iii) $\mathrm{height}(C'[C^n[s]]) = a' + a^n + c + b^n + b'$ for every non-negative integer $n \in \mathbb{N}$. Let us consider the particular tree $t = \sigma(t_1, t_2)$, where $t_1, t_2 \in T_\Sigma$ are trees of height $m + 1$ and $2m + 1$, respectively. Hence, $(S, t) = \mathrm{height}(t) = 2m + 2$. Moreover, there exists a path $w = w_1 w_2 \dots w_{m+1}$ with $w_1 = 1$ of the input tree t of length $l = m+1$ such that $w_2 \dots w_{m+1}$ is a (longest) path of the subtree t_1. Hence, there exists a decomposition of t along the path w satisfying (i), (ii), and (iii). Since w is a path of length $l = m + 1$, we deduce from (ii) that $i \geq 1$. Hence $w_1 \dots w_i \neq \varepsilon$ and $C[s]$ is a subtree of t_1 (cf. Figure 1). We now show that

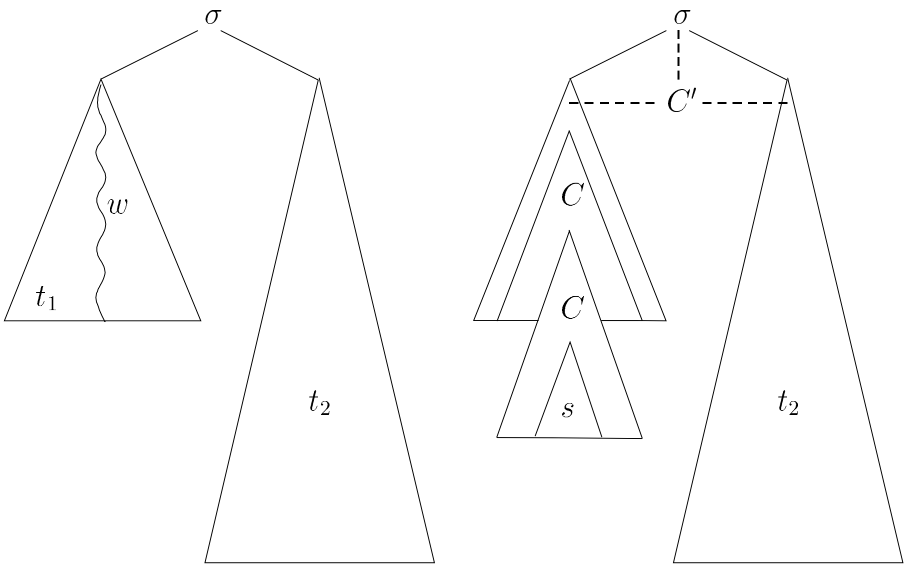

Figure 1: The trees $t = C'[C[s]]$ and $C'[C^2[s]]$.

pumping the context C once, does not increase the height of the tree: to do so let us consider the paths of the pumped tree $C'[C^2[s]]$. Every such path $w \in \mathrm{paths}(C'[C^2[s]])$ is either a path of the given tree $t = C'[C[s]]$ (i.e., its length is $\leq 2m+1$) or $w = 1.w_2 \dots w_i \dots w_j \overline{w}$ for some path $\overline{w}$ of the tree $C[s]$. Since $C[s]$ is a subtree of t_1, we deduce from Observation 2.2 that $\mathrm{length}(\overline{w}) \leq \mathrm{height}(t_1) - 1 = m$. Moreover, $j \leq m + 1$ by (ii), thus $\mathrm{length}(w) \leq j + \mathrm{length}(\overline{w}) \leq 2m + 1$. Hence every path of the pumped tree $C'[C^2[s]]$ is of length $\leq 2m + 1$. Thus $\mathrm{height}(C'[C[s]]) = \mathrm{height}(C'[C^2[s]]) = 2m + 2$. From (iii) we now deduce that $a' + a + c + b + b' = 2m + 2 = a' + 2a + c + 2b + b'$, i.e., $a = b = 0$. But then $\mathrm{height}(C'[C^n[s]]) = a' + n \cdot a + c + n \cdot b + b' = 2m + 2$ for every non-negative integer $n \in \mathbb{N}$, which means that pumping the context C arbitrary often does not increase the height of the pumped tree. Hence C must be the trivial context x_1. But this contradicts to (ii), since $t/w_1 \ldots w_j = s = C[s] = t/w_1 \ldots w_i$ and thus $i = j$. Hence S is not recognizable by a deterministic bu-w-fta over the Arctic semiring. $\diamond$

6 Decidability

In this section we investigate the following decidability problems for a given recognizable tree series $S \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$:

- **Constant-on-its-support problem:** Is the tree series S constant on its support?
- **Constant problem:** Is S a constant tree series?
- **Boolean problem:** Is S a boolean tree series?
- **Emptiness problem:** Is the support of S the empty set?
- **Finiteness problem:** Is the support of S finite?

Note that throughout this section decidable stands for effectively decidable and computable stands for effectively computable (in the sense of Church's hypothesis, cf. [HU79]). It turns out that the aforementioned five problems are decidable, provided that the given tree series S is recognized by a deterministic bu-w-fta over a commutative semiring. The decidability of the Finiteness problem is proven for zero-divisor free and commutative semirings. When talking about decidability problems, it is a common assumption to consider finitely represented input. So we do in the present paper. A recognizable tree series can be finitely represented by a bu-w-fta, which accepts this tree series. Hence we assume for the rest of this section, that besides the tree series S also a deterministic bu-w-fta $M = (Q, \Sigma, \nu, \mathcal{A}, \mu)$, which accepts S, is given. Note that by Lemma 4.7 all the results of this section also hold, if M is a non-deterministic bu-w-fta over a locally finite and commutative semiring $\mathcal{A}$. We also assume throughout this section that $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a computable semiring.

First we prove the decidability of both constant problems by applying the pumping lemma. The decidability of the Emptiness and Boolean problems are straightforward consequences of the decidability of the Constant-on-its-support problem. Finally we show that the Finiteness problem is decidable by reducing it to the Emptiness problem.

In order to shorten notation let us define the set P, in which we collect all those states, which are reachable by "small" trees s and there exists a context C, the height of which is bounded by $2 \cdot \mathrm{card}(Q) - 1$ such that $C[s] \in \mathrm{supp}(S)$:

$$\begin{aligned} P \;=\; \{\widetilde{\mu}(s) \in Q \mid (\exists s \in T_\Sigma), (\exists C \in C_\Sigma(X_1)) :\\ \mathrm{height}(s) \leq \mathrm{card}(Q), \mathrm{height}(C) \leq 2 \cdot \mathrm{card}(Q) - 1, C[s] \in \mathrm{supp}(S)\}. \end{aligned}$$

Observation 6.1
The set P is computable. $\diamond$

In the following lemma we show that dropping the requirement on the height of the context C in the definition of P does not effect the set P.

Lemma 6.2
If $\mathcal{A}$ is a commutative semiring, then it holds that $P = \{\widetilde{\mu}(s) \in Q \mid (\exists s \in T_\Sigma), (\exists C \in C_\Sigma(X_1)) : \text{height}(s) \leq \text{card}(Q), C[s] \in \text{supp}(S)\}$.

Proof:
Let $P' = \{\widetilde{\mu}(s) \in Q \mid (\exists s \in T_\Sigma), (\exists C \in C_\Sigma(X_1)) : \text{height}(s) \leq \text{card}(Q), C[s] \in \text{supp}(S)\}$. Clearly, $P \subseteq P'$. It remains to show $P' \subseteq P$. Let $p \in P'$, i.e., there exist a tree $s \in T_\Sigma$ of height $\leq \text{card}(Q)$ and a context $C \in C_\Sigma(X_1)$ such that $p = \widetilde{\mu}(s)$ and $C[s] \in \text{supp}(S)$. We show by contradiction that then also $p \in P$. Assume that $p \notin P$. Let s be fixed and assume without loss of generality that C is chosen such that $C[s]$ is of minimal size satisfying $C[s] \in \text{supp}(S)$. We set $t = C[s]$. Clearly, $\text{height}(C) \geq 2 \cdot \text{card}(Q)$. In order to apply the pumping lemma (Theorem 5.6) we specify a path $w = w_1 \dots w_l \in \text{paths}(C)$, where $w_1, \dots, w_l \in \mathbb{N}$, as follows (cf. Figure 2). Let $w' \in \text{paths}(C)$ denote the path such that the node of C at w' is labeled with x_1.

(α) If $\text{length}(w') \geq \text{card}(Q) + 1$, then set $w = w'$.

(β) If $\text{length}(w') \leq \text{card}(Q)$, then let w be an arbitrary path of length $2 \cdot \text{card}(Q) + 1$, which exists by $\text{height}(C) \geq 2 \cdot \text{card}(Q)$ and Observation 2.2.

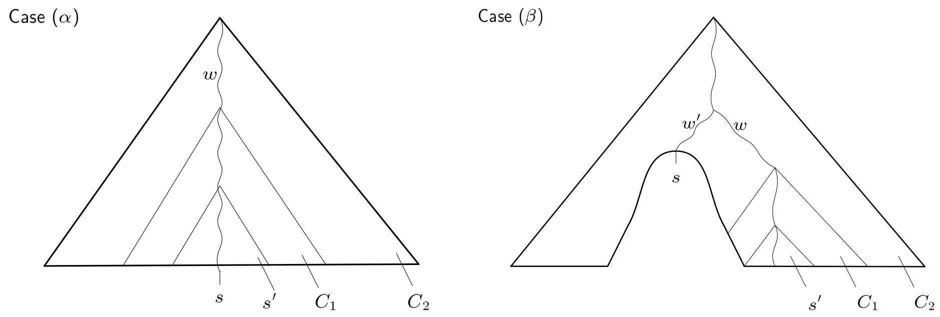

Figure 2: Decomposition of $C[s]$ in Cases (α) and (β).

By the pumping lemma (Theorem 5.6) there exists a decomposition of $t = C_2[C_1[s']]$ along the path w into contexts $C_1, C_2 \in C_\Sigma(X_1)$ and a tree $s' \in T_\Sigma$ such that

(i) there exist indices $i, j \in [0, l]$ with $i < j$ and $l - i \leq \text{card}(Q)$ such that $s' = t/w_1 \dots w_j$ and $C_1[s'] = t/w_1 \dots w_i$, and

(ii) there exist semiring elements $a, a', b, b', c \in A$ such that $(S, C_2[C_1^n[s]]) = a' \odot a^n \odot c \odot b^n \odot b'$ for every non-negative integer $n \in \mathbb{N}$.

From the choice of w it follows that s is a subtree of $C_2[s']$, which is formally proven as follows:

(α) By (i) s is a subtree of s' and thus s is a subtree of $C_2[s']$.

(β) By (i) it holds that $i \geq \text{card}(Q) + 1$ and thus $w_1 \dots w_i$ is not a prefix of w'. Clearly, $\text{paths}(C_2) = \{w \in \text{paths}(C) \mid w_1 \dots w_i \text{ is not a prefix of } w\}$, from which we deduce that $w' \in \text{paths}(C_2)$. Apparently s is a subtree of $C_2[s']$.

Thus there exists a context $C_3 \in C_\Sigma(X_1)$ such that $C_3[s] = C_2[s']$. Moreover, by (i) $C_1 \neq x_1$ and thus $\text{size}(C_3[s]) = \text{size}(C_2[s']) < \text{size}(C_2[C_1[s']]) = \text{size}(t)$. Let us denote this last statement by (‡). From (ii) we deduce that $(S,t) = a \odot b \odot (S, C_2[s']) = a \odot b \odot (S, C_3[s])$. We assumed that $t \in \mathrm{supp}(S)$, thus $(S,t) \neq \mathbf{0}$ and by the aforementioned equation $(S, C_3[s]) \neq \mathbf{0}$, i.e., $C_3[s] \in \mathrm{supp}(S)$. Since the context C was chosen such that $t = C[s]$ is of minimal size violating the statement of the lemma and by (‡) $\mathrm{size}(C_3[s]) < \mathrm{size}(t)$ it holds that $\widetilde{\mu}(s) \in P$. This contradicts to the assumption $\widetilde{\mu}(s) \notin P$. Hence $P = P'$. ■

We note that by a very similar proof also the assumption in P on the height of s can be dropped. Since this statement is not needed in the present paper, we leave the proof of this claim to the reader.

The decision procedures will be in terms of two sets B_1 and B_2. Roughly speaking, in B_1 we collect the costs of "small" contexts, which can be pumped, while B_2 is the set of all "small" trees. Formally,

$$\begin{aligned} B_1 &= \{a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \in A \mid (\exists s \in T_\Sigma), (\exists C \in C_\Sigma(X_1)) : \\ &\qquad \mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1, \widetilde{\mu}(s) = \widetilde{\mu}(C[s]) \in P\}, \\ B_2 &= \{s \in T_\Sigma \mid \mathrm{height}(s) \leq \mathrm{card}(Q)\}. \end{aligned}$$

Lemma 6.3
Let $\mathcal{A}$ be a commutative semiring and $d \in A$ a semiring element. For every input tree $t \in \mathrm{supp}(S)$ it holds that $(S,t) = d$, if and only if

- *(CPS1) for every semiring element $b \in B_1$ it holds that $b \odot d \in \{\mathbf{0}, d\}$ and*
- *(CPS2) for every tree $s \in B_2$ it holds that $(S,s) \in \{\mathbf{0}, d\}$.*

Proof:
First assume that $(S,t) = d$ for every input tree $t \in \mathrm{supp}(S)$. In particular, $d \neq \mathbf{0}$. We show that (CPS1) and (CPS2) are satisfied. Clearly, (CPS2) holds. Let us now prove that also (CPS1) is fulfilled. Let $b \in B_1$. By the definition of B_1 there exist a tree $s \in T_\Sigma$ and a context $C \in C_\Sigma(X_1)$ with $\mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1$ such that $b = a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)$ and $\widetilde{\mu}(s) = \widetilde{\mu}(C[s]) \in P$.

(α) If $C = x_1$ is the trivial context, then $a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) = b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) = \mathbf{1}$, thus $b = \mathbf{1}$ and $d \odot b = d \in \{\mathbf{0}, d\}$.

(β) If $C \neq x_1$, then $\mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1$ implies $\mathrm{height}(s) \leq \mathrm{card}(Q)$. Hence by Lemma 6.2 there exists a context $C' \in C_\Sigma(X_1)$ such that $C'[s] \in \mathrm{supp}(S)$ and thus $(S, C'[s]) = d$. Thus

$$\begin{aligned} & b \odot d \\ &= a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot (S, C'[s]) \\ &= a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot h_\mu(C'[s])_{\widetilde{\mu}(C'[s])} \odot \nu(\widetilde{\mu}(C'[s])) \\ &\quad \text{(by Observation 3.5(vi))} \\ &= a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot a_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[s]), C') \odot \\ &\odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C'[s]), C') \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot \nu(\widetilde{\mu}(C'[s])) \\ &\quad \text{(by Lemma 5.1)} \end{aligned}$$

$$= a_M(\widetilde{\mu}(s), \widetilde{\mu}(C[s]), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(C[s]), C) \odot a_M(\widetilde{\mu}(C[s]), \widetilde{\mu}(C'[C[s]]), C') \odot$$
$$\odot\, b_M(\widetilde{\mu}(C[s]), \widetilde{\mu}(C'[C[s]]), C') \odot h_\mu(s)_{\widetilde{\mu}(s)} \odot \nu(\widetilde{\mu}(C'[C[s]]))$$
(by $\widetilde{\mu}(s) = \widetilde{\mu}(C[s])$ and Corollary 3.6)
$$= h_\mu(C'[C[s]])_{\widetilde{\mu}(C'[C[s]])} \odot \nu(\widetilde{\mu}(C'[C[s]]))$$
(by Lemma 5.1)
$$= (S, C'[C[s]])$$
(by Observation 3.5(vi))
$$\in \{\mathbf{0}, d\}.$$

Hence also (CPS1) is satisfied.

Conversely assume that (CPS1) and (CPS2) hold. We show that $(S,t) = d$ for every input tree $t \in \mathrm{supp}(S)$. Without loss of generality we assume that $d \neq \mathbf{0}$. In fact, if $d = \mathbf{0}$, then it holds that $S = \widetilde{\mathbf{0}}$, i.e., $\mathrm{supp}(S) = \emptyset$ and thus $(S,t) = d$ for every input tree $t \in \mathrm{supp}(S)$ and semiring element $d \in A$. The prove of the claimed implication is by contradiction. In assuming the converse, let $t \in \mathrm{supp}(S)$ be a tree with $(S,t) \neq d$. Without loss of generality let $t \in \mathrm{supp}(S)$ be of minimal size violating $(S,t) = d$. Since by (CPS2) $(S,s) \in \{\mathbf{0}, d\}$ for every tree $s \in T_\Sigma$ of height $\leq \mathrm{card}(Q)$, we deduce $\mathrm{height}(t) \geq \mathrm{card}(Q) + 1$. By the pumping lemma (Corollary 5.8) there exists a decomposition $t = C'[C[s]]$ for some tree $s \in T_\Sigma$ and contexts $C, C' \in C_\Sigma(X_1)$ with $C \neq x_1$, $\mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1$, and $\widetilde{\mu}(s) = \widetilde{\mu}(C[s])$. Moreover, there exist semiring elements $a, a', b, b', c \in A$ such that $(S,t) = a' \odot a \odot c \odot b \odot b'$ and $(S, C'[s]) = a' \odot c \odot b'$. Consequently, $(S,t) = a \odot b \odot (S, C'[s])$, which we denote by (†). Also by the pumping lemma (Corollary 5.8) a and b can be set to

$$a = a_M(\widetilde{\mu}(s), \widetilde{\mu}(C[s]), C) \quad \text{and} \quad b = b_M(\widetilde{\mu}(s), \widetilde{\mu}(C[s]), C).$$

Since $t = C'[C[s]] \in \mathrm{supp}(S)$ and the fact that $\mathrm{height}(s) < \mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1$, we deduce from Lemma 6.2 that $\widetilde{\mu}(s) \in P$ and thus $a \odot b \in B_1$. Let us denote this last statement by (‡). Furthermore, $\mathrm{size}(C'[s]) < \mathrm{size}(t)$ by C not being the trivial context x_1. From this and the assumption that $t \in \mathrm{supp}(S)$ is of minimal size violating $(S,t) = d$ we deduce that $(S, C'[s]) \in \{\mathbf{0}, d\}$. Hence by (†), (‡), and Condition (CPS1) it follows that $(S,t) = a \odot b \odot (S, C'[s]) \in \{\mathbf{0}, d\}$, which contradicts to the assumptions $t \in \mathrm{supp}(S)$ and $(S,t) \neq d$. Hence $(S,t) = d$ for every tree $t \in \mathrm{supp}(S)$. ■

In order to state the decidability of the Constant-on-its-support problem it remains to show that B_1 and B_2 are computable.

Lemma 6.4
The sets B_1 and B_2 are computable.

Proof:
The claim is trivial for the set B_2. Let us show that also B_1 is computable. Since P is a computable set by Observation 6.1, also $B = \{(s, C) \in T_\Sigma \times C_\Sigma(X_1) \mid \mathrm{height}(C[s]) \leq \mathrm{card}(Q) + 1, \widetilde{\mu}(s) = \widetilde{\mu}(C[s]) \in P\}$ is computable. It remains to show that $a_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C) \odot b_M(\widetilde{\mu}(s), \widetilde{\mu}(s), C)$ is computable for every pair $(s, C) \in B$. Therefore it suffices to show that $a_M(p, q, C)$ and $b_M(p, q, C)$ are computable for every two states $p, q \in Q$ and context

$C \in C_\Sigma(X_1)$ of height $\leq \operatorname{card}(Q)+1$. We show the claim for $a_M(p,q,C)$ and note that the proof of $b_M(p,q,C)$ being computable is very similar. The prove of the computability of $a_M(p,q,C)$ is by induction on the structure of C:

Induction base: If C is the trivial context x_1, the claim is trivial.

Induction step: Now let $C = \sigma(t_1,\ldots,t_{i-1},C',t_{i+1},\ldots,t_k)$ for some positive integers $k \in \mathbb{N}_+$ and $i \in [k]$, k-ary input symbol $\sigma \in \Sigma^{(k)}$ and trees $t_j \in T_\Sigma$ for every index $j \in [k] \setminus \{i\}$. From $\operatorname{height}(C) \leq \operatorname{card}(Q)+1$ we deduce that, for every index $j \in [k] \setminus \{i\}$ it holds that $\operatorname{height}(t_j) \leq \operatorname{card}(Q)$. Hence the set $\widetilde{\mu}(t_j)$ is computable and thus it is decidable whether $\widetilde{\mu}(t_j) \in Q$ for every index $j \in [k] \setminus \{i\}$. Next we show that it is decidable whether there exists a state $r \in Q$ such that $r = \widetilde{\mu}(C'[t])$ for every tree $t \in T_\Sigma$ with $p = \widetilde{\mu}(t)$. We also show that this state r is computable, if it exists. It holds for every state $r \in Q$ that:

	For every tree $t \in T_\Sigma$ with $p = \widetilde{\mu}(t)$ it holds that $r = \widetilde{\mu}(C'[t])$.
$\Longleftrightarrow^{(*)}$	There exists a tree $t \in T_\Sigma$ with $p = \widetilde{\mu}(t)$ and $r = \widetilde{\mu}(C'[t])$.
$\Longleftrightarrow^{(**)}$	There exists a tree $t \in T_\Sigma$ of height $\leq \operatorname{card}(Q)$ with $p = \widetilde{\mu}(t)$ and $r = \widetilde{\mu}(C'[t])$.

Note that $\Rightarrow^{(*)}$ follows from the definition of P, $\Leftarrow^{(*)}$ is a consequence of Corollary 3.6(ii), $\Leftarrow^{(**)}$ follows by a repeated application of Corollary 5.8(v) with $n = 0$, and $\Leftarrow^{(**)}$ trivially holds. Clearly, the latter statement is decidable. From the above equivalences we deduce that, if r has the property that for every tree $t \in T_\Sigma$ with $p = \widetilde{\mu}(t)$ it holds that $r = \widetilde{\mu}(C'[t])$, then the state r is computable. Let us denote this last statement by (†). Moreover, by the above equivalences it is decidable whether there exists a state $r \in Q$ such that $r = \widetilde{\mu}(C'[t])$ for every tree $t \in T_\Sigma$ with $p = \widetilde{\mu}(t)$. Consequently it is decidability whether $\widetilde{\mu}(t_j) \in Q$ for every index $j \in [k] \setminus \{i\}$ and there exists a state $r \in Q$ such that $r = \widetilde{\mu}(C'[t])$ for every tree $t \in T_\Sigma$ with $p = \widetilde{\mu}(t)$. It remains to show that $h_\mu(t_1)_{\widetilde{\mu}(t_1)} \odot \cdots \odot h_\mu(t_{i-1})_{\widetilde{\mu}(t_{i-1})} \odot a_M(p,r,C')$ is computable. By induction hypothesis $a_M(p,r,C')$ is computable and, since $\operatorname{height}(C) \leq \operatorname{card}(Q)+1$ and thus $\operatorname{height}(t_j) \leq \operatorname{card}(Q)$ for every index $j \in [k] \setminus \{i\}$, also $h_\mu(t_j)$ and $\widetilde{\mu}(t_j)$ are computable. Hence $h_\mu(t_j)_{\widetilde{\mu}(t_j)}$ and $h_\mu(t_1)_{\widetilde{\mu}(t_1)} \odot \cdots \odot h_\mu(t_{i-1})_{\widetilde{\mu}(t_{i-1})} \odot a_M(p,r,C')$ are computable, which completes the proof. ■

Corollary 6.5 (Constant-on-its-support problem)
It is decidable whether the tree series S is constant on its support.

Proof:
Conditions (CPS1) and (CPS2) of Lemma 6.3 are decidable, since the sets B_1 and B_2 are computable by Lemma 6.4. It remains to find the appropriate semiring element $d \in A$, which we define according to Property (CPS2) as follows:

- If $\operatorname{card}(B_2 \setminus \{\mathbf{0}\}) = 0$, then $d \in A$ can be arbitrarily chosen.
- If $\operatorname{card}(B_2 \setminus \{\mathbf{0}\}) = 1$, then we set d such that $\{d\} = B_2 \setminus \{\mathbf{0}\}$.
- If $\operatorname{card}(B_2 \setminus \{\mathbf{0}\}) \geq 2$, then S does not have the desired property. ■

Let us now investigate the Constant problem, i.e., is it decidable whether $S = \widetilde{d}$ for some semiring element $d \in A$? We could prove this by deciding whether $(S,t) = d$ for every input tree $t \in \operatorname{supp}(S)$ (Constant-on-its-support problem) and $\operatorname{supp}(S) = T_\Sigma$. The latter decision problem could be solved as in classical theories (cf. [GS84], Theorem 10.3 of Chapter II, also using the well-known fact that T_Σ is a recognizable tree language) provided that the underlying semiring is zero-divisor free. We would like to avoid this additional assumption. Therefore we present two properties (CP1) and (CP2), which are equivalent to $S = \widetilde{d}$ for a given semiring element $d \in A$ (cf. Lemma 6.6), and then show that this semiring element d can be derived from the Properties (CP1) and (CP2) (cf. Corollary 6.7).

Lemma 6.6
Let $\mathcal{A}$ be a commutative semiring and $d \in A$ a semiring element. It holds that $S = \widetilde{d}$, if and only if

(CP1) for every semiring element $b \in B_1$ it holds that $b \odot d = d$ and

(CP2) for every tree $s \in B_2$ it holds that $(S, s) = d$.

Proof:
The proof is very similar to the proof of Lemma 6.3 and hence left to the reader. ∎

Corollary 6.7 (Constant problem)
If $\mathcal{A}$ is a commutative semiring, then it is decidable whether S is a constant tree series.

Proof:
The proof of Corollary 6.5 can be taken over word by word. ∎

From the results, which we have obtained in this section so far, we now derive several additional decidability results. A straightforward consequence of the decidability of the Constant problems is that the Emptiness problem is decidable.

Corollary 6.8 (Emptiness problem)
If $\mathcal{A}$ is a commutative semiring, then it is decidable whether $\mathrm{supp}(S) = \emptyset$.

Proof:
The claim follows from Lemma 6.3 with $d = \mathbf{0}$. ∎

Another interesting class of tree series are the boolean tree series. Is it decidable whether a recognizable tree series is boolean? The answer is yes, if the underlying semiring is commutative. The decision procedure uses the fact, that a tree series is boolean, if and only if $(S, t) = \mathbf{1}$ for every input tree $t \in \mathrm{supp}(S)$. We thereby reduce the decidability problem of S being boolean to the Constant-on-its-support problem.

Corollary 6.9 (Boolean problem)
If $\mathcal{A}$ is a commutative semiring, then it is decidable whether S is a boolean tree series.

Proof:
The claim follows from Lemma 6.3 with $d = \mathbf{1}$. ∎

Let us conclude this section by proving that the Finiteness problem is decidable provided that the underlying semiring is commutative and zero-divisor free. From the pumping lemma (cf. Corollary 5.8) and zero-divisor freeness we deduce that, if there exists a tree $t \in \mathrm{supp}(S)$ of height $\geq \mathrm{card}(Q)+1$, then $t = C'[C[s]]$ can be decomposed into contexts $C, C' \in C_\Sigma(X_1)$ and a tree $s \in T_\Sigma$ such that pumping the context C produces infinitely many trees $C'[C^n[s]] \in$

$\text{supp}(S)$, $n \in \mathbb{N}$. Hence, $\text{supp}(S)$ is finite, if and only if every tree $t \in \text{supp}(S)$ is of height $\leq \text{card}(Q)$. Let us therefore consider the tree series S', which is defined for every input tree $t \in T_\Sigma$ by

$$(S', t) = \begin{cases} (S, t) & \text{, if height}(t) \geq \text{card}(Q) + 1, \\ \mathbf{0} & \text{, otherwise.} \end{cases}$$

Thus S has finite support, if and only if the support of S' is the empty set. In order to apply the decidability of the Emptiness problem it remains to show that the tree series S' is recognizable by a deterministic bu-w-fta. For this purpose we define for every semiring $\mathcal{A}$ and non-negative integer $n \in \mathbb{N}$ the tree series $S_n^{\text{height}} \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$, which is defined for every input tree $t \in T_\Sigma$ by

$$(S_n^{\text{height}}, t) = \begin{cases} \mathbf{1} & \text{, if height}(t) \geq n + 1, \\ \mathbf{0} & \text{, otherwise.} \end{cases}$$

We observe that $S' = S \odot S_{\text{card}(Q)}^{\text{height}}$, where $\odot$ denotes the Hadamard product of the tree series S and $S_{\text{card}(Q)}^{\text{height}}$. By Corollary 3.9(ii) it remains to show that the tree series $S_{\text{card}(Q)}^{\text{height}}$ is recognized by some deterministic bu-w-fta in order to prove that S' is accepted by a deterministic bu-w-fta.

Lemma 6.10

Let $\mathcal{A}$ be a semiring and $n \in \mathbb{N}$ a non-negative integer. The tree series S_n^{height} is recognized by the deterministic bu-w-fta $M_n^{\text{height}} = (Q', \Sigma, \nu', \mathcal{A}, \mu')$, which is defined by $Q' = \{1, \ldots, n, n+1\}$, $\nu'(n+1) = \mathbf{1}$, $\nu(q') = \mathbf{0}$ for every state $q' \in Q' \setminus \{n+1\}$, and, for every non-negative integer $k \in \mathbb{N}$, input symbols $\alpha \in \Sigma^{(0)}$ and $\sigma \in \Sigma^{(k)}$, and states $q_1', \ldots, q_k', q' \in Q'$,

$$\mu_0'(\alpha)_{(),q'} = \begin{cases} \mathbf{1} & \text{, if } q' = 1, \\ \mathbf{0} & \text{, otherwise,} \end{cases}$$

$$\mu_k'(\sigma)_{(q_1',\ldots,q_k'),q'} = \begin{cases} \mathbf{1} & \text{, if } (q' = \max\{q_i' \mid i \in [k]\} + 1) \text{ or} \\ & \quad (q' = n+1 \text{ and } (\exists i \in [k]) : q_i' = n+1), \\ \mathbf{0} & \text{, otherwise.} \end{cases}$$

Proof (Sketch):

First we observe that M' is a deterministic bu-w-fta. A straightforward inductive proof shows that

(i) for every input tree $t \in T_\Sigma$ of height $\leq n$ it holds that $h_{\mu'}(t)_{\text{height}(t)} = \mathbf{1}$ and $h_{\mu'}(t)_{q'} = \mathbf{0}$ for every state $q' \in Q' \setminus \{\text{height}(t)\}$.

(ii) for every input tree $t \in T_\Sigma$ of height $> n$ it holds that $h_{\mu'}(t)_{n+1} = \mathbf{1}$ and $h_{\mu'}(t)_{q'} = \mathbf{0}$ for every state $q' \in Q' \setminus \{n+1\}$.

Hence, $(S_{M'}, t) = \sum_{q' \in Q'} h_{\mu'}(t)_{q'} \odot \nu'(q') = h_{\mu'}(t)_{n+1} = (S_n^{\text{height}}, t)$ for every input tree $t \in T_\Sigma$ and thus the tree series S_n^{height} is recognized by the deterministic bu-w-fta M'. ■

Theorem 6.11 (Finiteness problem)

If $\mathcal{A}$ is a zero-divisor free and commutative semiring, then it is decidable whether $\text{supp}(S)$ *is a finite set.*

Proof:
First we show that S has finite support, if and only if every input tree $t \in \mathrm{supp}(S)$ is of height $\leq \mathrm{card}(Q)$. We denote this statement by (*). Let us prove the non-trivial implication of (*) by contradiction, i.e., let $\mathrm{supp}(S)$ be a finite set and assume that there exists an input tree $t \in \mathrm{supp}(S)$ with $\mathrm{height}(t) \geq \mathrm{card}(Q) + 1$. By Corollary 5.8 there exists a decomposition $t = C'[C[s]]$ of t into contexts $C', C \in C_\Sigma(X_1)$ and a tree $s \in T_\Sigma$ such that C is not the trivial context x_1 and there exist semiring elements $a, a', b, b', c \in A$ with $(S, C'[C^n[s]]) = a' \odot a^n \odot c \odot b^n \odot b'$ for every non-negative integer $n \in \mathbb{N}$. In particular, $a \neq \mathbf{0}$, $b \neq \mathbf{0}$, and $(S, C'[C^n[s]]) = a^{n-1} \odot b^{n-1} \odot (S, t)$ for every positive integer $n \in \mathbb{N}_+$. By zero-divisor freeness and $t \in \mathrm{supp}(S)$ we have $(S, C'[C^n[s]]) \neq \mathbf{0}$ and thus $C'[C^n[s]] \in \mathrm{supp}(S)$ for every positive integer $n \in \mathbb{N}_+$, which either contradicts to $C \neq x_1$ or to1 $\mathrm{supp}(S)$ is a finite set.

By (*) the statement $\mathrm{supp}(S)$ is a finite set is equivalent to $S' = \widetilde{\mathbf{0}}$, i.e., $\mathrm{supp}(S') = \emptyset$. Let us denote this statement by (†). In order to apply the decidability result for the Emptiness problem (cf. Corollary 6.8, it remains to show that S' is recognized by a deterministic bu-w-fta. Since $S' = S \odot S^{\mathrm{height}}_{\mathrm{card}(Q)}$, we deduce from Lemmata 3.8 and 6.10 that the bu-w-fta $M \times M^{\mathrm{height}}_{\mathrm{card}(Q)}$ accepts S'. Moreover, since M and $M^{\mathrm{height}}_{\mathrm{card}(Q)}$ are deterministic bu-w-fta, also $M \times M^{\mathrm{height}}_{\mathrm{card}(Q)}$ is deterministic. Hence Corollary 6.8 is applicable and thus, by (†), it is decidable whether $\mathrm{supp}(S)$ is a finite set. ∎

Acknowledgment

The author wishes to thank the anonymous referees for helpful comments.

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