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DETERMINIZATION OF FINITE STATE WEIGHTED TREE AUTOMATA

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ABSTRACT

We investigate the determinization of nondeterministic bottom-up/top-down finite state weighted tree automata over some semiring A and compare the resulting four classes of formal tree series with each other. In particular, for every bottom-up finite state weighted tree automaton over some semifield (i. e., a semiring which contains a multiplicative inverse for every nonzero element) we construct a deterministic bottom-up weighted tree automaton. This construction is called partial determinization, because it may lead to an automaton with an infinite set of states. We prove sufficient conditions under which the partial determinization produces a bottom-up *finite state* weighted tree automaton which is equivalent to the original one. Using this partial determinization we generalize well known theorems on classes of tree languages (cf. [14] Chapter II, Theorems 2.6 and 2.10, Example 2.11), viz if A is a commutative and locally finite semifield, then (i) nondeterministic bottom-up, (ii) deterministic bottom-up, and (iii) nondeterministic top-down finite state weighted tree automata are equally powerful. Moreover, if the input alphabet is not trivial, then deterministic top-down finite state weighted tree automata are strictly less powerful than the aforementioned classes.

Keywords: Weighted tree automata, determinization, recognizable tree series

1. Introduction

Finite state weighted tree automata have been investigated in, e. g., [24, 18, 10]. Conceptually, they are obtained from finite state tree automata by replacing the Boolean semiring by an arbitrary semiring. Let us first make some remarks on finite state tree automata and the classes of tree languages accepted by them, before we discuss the concept of a determinization algorithm and its instances for various automata models.

The concept of finite state tree automata is a generalization of the concept of finite state (string) automata in the sense that tree automata accept trees rather than strings. According to the way in which a given input tree t is traversed, there are bottom-up tree automata [8, 25] (or: frontier-to-root automata) in which t is traversed by starting from its leaves and proceeding towards the root, and top-down tree automata [21, 19] (or: root-to-frontier automata) in which t is traversed by starting from its root and proceeding towards the leaves. For more details on finite state tree automata we refer the reader to [14, 15]. Orthogonal to the distinction between bottom-up and top-down, finite state tree automata can be nondeterministic or deterministic. These two distinction criteria give rise to four classes of tree languages:

- the class $\mathcal{L}^{n,bu}(T_\Sigma)$ of tree languages accepted by nondeterministic bottom-up tree automata,
- the class $\mathcal{L}^{d,bu}(T_\Sigma)$ of tree languages accepted by deterministic bottom-up tree automata,
- the class $\mathcal{L}^{n,td}(T_\Sigma)$ of tree languages accepted by nondeterministic top-down tree automata, and
- the class $\mathcal{L}^{d,td}(T_\Sigma)$ of tree languages accepted by deterministic top-down tree automata.

It is well known that

$$\mathcal{L}^{d,td}(T_\Sigma) \subsetneq \mathcal{L}^{n,td}(T_\Sigma) = \mathcal{L}^{n,bu}(T_\Sigma) = \mathcal{L}^{d,bu}(T_\Sigma)$$

(cf. e. g., [14] Chapter II, Theorems 2.6 and 2.10, Example 2.11). In particular, the inclusion $\mathcal{L}^{n,bu}(T_\Sigma) \subseteq \mathcal{L}^{d,bu}(T_\Sigma)$ is based on a determinization called power set construction.

In general, a device (e. g., automaton or transducer) is determinizable if there is an equivalent deterministic device. A determinization takes a nondeterministic device and constructs an equivalent deterministic device of the same sort. A determinization is partial if there are determinizable devices for which the algorithm does not construct an equivalent deterministic device. Since the determinization of finite state weighted tree automata is the main topic of the present paper, let us now briefly review what is known about determinization of finite state automata (for short: fsa) and generalizations of fsa (thereby also recalling the concepts of finite state weighted string/tree automata in some more detail).

The first systematic presentation of fsa is [22] (also cf. [11]). An fsa is a 5-tuple $M = (Q, \Sigma, I, F, \mu)$, where Q, I, F, and Σ are finite sets, viz the sets of all states, initial states, final states, and input symbols, respectively, and $\mu : \Sigma \to \mathbb{B}^{Q \times Q}$ is the transition function (where $\mathbb{B}$ is the Boolean semiring). The fsa M accepts words $w \in \Sigma^*$ over Σ. Hence its semantics is a subset $L_M \subseteq \Sigma^*$. One of the first goals in the theory of fsa was to determinize nondeterministic automata. An automaton is deterministic if, for every state $q \in Q$ and for every symbol $\sigma \in \Sigma$, there is at most one transition from q consuming a σ, i. e., there is at most one state $q' \in Q$ such that $\mu(\sigma)_{q,q'} \neq 0$. Here the determinization is a power set construction (also called subset construction, cf. [11]), because the new set of states is the power set of the original state set.

One possibility of generalizing fsa is to consider a ranked alphabet Σ rather than a string alphabet. This leads to trees and finite state tree automata (for short fta) which we have already considered above. Let us now go into more details. A bu-fta is a 4-tuple $M = (Q, \Sigma, F, \mu)$, where Q and F are the same as in an fsa, Σ is a finite ranked alphabet, and $\mu = (\mu_k \mid k \geq 0)$ is the family of transition functions, where $\mu_k : \Sigma^{(k)} \to \mathbb{B}^{Q^k \times Q}$ for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$. Note that the initial states for a leaf $\alpha \in \Sigma^{(0)}$ are all those states $q \in Q$ satisfying $\mu_0(\alpha)_{(),q} \neq 0$. The concept of td-fta is defined in a very similar way. Instead of a set of final states we deal with initial states, and the transition matrices are of type $\mu_k : \Sigma^{(k)} \to \mathbb{B}^{Q \times Q^k}$. Both, bu-fta and td-fta accept trees, i. e., their semantics are subsets $L_M \subseteq T_\Sigma$. It turns out that the power set construction can be extended to a determinization of bu-fta. (cf. [26, 7]). On the other hand, there exist nondeterministic td-fta such that there is no equivalent deterministic td-fta (cf. Theorem 2.4).

Another way of extending fsa is to consider string-to-string transducers (for short sst, cf. [16, 17, 11, 1]) rather than acceptors. To be more precise, we enrich an fsa by an output alphabet Δ and consider the 6-tuple $M = (Q, \Sigma, \Delta, I, F, \mu)$, where Q, I, F, and Σ are the same as in an fsa. Moreover, Δ is the finite set of output symbols, and $\mu : \Sigma \to \mathbb{B}\langle\langle\Delta^*\rangle\rangle^{Q \times Q}$ is the transition function, where $\mathbb{B}\langle\langle\Delta^*\rangle\rangle$ is the set of all formal power series $S : \Delta^* \to \mathbb{B}$. The semantics of an sst is a partial function $\tau_M : \Sigma^* \to \mathbb{B}\langle\langle\Delta^*\rangle\rangle$. The power set construction also applies to string-to-string transducers which was proven by [6]. Here the problem of handling the output occurs which is solved by considering longest common prefixes. We note that the determinization of sst is partial.

A third way of generalizing fsa is to enrich the automaton by weights which leads to the concept of finite state weighted automata (for short fwa, also string-to-weight transducer, cf. [23]). Roughly speaking, in an fwa every transition has a weight taken from some semiring $(A, \oplus, \odot, 0, 1)$. Along an accepting run for a word w, the weights (or: costs) are multiplied and if there are more accepting runs for w, then the costs of these runs are summed up. More precisely, an fwa is a 6-tuple $M = (Q, \Sigma, I, F, A, \mu)$, where again Q, I, and F are the finite sets of all, initial, and final states, respectively, Σ is an alphabet, A is a semiring, and $\mu : \Sigma \to A^{Q \times Q}$ is the mapping which maps every input symbol to its transition matrix. An fwa accepts words over the input alphabet Σ and outputs costs in the semiring A; hence its semantics is a formal power series $S_M \in A\langle\langle\Sigma^*\rangle\rangle$. Recently [20] and [4] presented a partial determinization for fwa over semifields. This construction can be regarded as the natural extension of the power set construction known from the theory of fsa. However, this construction is a partial determinization.

In the present paper we consider finite state weighted tree automata (for short w-fta). Such automata can be seen as a generalization of fta and fwa. As for fta we have to distinguish between bottom-up finite state weighted tree automata (bu-w-fta, cf. [24, 10]) and top-down finite state weighted tree automata (td-w-fta, cf. [18]). We define a bu-w-fta to be a 5-tuple $M = (Q, \Sigma, Q_d, A, \mu)$, where Q and Q_d are the finite sets of all states and of final states, respectively. Σ is a finite ranked alphabet of input symbols, A is a semiring, and $\mu = (\mu_k \mid k \geq 0)$, where $\mu_k : \Sigma^{(k)} \to A^{Q^k \times Q}$. Again, the concept of td-w-fta is very similar, the one and only difference is the form of the transition matrix which is $\mu_k : \Sigma^{(k)} \to A^{Q \times Q^k}$. Every family $(\mu_k \mid k \geq 0)$ induces a family $(\overline{\mu_k(\sigma)} \mid k \geq 0,\ \sigma \in \Sigma^{(k)})$ of operations on A^Q and thus A^Q becomes a Σ-algebra. Then if $h_\mu : T_\Sigma \to A^Q$ is the initial Σ-homomorphism (where T_Σ represents the initial Σ-algebra), the semantics of M is defined by $(S_M, t) = \sum_{q \in Q_d} h_\mu(t)_q$ for every $t \in T_\Sigma$; in particular, $S_M \in A\langle\langle T_\Sigma\rangle\rangle$.

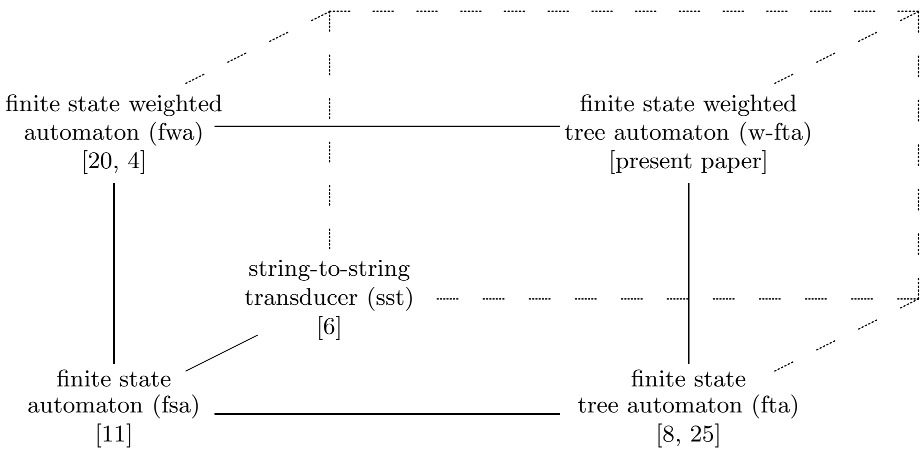

Figure 1: Determinization of finite state automata and of generalizations

Figure 1 summarizes results on the determinization of fsa and of various generalizations of fsa where moving north, east, and north-east means to add weights, to generalize from strings to trees, and to add output, respectively. Table 1 gives an overview of the type of the transition function and the semantics of the discussed concepts.

concept	type of transition function	type of semantics
finite state automaton (fsa)	$\mu : \Sigma \to \mathbb{B}^{Q\times Q}$	$L_M \subseteq \Sigma^*$
finite state tree automaton		
bottom-up (bu-fta)	$\mu_k : \Sigma^{(k)} \to \mathbb{B}^{Q^k\times Q}$	$L_M \subseteq T_\Sigma$
top-down (td-fta)	$\mu_k : \Sigma^{(k)} \to \mathbb{B}^{Q\times Q^k}$	$L_M \subseteq T_\Sigma$
string-to-string transducer (sst)	$\mu : \Sigma \to \mathbb{B}\langle\langle\Delta^*\rangle\rangle^{Q\times Q}$	$\tau_M : \Sigma^* \to \mathbb{B}\langle\langle\Delta^*\rangle\rangle$
finite state weighted automaton (fwa)	$\mu : \Sigma \to A^{Q\times Q}$	$S_M \in A\langle\langle\Sigma^*\rangle\rangle$
finite state weighted tree automaton (w-fta)		
bottom-up (bu-w-fta)	$\mu_k : \Sigma^{(k)} \to A^{Q^k\times Q}$	$S_M \in A\langle\langle T_\Sigma\rangle\rangle$
top-down (td-w-fta)	$\mu_k : \Sigma^{(k)} \to A^{Q\times Q^k}$	$S_M \in A\langle\langle T_\Sigma\rangle\rangle$

Table 1: Type of transition functions and of semantics for fsa and for its generalizations

Here we investigate the partial determinization of bu-w-fta. Recall that bu-w-fta generalize fsa and bu-fta and that the power set construction is applicable to both of these concepts. Now one might ask whether the power set construction is also applicable to bu-w-fta? The answer is no: Consider the bu-w-fta $M_1 = (Q, \Sigma, Q_d^{(1)}, A, \mu)$ and $M_2 = (Q, \Sigma, Q_d^{(2)}, A, \mu)$, where $Q = \{q_1, q_2\}$, $\Sigma = \{\alpha^{(0)}\}$, $Q_d^{(1)} = \{q_1\}$, $Q_d^{(2)} = \{q_1, q_2\}$, A is an arbitrary semiring, and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} a & b \end{pmatrix} \end{array},$$

where $a, b \in A$, $b \neq 0$. Clearly α is accepted by M_1 with costs a and by M_2 with costs $a \oplus b$. The power set construction generates the bu-w-fta $M_1' = (Q', \Sigma, Q'^{(1)}_d, A, \mu')$ and $M_2' = (Q', \Sigma, Q'^{(2)}_d, A, \mu')$, respectively, where $Q' = \{\emptyset, \{q_1\}, \{q_2\}, \{q_1, q_2\}\}$, $Q'^{(1)}_d = \{\{q_1\}, \{q_1, q_2\}\}$, $Q'^{(2)}_d = \{\{q_1\}, \{q_2\}, \{q_1, q_2\}\}$, and

$$\mu_0'(\alpha) = \begin{array}{c} \begin{array}{cccc} \emptyset & \{q_1\} & \{q_2\} & \{q_1, q_2\} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & a \oplus b \end{pmatrix} \end{array}.$$

But now α is accepted by M_1' and M_2' with costs $a \oplus b$, i. e., M_1 and M_1' are not equivalent. On the other hand, M_2 and M_2' are equivalent. Thus the determinization of bu-w-fta has to be more elaborated than the power set construction. The states of the determinization which is presented in this paper, are equivalence classes. Roughly speaking, the input trees t and t' are equivalent if they have the same state-coefficient behavior, i. e., if $h_\mu(t) = h_\mu(t')$. Then the weight of a transition on a k-ary input symbol σ is calculated by dividing the cost of the current input subtree $\sigma(t_1, \ldots, t_k)$ by the costs for accepting the subtrees $t_1, \ldots, t_k$, where the costs of $\sigma(t_1, \ldots, t_k)$ and $t_1, \ldots, t_k$ can be retrieved from the current states. This requires that A is a semifield, i. e., a semiring which contains a multiplicative inverse for every nonzero element.

One could have also considered bu-w-fta with final weights. This would be natural in the sense that it is a generalization of fwa (cf. [23]) and it reflects the formal frame work of [2]. We expect that the determinization algorithm of [20] can be generalized in a natural way to a determinization algorithm for bu-w-fta with final weights. On the other hand, neither the R-cost automata of [24] nor the bottom-up weighted tree automata of [10] have final weights. The same holds for all the tree series transducer models investigated in [12, 13] (note that a bottom-up weighted tree automaton can be considered as a particular bottom-up tree series transducer which computes a partial identity). Finally, we conjecture that deterministic bu-w-fta with final weights are strictly more powerful than deterministic bu-w-fta without final weights. For this, consider, e. g., the tree series

$$\varphi = \sum_{t \in T_\Sigma} \left(2^{\sharp_\alpha(t)} \cdot 3^{\sharp_\sigma(t)} \cdot 5\right) t$$

over the semifield $(\mathbb{R} \cup \{+\infty\}, +, \cdot, ^{-1}, 0, 1)$ and the ranked alphabet $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$; $\sharp_\alpha(t)$ denotes the number of occurrences of α in t, and similarly $\sharp_\sigma(t)$ is defined. Then φ can be accepted by a deterministic bu-w-fta with final weights, because the factor 5 can be handled as final weight. However we claim that φ can not be accepted by a deterministic bu-w-fta without final weights, because the factor 5 can neither be handled at the end of the computation (no final weights) nor at the beginning of the computation (there may be more than one occurrence of α). Thus we feel enough motivation to investigate bu-w-fta without final weights in this paper.

This paper is organized as follows: In Section 2 we recall a few algebraic definitions and well known results. In Section 3 we introduce the concept of w-fta. We also compare bu-w-fta with Seidl's R-cost automata and td-w-fta with Kuich's A'-tree automata. Section 4 deals with the determinization of bu-w-fta. First we introduce a general version and then we present two instances of it. In Section 5 we consider these two instances for bu-w-fta over the Boolean semiring and compare them with the power set construction of bu-fta. Finally we compare in Section 6 the four classes of deterministic/nondeterministic bu-w-fta and td-w-fta which leads to a generalization of well known theorems on tree languages (cf. Theorem 2.4). In the appendix we prove the idempotency of a second instance of our partial determinization.

2. Preliminaries

2.1. General Notions and Trees

The set of nonnegative integers is denoted by $\mathbb{N}$, i. e., $\mathbb{N} = \{0, 1, 2, \ldots\}$. For a set H, its cardinality and its power set are denoted by $card(H)$ and $\mathfrak{P}(H)$. The empty set is denoted by $\emptyset$. A *ranked alphabet* is a tuple (Σ, rk) where Σ is a finite set and $rk : \Sigma \to \mathbb{N}$ is the rank function. For every $k \geq 0$, we define $\Sigma^{(k)} = \{\sigma \in \Sigma \mid rk(\sigma) = k\}$, the set of all symbols of Σ which have rank k. An element $\sigma \in \Sigma$ is also written as $\sigma^{(k)}$, where $k = rk(\sigma)$.

In the rest of this paper let Σ be a ranked alphabet. Moreover, let H be a set disjoint with Σ. The set of trees over Σ indexed by H, denoted by $T_\Sigma(H)$, is the smallest set $T_\Sigma(H) \subseteq (\Sigma \cup H \cup \{(,),,\})^*$ such that (i) $H \cup \Sigma^{(0)} \subseteq T_\Sigma(H)$ and (ii) if $k \geq 1$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma(H)$, then $\sigma(s_1, \ldots, s_k) \in T_\Sigma(H)$. The set $T_\Sigma(\emptyset)$ is denoted by T_Σ. Every subset $L \subseteq T_\Sigma$ is called a (*formal*) *tree language* (*over* Σ).

If $t = \sigma(t_1, \ldots, t_k) \in T_\Sigma$ for some $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k \in T_\Sigma$, then σ is the *root of* t. All symbols α occurring in t with rank 0 are called the *leaves of* t. The height of trees is defined to be the function $height : T_\Sigma \to \mathbb{N}$, where $height(\sigma(t_1, \ldots, t_k)) = 1 + \max\{height(t_i) \mid 1 \leq i \leq k\}$. Moreover, the *size* of trees is a function $size : T_\Sigma \to \mathbb{N}$ which is defined for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ by $size(\sigma(t_1, \ldots, t_k)) = 1 + \sum_{1 \leq i \leq k} size(t_i)$.

2.2. Finite Tree Automata

In this section we recall the definition of a bu-fta (cf. [14, 15]).

Definition 2.1 *A* bottom-up finite state tree automaton (for short bu-fta) *is a tuple $M = (Q, \Sigma, \delta, Q_d)$, where Q is a finite set (of states), Σ is a ranked alphabet (of input symbols), $\delta = \{\delta_\sigma\}_{k \geq 0, \sigma \in \Sigma^{(k)}}$ is a family of mappings $\delta_\sigma : Q^k \to \mathfrak{P}(Q)$, and $Q_d \subseteq Q$ is a set (of final states). M is* deterministic *if, for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q$, $\delta_\sigma(q_1, \ldots, q_k)$ is either the empty set or a singleton.*

The family δ of mappings can be extended in a very natural way to obtain a mapping $\hat{\delta} : T_\Sigma \to \mathfrak{P}(Q)$ which is inductively defined for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ by $\hat{\delta}(\sigma(t_1, \ldots, t_k)) = \bigcup\{\delta_\sigma(q_1, \ldots, q_k) \mid q_1 \in \hat{\delta}(t_1), \ldots, q_k \in \hat{\delta}(t_k)\}$. The *tree language recognized by* M is $L(M) = \{t \in T_\Sigma \mid \hat{\delta}(t) \cap Q_d \neq \emptyset\}$. The classes of all tree languages over Σ which are recognized by bu-fta and by deterministic bu-fta are denoted by $\mathcal{L}^{n,bu}(T_\Sigma)$ and $\mathcal{L}^{d,bu}(T_\Sigma)$, respectively.

The *power set construction* of bu-fta is defined as follows: For a given bu-fta $M = (Q, \Sigma, \delta, Q_d)$, we construct the deterministic bu-fta $det(M) = (Q', \Sigma, \delta', Q'_d)$ by $Q' = \mathfrak{P}(Q)$, $Q'_d = \{\tilde{Q} \in Q' \mid \tilde{Q} \cap Q_d \neq \emptyset\}$, and, for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $Q_1, \ldots, Q_k \in Q'$, we define ${\delta'}_\sigma(Q_1, \ldots, Q_k) = \bigcup\{\delta_\sigma(q_1, \ldots, q_k) \mid q_1 \in Q_1, \ldots, q_k \in Q_k\}$. The bu-fta $det(M)$ is also called the determinization of M. In [14] (Chapter II, Theorem 2.6) a proof of the equivalence of M and $det(M)$ can be found. An intermediate result which we will apply later is the following.

Lemma 2.2 *Let $M = (Q, \Sigma, \delta, Q_d)$ be a bu-fta and $det(M) = (Q', \Sigma, \delta', Q'_d)$ be its determinization. Then, for every $t \in T_\Sigma$, we have $\hat{\delta}'(t) = \hat{\delta}(t)$.*

The top-down case is defined very similar.

Definition 2.3 *A* top-down finite state tree automaton (for short td-fta) *is a tuple $M = (Q, \Sigma, \delta, Q_d)$, where Q is a finite set (of states), Σ is a ranked alphabet (of input symbols), $\delta = \{\delta_\sigma\}_{k \geq 0, \sigma \in \Sigma^{(k)}}$ is a family of mappings $\delta_\sigma : Q \to \mathfrak{P}(Q^k)$, and $Q_d \subseteq Q$ is a set (of initial states). M is* deterministic *if Q_d is a singleton and, for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q \in Q$, the set $\delta_\sigma(q)$ is either the empty set or a singleton.*

Also here the family δ can be extended to a mapping $\hat{\delta} : T_\Sigma \to \mathfrak{P}(Q)$ which is inductively defined for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ by $\hat{\delta}(\sigma(t_1, \ldots, t_k)) = \{q \mid \delta_\sigma(q) \in \hat{\delta}(t_1) \times \cdots \times \hat{\delta}(t_k)\}$. The *tree language recognized by* M is $L(M) = \{t \in T_\Sigma \mid \hat{\delta}(t) \cap Q_d \neq \emptyset\}$. The classes of all tree languages over Σ which are recognized by td-fta and by deterministic td-fta are denoted by $\mathcal{L}^{n,td}(T_\Sigma)$ and $\mathcal{L}^{d,td}(T_\Sigma)$, respectively.

We conclude this section by a theorem on classes of tree languages which we will generalize (cf. Theorem 6.6). A proof of Theorem 2.4 can be found in [14] (Chapter II, Theorems 2.6 and 2.10, Example 2.11).

Theorem 2.4 $\mathcal{L}^{d,td}(T_\Sigma) \subsetneq \mathcal{L}^{n,td}(T_\Sigma) = \mathcal{L}^{n,bu}(T_\Sigma) = \mathcal{L}^{d,bu}(T_\Sigma)$.

2.3. Algebraic Structures

In this section we recall some definitions of algebraic structures. Let $\mathfrak{F}$ be a ranked alphabet (of function symbols). An $\mathfrak{F}$-*algebra with the carrier set* A is a tuple $\mathfrak{A} = (A, F)$, where A is a nonempty set and F is a family $(f^\mathfrak{A} \mid f \in \mathfrak{F})$ of operations such that for every $k \geq 0$ and $f \in \mathfrak{F}^{(k)}$ the operation $f^\mathfrak{A} \in F$ is k-ary. If $\mathfrak{F}$ is finite, say $\mathfrak{F} = \{f_1^{(r_1)}, \ldots, f_n^{(r_n)}\}$, then we write $(A, f_1, \ldots, f_n)$ rather than (A, F) and define the type of the algebra $\mathfrak{A}$ to be the n-tuple $(r_1, \ldots, r_k)$. Let $\mathfrak{A}_1 = (A_1, F_1)$ and $\mathfrak{A}_2 = (A_2, F_2)$ be two $\mathfrak{F}$-algebras. Then the mapping $\varphi : A_1 \to A_2$ is a *homomorphism* from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ if, for every $k \geq 0$, $f \in \mathfrak{F}^{(k)}$, and $a_1, \ldots, a_k \in A_1$, the equation $\varphi\left(f^{\mathfrak{A}_1}(a_1, \ldots, a_k)\right) = f^{\mathfrak{A}_2}(\varphi(a_1), \ldots, \varphi(a_k))$ holds. If φ is bijective (i. e., one to one and onto), then φ is called *isomorphism.*

An algebra is called *finite* if its carrier set is finite. Next we define algebras which are generated by a subset $\hat{A}$ of the carrier set A. The $\mathfrak{F}$-*algebra generated by* $\hat{A}$ is the smallest $\mathfrak{F}$-algebra $(\langle\hat{A}\rangle, F)$ such that $\hat{A} \subseteq \langle\hat{A}\rangle \subseteq A$. In other words, $\langle\hat{A}\rangle$ is the smallest superset of $\hat{A}$ such that $\langle\hat{A}\rangle$ is closed under F, i. e., for every $k \geq 0$, $f^{\mathfrak{A}} \in F^{(k)}$, and $a_1, \ldots, a_k \in \langle\hat{A}\rangle$, we have $f^{\mathfrak{A}}(a_1, \ldots, a_k) \in \langle\hat{A}\rangle$. An $\mathfrak{F}$-algebra (A, F) is *locally finite* if for every finite subset $\hat{A} \subseteq A$ the set $\langle\hat{A}\rangle$ is finite. Clearly, every finite $\mathfrak{F}$-algebra is locally finite.

We continue by recalling the definitions of some particular algebras. A *monoid* is an algebra $(A, \odot, 1)$ of type $(2, 0)$ such that, (i) for every $a, b, c \in A$, $a \odot (b \odot c) = (a \odot b) \odot c$ (associativity) and, (ii) for every $a \in A$, $a \odot 1 = 1 \odot a = a$ (1 is a neutral element). A monoid is *commutative* if, for every $a, b \in A$, $a \odot b = b \odot a$.

A *group* is an algebra $(A, \odot, ^{-1}, 1)$ of type (2,1,0) such that (i) $(A, \odot, 1)$ is a monoid and, (ii) for every $a \in A$, we have $a^{-1} \odot a = a \odot a^{-1} = 1$ (inverse elements). A group $(A, \odot, ^{-1}, 1)$ is *Abelian* (or: *commutative*) if $(A, \odot, 1)$ is a commutative monoid.

A *semiring* is an algebra $(A, \oplus, \odot, 0, 1)$ of type (2,2,0,0) such that (i) $(A, \oplus, 0)$ is a commutative monoid, (ii) $(A, \odot, 1)$ is a monoid, (iii) for every $a, b, c \in A$ we require $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ and $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ (distributivity), (iv) for every $a \in A$ the equations $a \odot 0 = 0 \odot a = 0$ hold, and (v) $0 \neq 1$. Whenever the operations $\oplus$, $\odot$, 0, and 1 are clear from the context, then we denote the semiring $(A, \oplus, \odot, 0, 1)$ just by A. Well known semirings are

- the *Boolean semiring* $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$, where the operations are disjunction and conjunction,
- the *semiring of natural numbers with* $+\infty$, denoted by $Nat_\infty = (\mathbb{N} \cup \{+\infty\}, +, \cdot, 0, 1)$, with the usual extension of addition and multiplication to $\mathbb{N} \cup \{+\infty\}$ (i. e., for every $a \in \mathbb{N} \cup \{+\infty\}$, $a + (+\infty) = (+\infty) + a = +\infty$ and $a \cdot (+\infty) = (+\infty) \cdot a = +\infty$),
- the *tropical semiring* $Trop = (\mathbb{R}_+ \cup \{+\infty\}, \min, +, +\infty, 0)$ with minimum and addition, where $\mathbb{R}_+$ is the set of all positive real numbers including 0 and, as usual, for every $a \in \mathbb{R}_+ \cup \{+\infty\}$, $\min(a, +\infty) = \min(+\infty, a) = a$, and $a + (+\infty) = (+\infty) + a = +\infty$,
- $([0, 1], \max, \cdot, 0, 1)$, i. e., the interval of real numbers ≥ 0 and ≤ 1 with maximum and multiplication.

Consider an arbitrary semiring $(A, \oplus, \odot, 0, 1)$. It is *commutative* if $(A, \odot, 1)$ is a commutative monoid; it is *zero divisor free* if, for every $a, b \in A$, the property $a \odot b = 0$ implies $a = 0$ or $b = 0$; finally, it is *zero sum free* if for every $a, b \in A$ the property $a \oplus b = 0$ implies $a = b = 0$.

A *semifield* is an algebra $(A, \oplus, \odot, ^{-1}, 0, 1)$ of type (2,2,1,0,0) such that (i) $(A, \oplus, \odot, 0, 1)$ is a semiring and (ii) $(A \backslash \{0\}, \odot, ^{-1}, 1)$ is a group. A semifield is *commutative* (*zero divisor free*, *zero sum free*) if the semiring $(A, \oplus, \odot, 0, 1)$ is commutative (zero divisor free, zero sum free, respectively). Typical semifields are

- the *Boolean semifield* $\mathbb{B}_{sf} = (\{0,1\}, \vee, \wedge, ^{-1}, 0, 1)$, where the operations are disjunction and conjunction, and the inverse which is defined by $0^{-1} = 0$ and $1^{-1} = 1$,
- the *tropical semifield* $Trop_{sf} = (\mathbb{R} \cup \{+\infty\}, \min, +, ^{-1}, +\infty, 0)$ with minimum, addition (defined as for *Trop*), and the inverse which is defined for every $a \in \mathbb{R}$ to be $-a$ and for $+\infty$ to be $+\infty$.

Lemma 2.5 [9, Chapter 4] *Let* $(A, \oplus, \odot, ^{-1}, 0, 1)$ *be a commutative semifield. Then the following statements are equivalent:*

(i) $(A, \oplus, \odot, ^{-1}, 0, 1)$ *is a locally finite semifield.*

(ii) $(A, \oplus, \odot, 0, 1)$ *is a locally finite semiring.*

(iii) $(A, \oplus, 0)$ *and* $(A, \odot, 1)$ *are locally finite monoids.*

Proof. In order to distinguish between the different closures of a subset $\hat{A} \subseteq A$, we introduce for every $F \subseteq \{\oplus, \odot, ^{-1}\}$ the notation $\langle \hat{A} \rangle_F$ which denotes the closure of $\hat{A}$ under all operations $f \in F$. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (ii) can be shown as follows: Let $(A, \oplus, 0)$ and $(A, \odot, 1)$ be locally finite monoids and $\hat{A} \subseteq A$ be a finite subset of A. By using the distributivity of the semiring, every expression over $\hat{A}$, $\oplus$, and $\odot$ can be transformed into the form $\sum_{1 \le i \le r} \prod_{1 \le j \le n_i} a_{i,j}$ for some $r, n_1, \ldots, n_r \in \mathbb{N}$ and $a_{i,j} \in \hat{A}$. Hence $\langle \hat{A} \rangle_{\oplus,\odot} = \langle\, \langle \hat{A} \rangle_{\odot} \,\rangle_{\oplus}$ which is a finite set by the assumption that both monoids $(A, \oplus, 0)$ and $(A, \odot, 1)$ are locally finite.

Finally we show (ii) $\Rightarrow$ (i). This can be done by proving that $\langle \hat{A} \rangle_{\oplus,\odot,^{-1}} \subseteq B$, where

$$B = \{x \odot y^{-1} \mid x, y \in \langle \hat{A} \rangle_{\oplus,\odot}\}$$

for every finite subset $\hat{A} \subseteq A$. A straightforward calculation proves that, for every $b_1, b_2 \in B$, also $b_1 \oplus b_2$, $b_1 \odot b_2$, $b_1^{-1} \in B$. Since the algebraic laws of semifields carry over from $(A, \oplus, \odot, ^{-1}, 0, 1)$ to $(B, \oplus, \odot, ^{-1}, 0, 1)$, the latter one is a semifield. Clearly, if $\langle \hat{A} \rangle_{\oplus,\odot} = \emptyset$, then $B = \emptyset = \langle \hat{A} \rangle_{\oplus,\odot}$. Moreover, from $\langle \hat{A} \rangle_{\oplus,\odot} \neq \emptyset$ we deduce that $1 = a \odot a^{-1} \in B$ for every $a \in \langle \hat{A} \rangle_{\oplus,\odot}$. Hence $a = a \odot 1 \in B$ and $a^{-1} \in B$, i. e., $\langle \hat{A} \rangle_{\oplus,\odot,^{-1}} \subseteq B$. On the other hand, since $\hat{A}$ is a finite set and $(A, \oplus, \odot, 0, 1)$ is a locally finite semiring, $\langle \hat{A} \rangle_{\oplus,\odot}$ is a finite set. Hence B is finite. From $\langle \hat{A} \rangle_{\oplus,\odot,^{-1}} \subseteq B$ we deduce that $\langle \hat{A} \rangle_{\oplus,\odot,^{-1}}$ is a finite set, hence $(A, \oplus, \odot, ^{-1}, 0, 1)$ is a locally finite semifield. □

Let us give three examples of classes of locally finite semifields:

- Every finite semifield is locally finite.
- Let $F_1 \subseteq F_2 \subseteq \cdots$ be an infinite sequence of finite semifields (for instance, choose $F_i = GF(p^{i!})$, i. e., the finite field with $p^{i!}$ elements for some fixed prime number p). Then $\bigcup_{i \in \mathbb{N}} F_i$ is an infinite semifield which is locally finite (cf. Chapter 3.8, Exercise 4 of [3]).
- The algebraic closure of a finite field is a locally finite field, hence it is a locally finite semifield.

Lemma 2.6 *Every semifield is zero divisor free.*

Proof. Let A be a semifield. Assume that there exist elements $a, b \in A \setminus \{0\}$ such that $a \odot b = 0$. Since $(A \setminus \{0\}, \odot, ^{-1}, 1)$ is a group, $a \odot b \in A \setminus \{0\}$ which contradicts to $a \odot b = 0$. $\square$

Now consider an arbitrary ranked alphabet Σ and let $k \geq 0$. The *term algebra* (*with respect to* Σ) is the algebra (T_Σ, Σ), where every $\sigma \in \Sigma^{(k)}$ is identified with the mapping $\sigma : {T_\Sigma}^k \to T_\Sigma$ (*top concatenation*) defined for every $t_1, \ldots, t_k \in T_\Sigma$ to be the tree $\sigma(t_1, \ldots, t_k)$.

2.4. Matrices

We now recall some definitions concerning matrices. An $I \times J$*-matrix* is a mapping $M : I \times J \to A$, where A, I, and J are arbitrary sets. As usual, $M(i, j)$ is also denoted by $M_{i,j}$. The set of all $I \times J$-matrices over A is denoted by $A^{I \times J}$.

We call $M \in A^{I \times J}$ a *row vector* if $I = \{i\}$ is a singleton. In this case, for every $j \in J$, we write A^J and M_j rather than $A^{I \times J}$ and $M_{i,j}$, respectively. On the other hand, $M \in S^{I \times J}$ is a *column vector* if $J = \{j\}$ is a singleton. In this case, for every $i \in I$, we write A^I and M_i rather than $A^{I \times J}$ and $M_{i,j}$, respectively.

Now let $(A, \odot, 1)$ be a monoid, $a \in A$, and I and J be arbitrary index sets. Then, for every $M \in A^{I \times J}$, the matrix $a \odot M$ is a mapping $a \odot M : I \times J \to A$ such that, for every $i \in I$ and $j \in J$, we define $(a \odot M)(i, j) = a \odot M(i, j)$. As usual we use aM as a shorthand for $a \odot M$.

A matrix $M \in A^{I \times J}$ is the *zero matrix*, denoted by $\underline{0}$ if $M_{i,j} = 0$ for every $i \in I$, $j \in J$. Moreover, two matrices $M, N \in A^{I \times J}$ are equal, denoted by $M = N$ if, for every $i \in I$ and $j \in J$, we have $M_{i,j} = N_{i,j}$.

2.5. Relations

Let S be a nonempty set. A *binary relation* ρ *on* S is a subset $\rho \subseteq S \times S$. We call ρ *reflexive* if, for every $a \in S$, we have $(a, a) \in \rho$, and *symmetric* if, for every $a, b \in S$ with $(a, b) \in \rho$, also $(b, a) \in \rho$. Moreover, ρ is *transitive* if, for every $a, b, c \in S$ with $(a, b), (b, c) \in \rho$, also $(a, c) \in \rho$. A binary relation is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Let ρ be an equivalence relation. If $(a, b) \in \rho$, then a and b are *equivalent.* For a given element $a \in S$, the set of all elements of S being equivalent to a, is called *equivalence class of a (with respect to* ρ*)*, denoted by $[a]_\rho = \{b \in S \mid (a, b) \in \rho\}$. The set of all equivalence classes is $S/_\rho = \{[a]_\rho \mid a \in S\}$ and the *index of* ρ, denoted by *index*(ρ), is the cardinality of $S/_\rho$.

Now let $\mathfrak{A} = (A, F)$ be an $\mathfrak{F}$-algebra and $\rho \subseteq A \times A$ be an equivalence relation. ρ is a *congruence relation (with respect to* $\mathfrak{A}$*)* if for every $k \geq 0$, $f \in \mathfrak{F}^{(k)}$, and $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$ with $(a_1, b_1), \ldots, (a_k, b_k) \in \rho$ also $(f^{\mathfrak{A}}(a_1, \ldots, a_k), f^{\mathfrak{A}}(b_1, \ldots, b_k)) \in \rho$.

Let $\mathfrak{A}_1 = (A_1, F_1)$ and $\mathfrak{A}_2 = (A_2, F_2)$ be two $\mathfrak{F}$-algebras and $h : A_1 \to A_2$ a homomorphism. The *kernel of* h, denoted by *ker*(h), is the set $\{(a, b) \in A_1 \times A_1 \mid$

$h(a) = h(b)\}$. By standard arguments one obtains that $ker(h)$ is a congruence relation with respect to $\mathfrak{A}_1$ (cf. e. g., Theorem 6.8 in Chapter II of [5]).

2.6. Formal Tree Series

Let A be a semiring and Σ be a ranked alphabet. A (*formal*) *tree series* (*over* Σ *and* A) is a mapping $S : T_\Sigma \to A$. For every $t \in T_\Sigma$, the image $S(t) \in A$ is called *coefficient* of t and we write (S, t) rather than $S(t)$. Then the tree series S can be written as the formal sum $\sum_{t\in T_\Sigma}(S,t)\,t$. The *support of* S is the set $supp(S) = \{t \in T_\Sigma \mid (S, t) \neq 0\}$. For a class $\mathfrak{C}$ of tree series, $supp(\mathfrak{C})$ is the class of all $supp(\varphi)$ for $\varphi \in \mathfrak{C}$. For every $a \in A$, we denote by $\tilde{a}$ the constant mapping $(\tilde{a}, t) = a$ for every tree $t \in T_\Sigma$.

The *set of all tree series over* Σ *and* A is denoted by $A\langle\langle T_\Sigma\rangle\rangle$. If $S_1, S_2 \in A\langle\langle T_\Sigma\rangle\rangle$ are tree series, then the *sum of* S_1 *and* S_2 is the tree series $S_1 + S_2$ which is defined for every $t \in T_\Sigma$ by $(S_1 + S_2, t) = (S_1, t) + (S_2, t)$. The following observation shows that tree languages are an instance of tree series.

Observation 2.7 *The monoids* $(\mathbb{B}\langle\langle T_\Sigma\rangle\rangle, +, \tilde{0})$ *and* $(\mathfrak{P}(T_\Sigma), \cup, \emptyset)$ *are isomorphic. The isomorphisms are* $\varphi : \mathbb{B}\langle\langle T_\Sigma\rangle\rangle \longrightarrow \mathfrak{P}(T_\Sigma)$ *with* $\varphi(S) = supp(S)$ *and* $\varphi^{-1} : \mathfrak{P}(T_\Sigma) \longrightarrow \mathbb{B}\langle\langle T_\Sigma\rangle\rangle$ *with* $\varphi^{-1}(L) = S_L$*, where, for every* $t \in T_\Sigma$*, we have* $(S_L, t) = 1$ *iff* $t \in L$*.*

For more details on formal tree series the reader may consult [18].

3. Weighted Tree Automata

In this section we recall finite state weighted tree automata. As in the unweighted case we distinguish between bottom-up and top-down finite state weighted tree automata which are defined in Sections 3.1 and 3.2, respectively. We also compare bottom-up finite state weighted tree automata with R-cost automata [24] and top-down finite state weighted tree automata with A'-tree automata [18], respectively.

3.1. Bottom-Up Finite State Weighted Tree Automata

In this section we enrich bu-fta with weights in some semiring $(A, \oplus, \odot, 0, 1)$. The following Definition 3.1, Observation 3.2, and Definition 3.3 can be considered as instances of the corresponding parts on pages 27 and 28 of [12].

Definition 3.1 *A* bottom-up tree representation (over Q, Σ, and A) *is a family* $\mu = (\mu_k \mid k \geq 0)$ *of mappings* $\mu_k : \Sigma^{(k)} \longrightarrow A^{Q^k \times Q}$*, where* Q *is an arbitrary, not necessarily finite set,* Σ *is a ranked alphabet, and* A *is a semiring.* μ *is* deterministic *if, for every* $k \geq 0$*,* $\sigma \in \Sigma^{(k)}$*, and* k*-tuple* $(q_1, \ldots, q_k) \in Q^k$*, there is at most one* $q \in Q$ *such that* $\mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq 0$*.*

Every tree representation μ over a *finite* set Q induces a family $(\overline{\mu_k(\sigma)} \mid k \geq 0,$ $\sigma \in \Sigma^{(k)})$ of mappings in the following way: for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$, we define a mapping $\overline{\mu_k(\sigma)} : A^Q \times \cdots \times A^Q \longrightarrow A^Q$ with k arguments such that, for every $V_1, \ldots, V_k \in A^Q$ and $q \in Q$,

$$\overline{\mu_k(\sigma)}(V_1, \ldots, V_k)_q = \sum_{(q_1,\ldots,q_k)\in Q^k} (V_1)_{q_1} \odot \cdots \odot (V_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q}.$$

Note that, since we have defined $\overline{\mu_k(\sigma)}(V_1, \ldots, V_k)$ only for a tree representation over a finite set Q, we need not require A to be a complete semiring.

Observation 3.2 *If Q is a finite set, then the algebraic structure $\big(A^Q, \big(\overline{\mu_k(\sigma)} \mid k \geq 0,\ \sigma \in \Sigma^{(k)}\big)\big)$ is a Σ-algebra. Its unique homomorphism $h_\mu : T_\Sigma \to A^Q$ is defined for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ by $h_\mu(\sigma(t_1, \ldots, t_k)) = \overline{\mu_k(\sigma)}(h_\mu(t_1), \ldots, h_\mu(t_k))$.*

Because of later convenience we first define a tree automaton with an arbitrary, not necessarily finite set Q of states.

Definition 3.3 *A* bottom-up weighted tree automaton over A (for short bu-w-ta) *is a 5-tuple $M = (Q, \Sigma, Q_d, A, \mu)$, where Q is a set, Σ is a ranked alphabet, $Q_d \subseteq Q$, A is a semiring, and μ is a bottom-up tree representation. A bu-w-ta is* deterministic *if its tree representation is deterministic.*

The bu-w-ta $M = (Q, \Sigma, Q_d, A, \mu)$ is a bottom-up finite state weighted tree automaton over A (for short bu-w-fta) *if Q is finite. In this case the* tree series $S_M \in A\langle\langle T_\Sigma\rangle\rangle$ accepted by M *is defined for every $t \in T_\Sigma$ by $(S_M, t) = \sum_{q\in Q_d} h_\mu(t)_q$. We also call S_M the semantics of M. Moreover, two bu-w-fta M and N are* equivalent *if $S_M = S_N$. The classes of all tree series accepted by bu-w-fta and by deterministic bu-w-fta are denoted by $A^{n,bu}\langle\langle T_\Sigma\rangle\rangle$ and $A^{d,bu}\langle\langle T_\Sigma\rangle\rangle$, respectively.*

Note that we have defined the semantics only in the finite case. Hence again the completeness of A is not required.

Example 3.4 [24, page 114] We show a bu-w-fta that can count the number of paths of a pointed series parallel graph which is a graph with one source and one sink. The class of pointed series parallel graphs $SP(\Delta)$ over an alphabet Δ is defined inductively: (i) for every $\delta \in \Delta$, we have $\delta \in SP(\Delta)$ and (ii) if $G_1, G_2 \in SP(\Delta)$, then both, the series composition $G_1 \cdot G_2$ and the parallel composition $G_1 \parallel G_2$ are elements of $SP(\Delta)$. The series composition of G_1 and G_2 is obtained by pasting the sink of G_1 with the source of G_2, where the source of $G_1 \cdot G_2$ is the source of G_1 and the sink of $G_1 \cdot G_2$ is the sink of G_2. The parallel composition is obtained by pasting the sources and the sinks, respectively. The source and the sink of $G_1 \parallel G_2$ are the obvious nodes. We identify $SP(\Delta)$ and T_Σ for $\Sigma = \{\delta^{(0)} \mid \delta \in \Delta\} \cup \{\cdot^{(2)}, \parallel^{(2)}\}$ in the obvious way.

The number of paths of a pointed series parallel graph is the mapping $\#_{\mathrm{path}} : SP(\Delta) \to \mathbb{N}$ given by $\#_{\mathrm{path}}(\delta) = 1$ for every $\delta \in \Delta$. Moreover, for every $G_1, G_2 \in SP(\Delta)$, $\#_{\mathrm{path}}(G_1 \cdot G_2) = \#_{\mathrm{path}}(G_1) \cdot \#_{\mathrm{path}}(G_2)$ and $\#_{\mathrm{path}}(G_1 \parallel G_2) = \#_{\mathrm{path}}(G_1) + \#_{\mathrm{path}}(G_2)$.

Now consider the bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$, where $Q = \{q_1, q_2\}$, $\Sigma = \{\delta^{(0)} \mid \delta \in \Delta\} \cup \{\cdot^{(2)}, \|^{(2)}\}$, $Q_d = \{q_1\}$, and $A = Nat_\infty$. Moreover, for every $\delta \in \Sigma^{(0)}$, let

$$\mu_0(\delta) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array},$$

$$\mu_2(\cdot) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{l} (q_1, q_1) \\ (q_1, q_2) \\ (q_2, q_1) \\ (q_2, q_2) \end{array} \end{array}, \qquad \text{and} \qquad \mu_2(\|) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{l} (q_1, q_1) \\ (q_1, q_2) \\ (q_2, q_1) \\ (q_2, q_2) \end{array} \end{array}.$$

We now prove that $h_\mu(G)_{q_1} = \#_{\text{path}}(G)$ and $h_\mu(G)_{q_2} = 1$ for every $G \in SP(\Delta) = T_\Sigma$ by induction on the structure of G. For $G = \delta \in \Sigma^{(0)}$ the claims are trivial. Now let $G_1, G_2 \in SP(\Delta)$. Then

$$\begin{aligned} h_\mu(\cdot(G_1, G_2))_{q_1} &= \sum_{(p_1,p_2) \in Q^2} h_\mu(G_1)_{p_1} \cdot h_\mu(G_2)_{p_2} \cdot \mu_2(\cdot)_{(p_1,p_2),q_1} \\ &= h_\mu(G_1)_{q_1} \cdot h_\mu(G_2)_{q_1} \cdot \mu_2(\cdot)_{(q_1,q_q),q_1} \\ &= \#_{\text{path}}(G_1) \cdot \#_{\text{path}}(G_2) \cdot 1 \\ &= \#_{\text{path}}(\cdot(G_1, G_2)) \end{aligned}$$

and

$$\begin{aligned} h_\mu(\| \, (G_1, G_2))_{q_1} &= \sum_{(p_1,p_2) \in Q^2} h_\mu(G_1)_{p_1} \cdot h_\mu(G_2)_{p_2} \cdot \mu_2(\|)_{(p_1,p_2),q_1} \\ &= h_\mu(G_1)_{q_1} \cdot h_\mu(G_1)_{q_2} \cdot \mu_2(\|)_{(q_1,q_2),q_1} \\ &\quad + \; h_\mu(G_1)_{q_2} \cdot h_\mu(G_2)_{q_1} \cdot \mu_2(\|)_{(q_2,q_1),q_1} \\ &= \#_{\text{path}}(G_1) \cdot 1 \cdot 1 + 1 \cdot \#_{\text{path}}(G_2) \cdot 1 \\ &= \#_{\text{path}}(\| \, (G_1, G_2)). \end{aligned}$$

The claim $h_\mu(G_1 \cdot G_2)_{q_2} = h_\mu(G_1 \parallel G_2)_{q_2} = 1$ can be shown similarly. Thus the tree series accepted by M is given for every $G \in SP(\Delta)$ by $(S_M, G) = \sum_{q \in Q_d} h_\mu(G)_q = \#_{\text{path}}(G)$. We note that if we calculate in the tropical semiring *Trop*, then the bu-w-fta M computes the length of the shortest path from the source to the sink.

Later on, for a given $t \in T_\Sigma$, we will work with the set of all those states $q \in Q$ such that there is a "nonzero derivation" of t in M ending up in q. Formally, for every $t \in T_\Sigma$, we define

$$\hat{\mu}(t) = \{q \in Q \mid h_\mu(t)_q \neq 0\}.$$

Note that if $\hat{\mu}(t) \cap Q_d = \emptyset$, then $(S_M, t) = 0$. The following lemma gives a characterization for $\hat{\mu}(t)$.

Lemma 3.5 *Let A be a zero divisor free semiring and $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta. If* (i) *M is deterministic or* (ii) *A is a zero sum free semiring, then, for* *every* $k \geq 0$, $\sigma \in \Sigma^{(k)}$, *and* $t_1, \ldots, t_k \in T_\Sigma$,

$$\hat{\mu}(\sigma(t_1, \ldots, t_k)) = \bigcup_{q_1 \in \hat{\mu}(t_1), \ldots, q_k \in \hat{\mu}(t_k)} \{q \in Q \mid \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq 0\}.$$

Proof. Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$. Then

$$\begin{aligned}
&\hat{\mu}(\sigma(t_1, \ldots, t_k)) \\
&= \{q \in Q \mid h_\mu(\sigma(t_1, \ldots, t_k))_q \neq 0\} \\
&= \left\{ q \in Q \;\middle|\; \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq 0 \right\} \\
&= \{q \in Q \mid \exists\, q_1, \ldots, q_k \in Q : h_\mu(t_1)_{q_1} \neq 0, \ldots, h_\mu(t_k)_{q_k} \neq 0, \\
&\quad \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq 0\} \\
&\qquad (A \text{ is zero divisor free and } M \text{ is deterministic or } A \text{ is zero sum free}) \\
&= \big\{q \in Q \mid \exists q_1 \in \hat{\mu}(t_1), \ldots, q_k \in \hat{\mu}(t_k) :\ \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq 0\big\} \\
&= \bigcup_{q_1 \in \hat{\mu}(t_1), \ldots, q_k \in \hat{\mu}(t_k)} \{q \in Q \mid \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \neq 0\}.
\end{aligned}$$

□

Lemma 3.6 *Let* $M = (Q, \Sigma, Q_d, A, \mu)$ *be a deterministic bu-w-fta. Then, for every tree* $t \in T_\Sigma$, *card* $(\hat{\mu}(t)) \leq 1$. *If* $\hat{\mu}(t) = \{q\}$ *for some* $q \in Q$, *then we will write* $\hat{\mu}(t) = q$.

Proof. The proof is by induction on the structure of $t \in T_\Sigma$. Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ such that $t = \sigma(t_1, \ldots, t_k)$. By induction hypothesis we have *card* $(\hat{\mu}(t_i)) \leq 1$ for $1 \leq i \leq k$. First assume that there exists an i with $1 \leq i \leq k$ and *card* $(\hat{\mu}(t_i)) = 0$, i.e., for every $q_i \in Q$, $h_\mu(t_i)_{q_i} = 0$. Then, for every $q \in Q$,

$$\begin{aligned}
h_\mu(t)_q &= \overline{\mu_k(\sigma)}(h_\mu(t_1), \ldots, h_\mu(t_k))_q \\
&= \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_i)_{q_i} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&= \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot 0 \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&= 0.
\end{aligned}$$

Hence *card* $(\hat{\mu}(t)) = 0$. Now let *card* $(\hat{\mu}(t_1)) = \cdots =$ *card* $(\hat{\mu}(t_k)) = 1$. Then

$$\begin{aligned}
h_\mu(t)_q &= \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&= h_\mu(t_1)_{\hat{\mu}(t_1)} \odot \cdots \odot h_\mu(t_k)_{\hat{\mu}(t_k)} \odot \mu_k(\sigma)_{(\hat{\mu}(t_1),\ldots,\hat{\mu}(t_k)),q}.
\end{aligned}$$

Since M is a deterministic bu-w-fta, μ is a deterministic bottom-up tree representation. Hence there exists at most one $\tilde{q} \in Q$ such that $\mu_k(\sigma)_{(\hat{\mu}(t_1),\ldots,\hat{\mu}(t_k)),\tilde{q}} \neq 0$. Thus, for every $q \in Q \setminus \tilde{q}$,

$$h_\mu(t)_q = h_\mu(t_1)_{\hat{\mu}(t_1)} \odot \cdots \odot h_\mu(t_k)_{\hat{\mu}(t_k)} \odot \mu_k(\sigma)_{(\hat{\mu}(t_1),\ldots,\hat{\mu}(t_k)),q} = 0.$$

Hence *card* $(\hat{\mu}(t)) \leq 1$. □

Lemma 3.7 *Let $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta. Also, let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$. If $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$, then $\hat{\mu}(t_i) \neq \emptyset$ for every i with $1 \leq i \leq k$.*

Proof. We prove the claim by contradiction. Assume $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$ and there is a subtree t_i with $\hat{\mu}(t_i) = \emptyset$. Then, for every state $q' \in Q$, $h_\mu(t_i)_{q'} = 0$. Now let $q \in Q$ be an arbitrary state. Then we can compute that $h_\mu(\sigma(t_1, \ldots, t_k))_q = 0$ in the same way as in the proof of Lemma 3.6. Hence $\hat{\mu}(\sigma(t_1, \ldots, t_k)) = \emptyset$ which contradicts to the assumption $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$. □

We now introduce the concept of isomorphisms of bu-w-fta.

Definition 3.8 *Let $M = (Q, \Sigma, Q_d, A, \mu)$ and $M' = (Q', \Sigma, Q'_d, A, \mu')$ be bu-w-fta and $\psi : Q \to Q'$ be a bijection such that $\psi(Q_d) = Q'_d$. M and M' are called* isomorphic *via ψ if, for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, q \in Q$, $\mu_k(\sigma)_{(q_1,\ldots,q_k),q} = \mu'_k(\sigma)_{(\psi(q_1),\ldots,\psi(q_k)),\psi(q)}$. The automaton M' which is isomorphic to M via ψ will be denoted by $\psi(M)$.*

Clearly, if M and M' are isomorphic bu-w-fta, then they are equivalent. Moreover, if M is deterministic, then also M' is deterministic.

Next we show that bu-w-fta over the Boolean semiring $\mathbb{B}$ and bu-fta coincide. The bu-w-fta $M = (Q, \Sigma, Q_d, \mathbb{B}, \mu)$ and the bu-fta $\tilde{M} = (Q, \Sigma, \delta, Q_d)$ are *related* if the equivalence $\mu_k(\sigma)_{(q_1,\ldots,q_k),q} = 1$ iff $q \in \delta_\sigma(q_1, \ldots, q_k)$ holds for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, q \in Q$.

Lemma 3.9 *Let $M = (Q, \Sigma, Q_d, \mathbb{B}, \mu)$ be a bu-w-fta and $\tilde{M} = (Q, \Sigma, \delta, Q_d)$ be a bu-fta such that M and $\tilde{M}$ are related. Then* (i) *for every $t \in T_\Sigma$, $\hat{\mu}(t) = \hat{\delta}(t)$ and* (ii) *$supp(S_M) = L(\tilde{M})$.*

Proof. Statement (i) can be proved easily by structural induction on t, where relatedness and some algebraic rules of $\mathbb{B}$ are applied. Statement (ii) follows from (i) straightforwardly. □

Theorem 3.10 *$supp(\mathbb{B}^{n,bu}\langle\langle T_\Sigma\rangle\rangle) = \mathcal{L}^{n,bu}\langle\langle T_\Sigma\rangle\rangle$ and $supp(\mathbb{B}^{d,bu}\langle\langle T_\Sigma\rangle\rangle) = \mathcal{L}^{d,bu}\langle\langle T_\Sigma\rangle\rangle$.*

We conclude this section by comparing bu-w-fta with R-cost automata of [24]. An R-cost automaton is based on a tree representation $\mu = (\mu_k \mid k \geq 0)$ where $\mu_k : \Sigma^{(k)} \to A[X_k]^{Q^k \times Q}$ and $A[X_k]$ is the set of polynomials over A and $X_k = \{x_1, \ldots, x_k\}$. Such a tree representation induces a family $(\overline{\mu_k(\sigma)} \mid k \geq 0,\ \sigma \in \Sigma^{(k)})$ of mappings $\overline{\mu_k(\sigma)} : A^Q \times \cdots \times A^Q \to A^Q$ such that

$$\overline{\mu_k(\sigma)}(V_1, \ldots, V_k)_q = \sum_{(q_1,\ldots,q_k)\in Q^k} \mu_k(\sigma)\big[(V_1)_{q_1}, \ldots, (V_k)_{q_k}\big]$$

and, for every polynomial $\tau \in A[X_k]$ and $a_1, \ldots, a_k \in A$, $\tau[a_1, \ldots, a_k] \in A$ is defined by replacing in τ every x_i by a_i and calculating an element of A.

Thus our bu-w-fta are particular R-cost automata. Vice versa, if we restrict the polynomials $\mu_k(\sigma)_{(q_1,\ldots,q_k),q}$ to have the form $a \odot x_1 \odot \cdots \odot x_k$, then we reobtain the bu-w-fta.

3.2. Top-Down Finite State Weighted Tree Automata

We also can traverse the trees from the root to the frontier. This is formalized by a top-down tree representation and td-w-fta. Here we define immediately the finite case.

Definition 3.11 *A* top-down tree representation (over Q, Σ, and A) *is a family* $\mu = (\mu_k \mid k \geq 0)$ *of mappings* $\mu_k : \Sigma^{(k)} \longrightarrow A^{Q \times Q^k}$*, where* Q *is a finite set,* Σ *is a ranked alphabet, and* A *is a semiring.* μ *is* deterministic *if, for every* $k \geq 0$*,* $\sigma \in \Sigma^{(k)}$*, and* $q \in Q$*, there is at most one* k*-tuple* $(q_1, \ldots, q_k) \in Q^k$ *such that* $\mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \neq 0$.

Every top-down tree representation μ induces a family $(\overline{\mu_k(\sigma)} \mid k \geq 0,\ \sigma \in \Sigma^{(k)})$ of mappings: for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$, we define a mapping $\overline{\mu_k(\sigma)} : A^Q \times \cdots \times A^Q \longrightarrow A^Q$ with k arguments such that, for every state $q \in Q$,

$$\overline{\mu_k(\sigma)}(V_1, \ldots, V_k)_q = \sum_{(q_1,\ldots,q_k) \in Q^k} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \odot (V_1)_{q_1} \odot \cdots \odot (V_k)_{q_k}.$$

Note that the order of the factors differs from the bottom-up case. If we traverse a tree from the root towards the leaves, then at each stage we first apply the rule to obtain a k-tuple of states, for each subtree one. After that we traverse each of the subtrees.

Observation 3.12 *The algebraic structure* $\left(A^Q, \left(\overline{\mu_k(\sigma)} \mid k \geq 0,\ \sigma \in \Sigma^{(k)}\right)\right)$ *is a* Σ*-algebra. The unique homomorphism* $h_\mu : T_\Sigma \to A^Q$ *is defined for every* $k \geq 0$*,* $\sigma \in \Sigma^{(k)}$*, and* $t_1, \ldots, t_k \in T_\Sigma$ *by* $h_\mu(\sigma(t_1, \ldots, t_k)) = \overline{\mu_k(\sigma)}(h_\mu(t_1), \ldots, h_\mu(t_k))$.

Definition 3.13 *A* top-down finite state weighted tree automaton over A (for short td-w-fta) *is a 5-tuple* $M = (Q, \Sigma, Q_d, A, \mu)$*, where* Q *is a finite set,* Σ *is a ranked alphabet,* $Q_d \subseteq Q$ *is the set of final states,* A *is a semiring, and* μ *is a top-down tree representation. A td-w-fta is* deterministic *if its tree representation is deterministic and* Q_d *is a singleton.*

The tree series S_M accepted by M *is defined for every* $t \in T_\Sigma$ *by* $(S_M, t) = \sum_{q \in Q_d} h_\mu(t)_q$*. We also call* S_M *the* semantics of M*. Two td-w-fta* M *and* N *are* equivalent *if* $S_M = S_N$*. The classes of tree series accepted by td-w-fta and deterministic td-w-fta are denoted by* $A^{n,td}\langle\langle T_\Sigma \rangle\rangle$ *and* $A^{d,td}\langle\langle T_\Sigma \rangle\rangle$*, respectively.*

Example 3.14 Let $M = (Q, \Sigma, Q_d, A, \mu)$ be the td-w-fta defined by $Q = Q_d = \{q\}$, $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$, $A = \mathit{Trop}$, and $\mu_0(\alpha) = \mu_2(\sigma) = (1)$. One can easily prove that $h_\mu(t)_q = \mathit{size}(t)$ for every $t \in T_\Sigma$. Hence, $(S_M, t) = \sum_{p \in Q_d} h_\mu(t)_p = h_\mu(t)_q = \mathit{size}(t)$.

As in the bottom-up case one can prove that td-w-fta over the Boolean semiring and td-fta coincide.

Theorem 3.15 $\mathit{supp}(\mathbb{B}^{n,td}\langle\langle T_\Sigma \rangle\rangle) = \mathcal{L}^{n,td}\langle\langle T_\Sigma \rangle\rangle$ *and* $\mathit{supp}(\mathbb{B}^{d,td}\langle\langle T_\Sigma \rangle\rangle) = \mathcal{L}^{d,td}\langle\langle T_\Sigma \rangle\rangle$.

In [18] a very general concept of tree automata is defined in terms of systems of equations. We now briefly recall this concept and show that it covers our model of td-w-fta under the assumption that the underlying semiring is naturally ordered, complete, and continuous. The involved algebraic structures are a continuous distributive multioperator (DM-) monoid $(A, \oplus, 0, \Omega)$, a subset $A' \subseteq A$, a naturally ordered, complete, and continuous semiring $(V, \oplus, \odot, 0, 1)$ such that A is a complete V-semimodule, and a finite set $Y_k = \{y_1, \ldots, y_k\}$ of variables.[2] Then a finite A'-tree automaton is a quadruple $M' = (Q', \nu', S', P')$, where Q' is the finite set of states, $\nu' = (\nu'_k \mid k \geq 1)$ is the sequence of transition matrices $\nu'_k \in (V\langle\langle T_\Delta(A' \cup Y_k)\rangle\rangle)^{Q' \times Q'^k}$, $S' \in (V\langle\langle T_\Delta(A' \cup Y_1)\rangle\rangle)^{Q'}$ is the initial state vector, and $P' \in (V\langle\langle T_\Delta(A')\rangle\rangle)^{Q'}$ is the final state vector.

The semantics of M' is defined in terms of an approximation sequence $(\delta'^{(j)} \in A^{Q'} \mid j \in \mathbb{N})$ for every $j \in \mathbb{N}$ starting with $\delta'^{(0)} = \underline{0}$. Moreover, for every $j \geq 1$ and $q' \in Q'$,

$$\delta'^{(j+1)} = \sum_{k \geq 1} \nu'_k(\delta'^{(j)}, \ldots, \delta'^{(j)}) + P, \qquad \text{where}$$

$$\nu'_k(\delta'^{(j)}, \ldots, \delta'^{(j)})_q = \sum_{\substack{(q'_1, \ldots, q'_k) \in Q'^k \\ t \in T_\Delta(Y_k)}} ((\nu'_k)_{q', (q'_1, \ldots, q'_k)}, t)\; t((\delta'^{(j)})_{q'_1}, \ldots, (\delta'^{(j)})_{q'_k}).$$

The behavior of M' is defined to be $\| M' \| = \sum_{q' \in Q'} S'_{q'}(sup(\delta'^{(j)} \mid j \geq 0)_{q'})$, where $sup(B)$ is the supremum of B.

For a given td-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$ over a naturally ordered, complete, and continuous semiring A, we construct an $A\langle\langle T_\Sigma\rangle\rangle$-tree automaton $M' = (Q', \nu', S', P')$ with behavior $\| M' \| = S_M$, where the underlying DM-monoid is $(A\langle\langle T_\Sigma\rangle\rangle, \oplus, \tilde{0}, \bar{\Sigma})$, in the following way. For every $k \geq 1$ and $q', q'_1, \ldots, q'_k \in Q'$, let

$$(\nu'_k)_{q', (q'_1, \ldots, q'_k)} = \sum_{\sigma \in \Sigma^{(k)}} \mu_k(\sigma)_{q, (q_1, \ldots, q_k)}\; \sigma(y_1, \ldots, y_k),$$

$$P'_q = \sum_{\sigma \in \Sigma^{(0)}} \mu_0(\sigma)_{q, ()}\; \sigma,$$

$$S'_q = \begin{cases} y_1 & \text{if } q \in Q_d, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, there are $A\langle\langle T_\Sigma\rangle\rangle$-tree automata such that there is no td-w-fta with the same semantics. This is due to

- the special form of the underlying DM-monoid $(A\langle\langle T_\Sigma\rangle\rangle, \oplus, \tilde{0}, \bar{\Sigma})$,
- the special form of ν' ($supp((\nu_k)_{q, (q_1, \ldots, q_k)}) \in \Sigma(Y)$ for every $q, q_1, \ldots, q_k \in Q$), and
- the special form of S_q

while simulating td-w-fta by $A\langle\langle T_\Sigma\rangle\rangle$-tree automata.

[2] For the definitions of DM-monoids and complete V-semimodules we refer the reader to [18].

4. Partial Determinization of bu-w-fta

In this section we construct, for a given nondeterministic bu-w-fta M, a deterministic bu-w-ta M' which if it is finite is equivalent to M. In fact, this transformation is a partial determinization algorithm in the sense that it does not transform every nondeterministic bu-w-fta into an equivalent deterministic bu-w-fta. However, we will give sufficient conditions on the involved semifield such that the resulting bu-w-ta M' has a finite number of states and that it is equivalent to M.

More precisely, in Section 4.1 we introduce a partial determinization of bu-w-fta over a semifield. The partial determinization has two parameters: (i) a formal tree series φ and (ii) a congruence relation ρ. It constructs for a given bu-w-fta M a bottom-up weighted tree automaton $det(\varphi, \rho, M)$ which need not be finite. Moreover, if it is finite, then $det(\varphi, \rho, M)$ is equivalent to M provided that φ and ρ satisfy certain conditions ((P1) and (P2)–(P4), respectively). We also present an algorithm which explicitly constructs an equivalent deterministic automaton. In Sections 4.2 we present two instances of the partial determinization. Throughout this section let $M = (Q, \Sigma, Q_d, A, \mu)$ denote an arbitrary bu-w-fta over an arbitrary semifield A if not stated otherwise.

4.1. The Concept of Partial Determinization

The concept which we introduce now has two parameters, a tree series $\varphi \in A\langle\langle T_\Sigma\rangle\rangle$ and a congruence relation ρ with respect to the term algebra (T_Σ, Σ). Both, φ and ρ, are supposed to satisfy certain properties in order to ensure that the constructed automaton $det(\varphi, \rho, M)$ is well defined. After proving some properties of our partial determinization, we present an algorithm which generates $det(\varphi, \rho, M)$.

Let us start with the partial determinization. As mentioned above, we consider a given tree series $\varphi \in A\langle\langle T_\Sigma\rangle\rangle$ with the property that,

(P1) for every $t \in T_\Sigma$ with $t \notin supp(S_M)$ and $\hat{\mu}(t) \neq \emptyset$, we have $(\varphi, t) \neq 0$.

Now we define a new tree series $S_M^{(\varphi)} \in A\langle\langle T_\Sigma\rangle\rangle$ which is given for every $t \in T_\Sigma$ by

$$(S_M^{(\varphi)}, t) = \begin{cases} (S_M, t) & \text{if } t \in supp(S_M), \\ (\varphi, t) & \text{if } t \notin supp(S_M) \text{ and } \hat{\mu}(t) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Observation 4.1 *Let $\varphi \in A\langle\langle T_\Sigma\rangle\rangle$ be a formal tree series satisfying (P1). Then, for every $t \in T_\Sigma$, we have $(S_M^{(\varphi)}, t) \neq 0$ and hence $(S_M^{(\varphi)}, t)^{-1}$ exists.*

Furthermore, for our partial determinization we need a congruence relation ρ with respect to the term algebra (T_Σ, Σ) satisfying the following conditions:

(P2) For every $t, t' \in T_\Sigma$ with $(t, t') \in \rho$, we have $\hat{\mu}(t) \neq \emptyset$ iff $\hat{\mu}(t') \neq \emptyset$.

(P3) For every $t, t' \in T_\Sigma$ with $(t, t') \in \rho$, we have $t \in supp(S_M)$ iff $t' \in supp(S_M)$.

(P4) For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, t_1', \ldots, t_k, t_k' \in T_\Sigma$ such that $(t_1, t_1'), \ldots, (t_k, t_k') \in \rho$ and $\hat{\mu}(t_1) \neq \emptyset$, ..., $\hat{\mu}(t_k) \neq \emptyset$, $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$, we have

$$\begin{aligned} &(S_M^{(\varphi)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1)^{-1} \odot (S_M^{(\varphi)}, \sigma(t_1, \ldots, t_k)) \\ = &(S_M^{(\varphi)}, t_k')^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1')^{-1} \odot (S_M^{(\varphi)}, \sigma(t_1', \ldots, t_k')). \end{aligned}$$

Now we define the determinization for a given bu-w-fta M with the parameters φ and ρ. Note that the constructed automaton might have an infinite set of states, i. e., we only generate a bu-w-ta (cf. Definition 3.3). Later on we will prove that if

- ρ has finite index (cf. Corollary 4.9) or
- A is a locally finite semifield (cf. Lemma 6.1),

then the set of states of $det(\varphi, \rho, M)$ is finite which implies that $det(\varphi, \rho, M)$ is a bu-w-fta.

Definition 4.2 *Let $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta over a semifield A. Moreover, let*

- *$\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ be a formal tree series satisfying* (P1) *and*
- *ρ be a congruence relation satisfying* (P2)–(P4).

Then the determinization of M with respect to φ and ρ *is the bu-w-ta $det(\varphi, \rho, M) = (Q', \Sigma, Q_d', A, \mu')$ given by*

$$Q' = \{[t]_\rho \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}, \qquad Q_d' = \{[t]_\rho \mid t \in supp(S_M)\},$$

and, for every $k \geq 0$, $q_1', \ldots, q_k', q' \in Q'$, and $\sigma \in \Sigma^{(k)}$,

$$\mu_k'(\sigma)_{(q_1', \ldots, q_k'), q'} = \begin{cases} (S_M^{(\varphi)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1)^{-1} \odot (S_M^{(\varphi)}, \sigma(t_1, \ldots, t_k)) \\ \quad \text{if } \exists t_1, \ldots, t_k \in T_\Sigma,\ q_1' = [t_1]_\rho, \ldots, q_k' = [t_k]_\rho, \text{ and} \\ \quad q' = [\sigma(t_1, \ldots, t_k)]_\rho, \\ 0 \quad \text{otherwise.} \end{cases}$$

In order to shorten the notations, we will assume in the rest of this section that

- $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ is a formal tree series satisfying (P1),
- ρ is a congruence relation with respect to the term algebra (T_Σ, Σ) satisfying (P2)–(P4), and
- $det(\varphi, \rho, M) = (Q', \Sigma, Q_d', A, \mu')$.

Lemma 4.3 *The bu-w-ta $det(\varphi, \rho, M)$ is well defined.*

Proof. First we consider Q_d'. We have to show that, for every $t, t' \in T_\Sigma$, from $t \in supp(S_M)$ and $t' \in [t]_\rho$ also $t' \in supp(S_M)$ follows. In fact, this we require in (P3). Hence Q_d' is well defined. Similarly, condition (P2) implies that if $\hat{\mu}(t) \neq \emptyset$ and $t' \in [t]_\rho$, then also $\hat{\mu}(t') \neq \emptyset$. Thus Q' is well defined.

Finally we consider $\mu'_k(\sigma)$ for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$, where we have to show that the definition of $\mu'_k(\sigma)$ is independent from the choice of the representants of the equivalence classes $[t_1]_\rho, \ldots, [t_k]_\rho$. Let $t_1, t'_1, \ldots, t_k, t'_k \in T_\Sigma$ with $(t_1, t'_1), \ldots, (t_k, t'_k) \in \rho$. Since ρ is a congruence relation, we deduce that $(\sigma(t_1, \ldots, t_k), \sigma(t'_1, \ldots, t'_k)) \in \rho$. In other words, the equivalence class $[\sigma(t_1, \ldots, t_k)]_\rho$ and hence the associated state $[\sigma(t_1, \ldots, t_k)]_\rho$ (if defined) are independent from the choice of the representants $t_1, \ldots, t_k$ of the equivalence classes $[t_1]_\rho, \ldots, [t_k]_\rho$. Furthermore, by (P4), the value of $(S_M^{(\varphi)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1)^{-1} \odot (S_M^{(\varphi)}, \sigma(t_1, \ldots, t_k))$ does not depend on the chosen representants. From (P1) we deduce that $(S_M^{(\varphi)}, t)^{-1}$ exists for every $t \in T_\Sigma$. $\square$

Clearly, we have to prove the correctness of our partial determinization, i. e., we have to show that for a given bu-w-fta M the resulting bu-w-ta $det(\varphi, \rho, M)$ is deterministic (cf. Observation 4.4), and if $det(\varphi, \rho, M)$ has a finite set of states, then M and $det(\varphi, \rho, M)$ are equivalent (cf. Lemma 4.7).

Observation 4.4 *The bu-w-ta $det(\varphi, \rho, M)$ is deterministic.*

Proof. Let $k \geq 0$, $q'_1, \ldots, q'_k \in Q'$, and $\sigma \in \Sigma^{(k)}$. By Definition 4.2 there exist trees $t_1, \ldots, t_k \in T_\Sigma$ such that $q'_1 = [t_1]_\rho, \ldots, q'_k = [t_k]_\rho$. Clearly, there is at most one state $q' = [\sigma(t_1, \ldots, t_k)]_\rho$ (if defined) such that $\mu_k(\sigma)_{(q'_1, \ldots, q'_k), q'} \neq 0$. $\square$

We continue with the proof of the equivalence of M and $det(\varphi, \rho, M)$ in the finite case. For this purpose we need two lemmas.

Lemma 4.5 *Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ such that $\hat{\mu}(t_1) \neq \emptyset, \ldots, \hat{\mu}(t_k) \neq \emptyset$, and $\hat{\mu}(\sigma(t_1, \ldots, t_k)) = \emptyset$. Then, for every $q' \in Q'$, $\mu'_k(\sigma)_{([t_1]_\rho, \ldots, [t_k]_\rho), q'} = 0$.*

Proof. Assume, there is a $q' \in Q'$ with $\mu'_k(\sigma)_{([t_1]_\rho, \ldots, [t_k]_\rho), q'} \neq 0$. By Definition 4.2 there exist trees $t'_1, \ldots, t'_k \in T_\Sigma$ such that $[t_1]_\rho = [t'_1]_\rho, \ldots, [t_k]_\rho = [t'_k]_\rho$, and $q' = [\sigma(t'_1, \ldots, t'_k)]_\rho$. Hence $(t_1, t'_1), \ldots, (t_k, t'_k) \in \rho$. Since ρ is a congruence relation with respect to the term algebra (T_Σ, Σ), also $(\sigma(t_1, \ldots, t_k), \sigma(t'_1, \ldots, t'_k)) \in \rho$. Then, from (P2) and $\hat{\mu}(\sigma(t_1, \ldots, t_k)) = \emptyset$, we deduce that $\hat{\mu}(\sigma(t'_1, \ldots, t'_k)) = \emptyset$. Thus $q' = [\sigma(t'_1, \ldots, t'_k)]_\rho \notin Q'$ which contradicts $q' \in Q'$. $\square$

Lemma 4.6 *If the set of states of $det(\varphi, \rho, M)$ is finite, then,*

(i) *for every tree $t \in T_\Sigma$,*

$$\hat{\mu'}(t) = \begin{cases} [t]_\rho & \text{if } \hat{\mu}(t) \neq \emptyset, \\ \emptyset & \text{otherwise}, \end{cases}$$

(ii) *for every tree $t \in T_\Sigma$ with $\hat{\mu}(t) \neq \emptyset$, $h_{\mu'}(t)_{[t]_\rho} = (S_M^{(\varphi)}, t)$.*

Proof. We show both statements simultaneously by induction on the structure of $t \in T_\Sigma$, where the induction base is covered by the induction step. Let $t = \sigma(t_1, \ldots, t_k)$ for some $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$.

First we prove claim (ii). Let $\hat{\mu}(t) \neq \emptyset$. By Lemma 3.7 also $\hat{\mu}(t_i) \neq \emptyset$ for every i with $1 \leq i \leq k$ and hence $[t]_\rho, [t_1]_\rho, \ldots, [t_k]_\rho \in Q'$. Moreover, by induction hypothesis (i), $\hat{\mu'}(t_i) = [t_i]_\rho$ for every i with $1 \leq i \leq k$. Thus $\{q' \in Q' \mid h_{\mu'}(t_i)_{q'} \neq 0\} = \{[t_i]_\rho\}$. Let us denote this last equation by $(*)$.

$$\begin{aligned}
&h_{\mu'}(t)_{[t]_\rho} \\
&= \sum_{(q'_1,\ldots,q'_k)\in(Q')^k} h_{\mu'}(t_1)_{q'_1} \odot \cdots \odot h_{\mu'}(t_k)_{q'_k} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),[t]_\rho} \\
&= h_{\mu'}(t_1)_{[t_1]_\rho} \odot \cdots \odot h_{\mu'}(t_k)_{[t_k]_\rho} \odot \mu'_k(\sigma)_{([t_1]_\rho,\ldots,[t_k]_\rho),[t]_\rho} \qquad \text{(by } (*)\text{)} \\
&= (S_M^{(\varphi)}, t_1) \odot \cdots \odot (S_M^{(\varphi)}, t_k) \odot \big((S_M^{(\varphi)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1)^{-1} \odot (S_M^{(\varphi)}, t)\big) \\
&\qquad \text{(by induction hypothesis (ii) which is applicable by Lemma 3.7)} \\
&= (S_M^{(\varphi)}, t)
\end{aligned}$$

which proves claim (ii).

Next we prove claim (i). Let $\hat{\mu}(t) \neq \emptyset$. By the above calculation and Observation 4.1 we have $h_{\mu'}(t)_{[t]_\rho} = (S_M^{(\varphi)}, t) \neq 0$, hence $[t]_\rho \in \hat{\mu'}(t)$. Since Q' is finite, we can apply Lemma 3.6 to obtain $card(\hat{\mu'}(t)) \leq 1$. Thus $\hat{\mu'}(t) = [t]_\rho$. Now let $\hat{\mu}(t) = \emptyset$. Here we have to consider two cases. First assume that there exists an i with $1 \leq i \leq k$ such that $\hat{\mu}(t_i) = \emptyset$. Then, for every $q' \in Q'$,

$$\begin{aligned}
&h_{\mu'}(t)_{q'} \\
&= \sum_{(q'_1,\ldots,q'_k)\in(Q')^k} h_{\mu'}(t_1)_{q'_1} \odot \cdots \odot h_{\mu'}(t_i)_{q'_i} \odot \cdots \odot h_{\mu'}(t_k)_{q'_k} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'} \\
&= \sum_{(q'_1,\ldots,q'_k)\in(Q')^k} h_{\mu'}(t_1)_{q'_1} \odot \cdots \odot 0 \odot \cdots \odot h_{\mu'}(t_k)_{q'_k} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'} \\
&= 0.
\end{aligned}$$

Hence $\hat{\mu'}(t) = \emptyset$ in this case. Second assume that $\hat{\mu}(t_i) \neq \emptyset$ for every $1 \leq i \leq k$. The induction hypothesis (i) gives $\hat{\mu'}(t_i) = [t_i]_\rho \in Q'$. Then, for every $q' \in Q'$,

$$\begin{aligned}
&h_{\mu'}(t)_{q'} \\
&= \sum_{(q'_1,\ldots,q'_k)\in(Q')^k} h_{\mu'}(t_1)_{q'_1} \odot \cdots \odot h_{\mu'}(t_k)_{q'_k} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'} \\
&= h_{\mu'}(t_1)_{[t_1]_\rho} \odot \cdots \odot h_{\mu'}(t_k)_{[t_k]_\rho} \odot \mu'_k(\sigma)_{([t_1]_\rho,\ldots,[t_k]_\rho),q'} \\
&\qquad \text{(by induction hypothesis (i))} \\
&= h_{\mu'}(t_1)_{[t_1]_\rho} \odot \cdots \odot h_{\mu'}(t_k)_{[t_k]_\rho} \odot 0 \qquad \text{(by Lemma 4.5)} \\
&= 0.
\end{aligned}$$

Hence again $\hat{\mu}(t) = \emptyset$ which completes the proof of claim (i). □

From Lemma 4.6 we deduce the equivalence of M and $det(\varphi, \rho, M)$ as follows.

Lemma 4.7 *If the set of states of $det(\varphi, \rho, M)$ is finite, then M and $det(\varphi, \rho, M)$ are equivalent.*

Proof. Note that $det(\varphi, \rho, M)$ is deterministic by Observation 4.4. First let $t \in supp(S_M)$, i. e., $\hat{\mu}(t) \neq \emptyset$. By Lemma 4.6 (i) and by Definition 4.2 we have $\hat{\mu'}(t) = [t]_\rho \in Q'_d$. Furthermore, $(S_{det(\varphi,\rho,M)}, t) = \sum_{q' \in Q'_d} h_{\mu'}(t)_{q'} = h_{\mu'}(t)_{[t]_\rho}$ (since $\hat{\mu'}(t) = [t]_\rho \in Q'_d) = (S_M^{(\varphi)}, t)$ (by Lemma 4.6 (ii)) $= (S_M, t)$ (since $t \in supp(S_M)$).

If $t \notin supp(S_M)$, then, by Lemma 4.6 (i) and Definition 4.2, either $\hat{\mu'}(t) = [t]_\rho \notin Q'_d$ or $\hat{\mu'}(t) = \emptyset$. We deduce in both cases that $h_{\mu'}(t)_{q'} = 0$ for every $q' \in Q'_d$, hence $(S_{det(\varphi,\rho,M)}, t) = \sum_{q' \in Q'_d} h_{\mu'}(t)_{q'} = 0 = (S_M, t)$ which proves our lemma. □

Theorem 4.8 *If the set of states of $det(\varphi, \rho, M)$ is finite, then $det(\varphi, \rho, M)$ is a finite deterministic bu-w-fta being equivalent to M.*

Corollary 4.9 *If ρ has finite index, then $det(\varphi, \rho, M)$ is a deterministic bu-w-fta being equivalent to M.*

Finally we consider the partial determinization from the algorithmic point of view. For a given bu-w-fta M, a given tree series φ, and a given relation ρ, the partial determinization algorithm (cf. Figure 2) generates a bu-w-fta $det_t(\varphi, \rho, M)$ which is isomorphic to $det(\varphi, \rho, M)$ via the mapping ψ which is defined below. Contrary to Definition 4.2 the states of $det_t(\varphi, \rho, M)$ are trees rather than equivalence classes. For the rest of this section let

- $det_t(\varphi, \rho, M) = (P, \Sigma, P_d, A, \nu)$ and
- $\psi : P \to Q'$, where $\psi(t) = [t]_\rho$ for every $t \in P$.

Note that the partial determinization algorithm follows the scheme of dynamic programming. Moreover, we introduce the variable *count* to make the correctness proof more clear.

Observation 4.10 *Let $t, t' \in T_\Sigma$ such that $(t, t') \in \rho$. If $t \in P$, then $height(t) \leq height(t')$.*

In Lemma 4.15 we will prove that $det(\varphi, \rho, M)$ and $det_t(\varphi, \rho, M)$ are isomorphic via ψ. As preparation, let us investigate $\$P_{count}$. For every reachable $count \geq 1$, $\$P_{count}$ denotes the set $\$P$ which the partial determinization algorithm produces at the end of the $count^{th}$ run through the repeat-until-loop.

Observation 4.11 *Let $count \geq 1$ and $t \in P$. Then $height(t) \leq count$ iff $t \in \$P_{count}$.*

Observation 4.12 *Let $t \in P$. Then the partial determinization algorithm traverses at least $height(t) + 1$ times the repeat-until-loop.*

Observation 4.13 *The sequence $(\$P_{count})_{count \geq 1}$ is monotonically increasing, i. e., $\$P_{count} \subseteq \$P_{count+1}$. Its limit is P.*

Partial determinization algorithm with respect to φ and ρ

Input:
- bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$ over a semifield A
- tree series $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ satisfying (P1)
- congruence relation $\rho \subseteq T_\Sigma \times T_\Sigma$ satisfying (P2)–(P4)

Output:
- bu-w-fta $det_t(\varphi, \rho, M) = (P, \Sigma, P_d, A, \nu)$ being isomorphic to $det(\varphi, \rho, M)$

Variables:
- $k, count \in \mathbb{N}$
- $\sigma \in \Sigma$
- $t_1, \ldots, t_k, t \in T_\Sigma$
- $\$P, \$P_d, newstates \in \mathfrak{P}(T_\Sigma)$
- $answer \in \{yes, no\}$

$\$P = \emptyset$
$\$P_d = \emptyset$
$count = 0$
repeat
    $count = count + 1$
    $newstates = \emptyset$
    for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k \in \$P$ do
        $t = \sigma(t_1, \ldots, t_k)$
        if $\hat{\mu}(t) = \emptyset$ then
            $answer = no$
        else
            $answer = yes$
            for every $t' \in \$P \cup newstates$ do
                if $(t, t') \in \rho$ then
                    $answer = no$
        if $(answer = yes)$ then
            $newstates = newstates \cup \{t\}$
    $\$P = \$P \cup newstates$
    $\$P_d = \$P_d \cup (newstates \cap supp(S_M))$
until $newstates = \emptyset$
$P = \$P$
$P_d = \$P_d$

for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k, t \in P$ do
    if $(t, \sigma(t_1, \ldots, t_k)) \in \rho$ then
        $\nu_k(\sigma)_{(t_1,\ldots,t_k),t} = (S_M^{(\varphi)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1)^{-1} \odot (S_M^{(\varphi)}, t)$
    else
        $\nu_k(\sigma)_{(t_1,\ldots,t_k),t} = 0$

Figure 2: Partial determinization algorithm with respect to φ and ρ

Now we show that ψ is a bijection between the sets of states of $det(\varphi, \rho, M)$ and $det_t(\varphi, \rho, M)$.

Lemma 4.14 *The mapping ψ is a bijection.*

Proof. First we show that ψ is well defined. Recall that $Q' = \{[t]_\rho \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}$. From lines 9, 10, 16–18, and 21 of the partial determinization algorithm we deduce that $\hat{\mu}(t) \neq \emptyset$ for every $t \in P$. Thus $\psi(t) \in Q'$, hence ψ is well defined.

Now we show that ψ is a bijection. It suffices to prove that, (i) for every $t, t' \in P$, we have $(t, t') \notin \rho$ and (ii) for every $[t]_\rho \in Q'$ there is an $s \in P$ such that $(s, t) \in \rho$. Clearly, (i) and (ii) ensure that ψ is a bijection. Claim (i) follows directly from lines 12–18 and 21.

Claim (ii) can be proven as follows: We define a function $minheight : Q' \to \mathbb{N}$ for every $[t]_\rho \in Q'$ by

$$minheight([t]_\rho) = \min\{height(s) \mid (s, t) \in \rho\}$$

and show claim (ii) by induction on $minheight([t]_\rho)$. First let $[t]_\rho \in Q'$ such that $minheight([t]_\rho) = 1$, i.e., there exists an $\alpha \in \Sigma^{(0)}$ with $[t]_\rho = [\alpha]_\rho$. From $[t]_\rho \in Q'$ we deduce that $\hat{\mu}(t) \neq \emptyset$, hence, by Property (P2), $\hat{\mu}(\alpha) \neq \emptyset$. But then lines 7 and 11–18 imply either $\alpha \in \$P_1 \subseteq P$ or there exists a $\beta \in \$P_1 \subseteq P$ with $(\alpha, \beta) \in \rho$ (i.e., $[t]_\rho = [\alpha]_\rho = [\beta]_\rho$) which proves the induction base.

Now let $[t]_\rho \in Q'$ with $minheight([t]_\rho) = n + 1$ for an $n \in \mathbb{N}$. Clearly, there exist $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ such that $(t, \sigma(t_1, \ldots, t_k)) \in \rho$ and $height(\sigma(t_1, \ldots, t_k)) = minheight([t]_\rho) = n + 1$. First we show that there exist trees $t'_1, \ldots, t'_k \in \$P_n$ such that $(t_1, t'_1), \ldots, (t_k, t'_k) \in \rho$. From $[\sigma(t_1, \ldots, t_k)]_\rho = [t]_\rho \in Q'$ we deduce that $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$. From Lemma 3.7 then follows $\hat{\mu}(t_i) \neq \emptyset$ for every i with $1 \leq i \leq k$. Hence $[t_1]_\rho, \ldots, [t_k]_\rho \in Q'$. Since $minheight([t_i]_\rho) \leq height(t_i) \leq n$ for every such i, we can apply the induction hypothesis to obtain the existence of $t'_1, \ldots, t'_k \in P$ such that $[t_1]_\rho = [t'_1]_\rho, \ldots, [t_k]_\rho = [t'_k]_\rho$. From $t'_i \in P$ and Observation 4.10 we deduce that $height(t'_i) \leq height(t_i) = n$ for every i with $1 \leq i \leq k$. By Observation 4.11 then $t'_1, \ldots, t'_k \in \$P_n$. Hence, in particular, for every i with $1 \leq i \leq k$, $height(t'_j) = minheight([t_j]_\rho)$.

Since ρ is a congruence relation with respect to the term algebra (T_Σ, Σ) and $(t_1, t'_1), \ldots, (t_k, t'_k) \in \rho$, we also deduce that $(\sigma(t_1, \ldots, t_k), \sigma(t'_1, \ldots, t'_k)) \in \rho$, i.e., $[t]_\rho = [\sigma(t'_1, \ldots, t'_k)]_\rho$.

We now show that there is a j with $1 \leq j \leq k$ such that $minheight([t_j]_\rho) = n$. Assume that $minheight([t_i]_\rho) = height(t'_i) < n$ for every i with $1 \leq i \leq k$. Then

$$\begin{aligned} minheight([\sigma(t_1, \ldots, t_k)]_\rho) &\leq height(\sigma(t'_1, \ldots, t'_k)) \\ &= 1 + \max\{height(t'_i) \mid 1 \leq i \leq k\} \\ &< n + 1, \end{aligned}$$

contradicting to $minheight([\sigma(t_1, \ldots, t_k)]_\rho) = n + 1$. Hence there exists a j with $1 \leq j \leq k$ and $minheight([t_j]_\rho) = n$.

From $t'_j \in \$P_n \subseteq P$ and Observation 4.12, we deduce that our partial determinization algorithm runs at least $height(t'_j) + 1 = minheight([t_j]_\rho) + 1 = n + 1$ times through the repeat-until-loop. We show that there exists an $s \in \$P_{n+1}$ with $[s]_\rho = [t]_\rho$. Recall that $t'_1, \ldots, t'_k \in \$P_n$ and $[\sigma(t'_1, \ldots, t'_k)]_\rho = [\sigma(t_1, \ldots, t_k)] = [t]_\rho \in Q'$. Thus $\hat{\mu}(\sigma(t'_1, \ldots, t'_k)) \neq \emptyset$. From lines 7–18 we deduce that either there is a tree $t' \in \$P_n \subseteq P$ with $(\sigma(t'_1, \ldots, t'_k), t) \in \rho$ (i. e., $[t']_\rho = [\sigma(t'_1, \ldots, t'_k)] = [t]_\rho$) or $\sigma(t'_1, \ldots, t'_k) \in \$P_{n+1}$ (i. e., $[\sigma(t'_1, \ldots, t'_k)] = [t]_\rho$). This proves the induction step. □

Lemma 4.15 *Let ρ be of finite index. Then $det_t(\varphi, \rho, M)$ and $det(\varphi, \rho, M)$ are isomorphic via ψ.*

Proof. In Lemma 4.14 we have shown that ψ is a bijective mapping. Similar one can prove that $\psi|_{P_d}$ is a bijection between the sets P_d and Q'_d. Finally we show that $\nu_k(\sigma)_{(t_1,\ldots,t_k),t} = \mu'_k(\sigma)_{(\psi(t_1),\ldots,\psi(t_k)),\psi(t)}$ for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k$, $t \in P$.

$$\begin{aligned}
\nu_k(\sigma)_{(t_1,\ldots,t_k),t} &= \begin{cases} (S_M^{(\varphi)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t_1)^{-1} \odot (S_M^{(\varphi)}, \sigma(t_1, \ldots, t_k)) \\ \quad \text{if } (t, \sigma(t_1, \ldots, t_k)) \in \rho, \\ 0 \quad \text{otherwise}, \end{cases} \\
&= \begin{cases} (S_M^{(\varphi)}, t'_k)^{-1} \odot \cdots \odot (S_M^{(\varphi)}, t'_1)^{-1} \odot (S_M^{(\varphi)}, \sigma(t'_1, \ldots, t'_k)) \\ \quad \text{if } \exists t'_1, \ldots, t'_k \in T_\Sigma,\ \psi(t_1) = [t'_1]_\rho, \ldots, \psi(t_k) = [t'_k]_\rho, \text{ and} \\ \quad \psi(t) = [\sigma(t'_1, \ldots, t'_k)]_\rho, \\ 0 \quad \text{otherwise}, \end{cases} \\
&= \mu'_k(\sigma)_{(\psi(t_1),\ldots,\psi(t_k)),\psi(t)},
\end{aligned}$$

where the second equality follows by setting $t_1 = t'_1, \ldots, t_k = t'_k$. Hence ψ is an isomorphism between the bu-w-fta $det(\varphi, \rho, M)$ and $det_t(\varphi, \rho, M)$. □

Theorem 4.16 *Let ρ be of finite index. Then $det_t(\varphi, \rho, M)$ is a finite, deterministic bu-w-fta being equivalent to M.*

4.2. Instances of the Partial Determinization

In Section 4.1 we have presented a general version of the determinization which has two parameters: a tree series φ and an equivalence relation ρ. In this section we will instantiate this concept by considering two pairs of parameters which satisfy the required properties. The first pair of parameters is eligible for arbitrary semifields. But it turns out that the partial determinization with respect to this pair of parameters has some unnatural properties to be discussed on the fly. This we remedy by considering a second pair which is eligible only in the case of commutative semifields.

Let $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta over an arbitrary semifield A, until stated otherwise. Consider the particular tree series $\varphi = \tilde{1}$.

Observation 4.17 $\tilde{1}$ *has Property* (P1).

Then, for every tree $t \in T_\Sigma$,

$$(S_M^{(\tilde{1})}, t) = \begin{cases} (S_M, t) & \text{if } t \in \mathit{supp}(S_M), \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, we define the binary relation $\rho_M \subseteq T_\Sigma \times T_\Sigma$ by

$$\rho_M = \{(t, t') \in T_\Sigma \times T_\Sigma \mid h_\mu(t) = h_\mu(t')\}.$$

We now prove that ρ_M is eligible for the determinization in the sense that ρ_M is a congruence relation with respect to the term algebra (T_Σ, Σ) satisfying (P2) - (P4). First we prove that ρ_M is a congruence relation. Clearly, ρ_M is an equivalence relation for every bu-w-fta M over a semifield.

Observation 4.18 *ρ_M is a congruence relation with respect to the term algebra (T_Σ, Σ).*

Proof. Since ρ_M is the kernel of the homomorphism $h_\mu : T_\Sigma \to A^Q$, it follows that ρ_M is a congruence relation (cf. Section 2.5). □

We now show that ρ_M satisfies (P2), (P3), and (P4). This is done in Lemma 4.19 and Corollaries 4.21 and 4.23, respectively.

Lemma 4.19 *ρ_M satisfies Property* (P2).

Proof. Let $t, t' \in T_\Sigma$ such that $(t, t') \in \rho_M$. Then $h_\mu(t) = h_\mu(t')$, i. e., $h_\mu(t)_q = h_\mu(t')_q$ for every $q \in Q$. But then $h_\mu(t)_q \neq 0$ iff $h_\mu(t')_q \neq 0$. Hence $\hat{\mu}(t) \neq \emptyset$ iff $\hat{\mu}(t') \neq \emptyset$. □

We continue with (P3) by proving a stronger claim.

Lemma 4.20 *Let $t, t' \in T_\Sigma$ such that $(t, t') \in \rho_M$. Then $(S_M, t) = (S_M, t')$.*

Proof. Since $(t, t') \in \rho_M$, we have $h_\mu(t) = h_\mu(t')$, i. e., for every $q \in Q$, $h_\mu(t)_q = h_\mu(t')_q$. Then the following calculation proves the claim: $(S_M, t) = \sum_{q \in Q_d} h_\mu(t)_q = \sum_{q \in Q_d} h_\mu(t')_q = (S_M, t')$. □

Corollary 4.21 *ρ_M has Property* (P3).

We prepare the proof of Property (P4) as follows.

Lemma 4.22 *Let $t, t' \in T_\Sigma$ such that $(t, t') \in \rho_M$. Then $(S_M^{(\tilde{1})}, t) = (S_M^{(\tilde{1})}, t')$.*

Proof. First let $t \in \mathit{supp}(S_M)$. Since $(t, t') \in \rho_M$, we deduce from Corollary 4.21 that $t' \in \mathit{supp}(S_M)$. Hence, by Lemma 4.20, $(S_M^{(\tilde{1})}, t) = (S_M, t) = (S_M, t') = (S_M^{(\tilde{1})}, t')$. Now let $t \notin \mathit{supp}(S_M)$. Again by Corollary 4.21 we have $t' \notin \mathit{supp}(S_M)$. Hence $(S_M^{(\tilde{1})}, t) = 1 = (S_M^{(\tilde{1})}, t')$ which completes the proof. □

Corollary 4.23 *Let $t_1, t_1', \ldots, t_k, t_k' \in T_\Sigma$ such that $(t_1, t_1'), \ldots, (t_k, t_k') \in \rho_M$. Then*

$$\begin{aligned}&(S_M^{(\tilde{1})}, t_k)^{-1} \odot \cdots \odot (S_M^{(\tilde{1})}, t_1)^{-1} \odot (S_M^{(\tilde{1})}, \sigma(t_1, \ldots, t_k))\\ =\ &(S_M^{(\tilde{1})}, t_k')^{-1} \odot \cdots \odot (S_M^{(\tilde{1})}, t_1')^{-1} \odot (S_M^{(\tilde{1})}, \sigma(t_1', \ldots, t_k')).\end{aligned}$$

In particular, ρ_M has Property (P4).

Theorem 4.24 *If the set of states of $det(\tilde{1}, \rho_M, M)$ is finite, then $det(\tilde{1}, \rho_M, M)$ is a deterministic bu-w-fta being equivalent to M.*

Proof. By Observations 4.17 and 4.18, Lemma 4.19, and Corollaries 4.21 and 4.23, $\tilde{1}$ has Property (P1) and ρ_M satisfies Properties (P2)–(P4). Then the claimed statement is a corollary of Theorem 4.8. □

Thus we have found a particular partial determinization which, for a given nondeterministic bu-w-fta over an arbitrary semifield, generates a bu-w-ta which if it is finite is equivalent to M.

Example 4.25 Consider the nondeterministic bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$, where $Q = Q_d = \{q_1, q_2\}$, $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$, $A = (\mathbb{R} \cup \{+\infty\}, +, \cdot, ^{-1}, 0, 1)$, and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1 & 2 \end{pmatrix} \end{array}, \qquad \mu_2(\sigma) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{l} (q_1, q_1) \\ (q_1, q_2) \\ (q_2, q_1) \\ (q_2, q_2) \end{array} \end{array}.$$

It is an easy exercise to show that

$$\begin{aligned}
h_\mu(\alpha) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{array}{l} q_1 \\ q_2 \end{array}, \qquad (S_M, \alpha) = 3, \qquad \text{and}\\
h_\mu(t) &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{array}{l} q_1 \\ q_2 \end{array}, \qquad (S_M, t) = 4 \qquad \text{for every } t \in T_\Sigma \setminus \{\alpha\}.
\end{aligned}$$

Hence ρ_M has precisely two equivalence classes $\{\alpha\}$ and $T_\Sigma \setminus \{\alpha\}$. Applying the partial determinization gives the deterministic bu-w-fta $det(\tilde{1}, \rho_M, M) = (Q', \Sigma, Q_d', A, \mu')$ with $Q = Q_d' = \{[\alpha]_{\rho_M}, [\sigma(\alpha, \alpha)]_{\rho_M}\}$ and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} [\alpha]_{\rho_M} & [\sigma(\alpha,\alpha)]_{\rho_M} \end{array} \\ \begin{pmatrix} 3 & 0 \end{pmatrix} \end{array}, \qquad \mu_2(\sigma) = \begin{array}{cc} \begin{array}{cc} [\alpha]_{\rho_M} & [\sigma(\alpha,\alpha)]_{\rho_M} \end{array} & \\ \begin{pmatrix} 0 & 4/9 \\ 0 & 1/3 \\ 0 & 1/3 \\ 0 & 1/4 \end{pmatrix} & \begin{array}{l} ([\alpha]_{\rho_M}, [\alpha]_{\rho_M}) \\ ([\alpha]_{\rho_M}, [\sigma(\alpha,\alpha)]_{\rho_M}) \\ ([\sigma(\alpha,\alpha)]_{\rho_M}, [\alpha]_{\rho_M}) \\ ([\sigma(\alpha,\alpha)]_{\rho_M}, [\sigma(\alpha,\alpha)]_{\rho_M}) \end{array} \end{array}.$$

Unfortunately the determinization with respect to $\tilde{1}$ and ρ_M has two disadvantages. There is a nondeterministic bu-w-fta M which is equivalent to a deterministic one, but the determinization with respect to $\tilde{1}$ and ρ_M does not produce a finite automaton.

Example 4.26 Consider the bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$ such that $Q = Q_d = \{q_1, q_2\}$, $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$, $A = \mathit{Trop}_{sf}$, and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}, \qquad \mu_2(\sigma) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{array}{c} (q_1,q_1) \\ (q_1,q_2) \\ (q_2,q_1) \\ (q_2,q_2) \end{array} \end{array}.$$

Clearly, M is nondeterministic. We claim that

$$\forall t \in T_\Sigma \; \forall q \in Q : \; h_\mu(t)_q = \mathit{size}(t) \qquad (\dagger)$$

and prove this by induction on the structure of $t \in T_\Sigma$: If $t = \alpha \in \Sigma^{(0)}$, then $h_\mu(\alpha)_q = \mu_0(\alpha)_q = 1 = \mathit{size}(\alpha)$. Now let $t_1, t_2 \in T_\Sigma$. For every $q \in Q$,

$$\begin{aligned} h_\mu(\sigma(t_1,t_2))_q &= \min_{p_1,p_2 \in Q} \{h_\mu(t_1)_{p_1} + h_\mu(t_2)_{p_2} + \mu_2(\sigma)_{(p_1,p_2),q}\} \\ &= \min_{p_1,p_2 \in Q} \{\mathit{size}(t_1) + \mathit{size}(t_2) + 1\} \\ &= \mathit{size}(t_1) + \mathit{size}(t_2) + 1 \\ &= \mathit{size}(\sigma(\alpha,\alpha)). \end{aligned}$$

Then $(S_M, t) = \min_{q \in Q_d}\{h_\mu(t)_q\} = \min_{q \in Q_d}\{\mathit{size}(t)\} = \mathit{size}(t)$. Hence M generates for every tree its size.

Let us now prove that $\mathit{det}(\tilde{1}, \rho_M, M)$ is not a finite automaton. We show that the set Q' of states of $\mathit{det}(\tilde{1}, \rho_M, M)$ is an infinite set. By (†) we have

$$h_\mu(t) = \begin{pmatrix} \mathit{size}(t) \\ \mathit{size}(t) \end{pmatrix}$$

for every $t \in T_\Sigma$. Hence, for every $t, t' \in T_\Sigma$, $(t, t') \in \rho_M$ iff $\mathit{size}(t) = \mathit{size}(t')$ from which follows that ρ_M is of infinite index. Moreover, by (†), $\hat{\mu}(t) = \{q_1, q_2\}$ for every $t \in T_\Sigma$. Consequently, $Q' = \{[t]_{\rho_M} \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\} = \{[t]_{\rho_M} \mid t \in T_\Sigma\} = T_\Sigma/_{\rho_M}$. Because ρ_M has infinite index, Q' is an infinite set and hence $\mathit{det}(\tilde{1}, \rho_M, M)$ is not a finite automaton.

On the other hand, consider the deterministic bu-w-fta $M' = (P', \Sigma, P'_d, A, \mu')$ with $P' = P'_d = \{q'\}$, $A = \mathit{Trop}_{sf}$, $\mu'_0(\alpha) = (1)$, and $\mu'_2(\sigma) = (1)$. One can easily prove that $(S_{M'}, t) = \mathit{size}(t)$ for every $t \in T_\Sigma$, i. e., M and M' are equivalent.

Another disadvantage of the determinization with respect to $\tilde{1}$ and ρ_M is that even for a deterministic bu-w-fta M, $\mathit{det}(\tilde{1}, \varphi_M, M)$ need not to be a finite automaton.

Example 4.27 Consider the deterministic bu-w-fta $M' = (P', \Sigma, P'_d, A, \mu')$ from the end of Example 4.26. Note that $h_{\mu'}(t) = \mathit{size}(t)$ for every $t \in T_\Sigma$. Now one can prove similarly to Example 4.26 that the set of states of $\mathit{det}(\tilde{1}, \varphi_{M'}, M')$ is infinite. Hence $\mathit{det}(\tilde{1}, \varphi_{M'}, M')$ is not a finite bu-w-ta.

Now let us sum up the above phenomena by considering four syntactic (sub-)classes of bu-w-fta over an arbitrary but fixed semifield A. We start with the class $\mathfrak{C}^{n,bu}(A)$

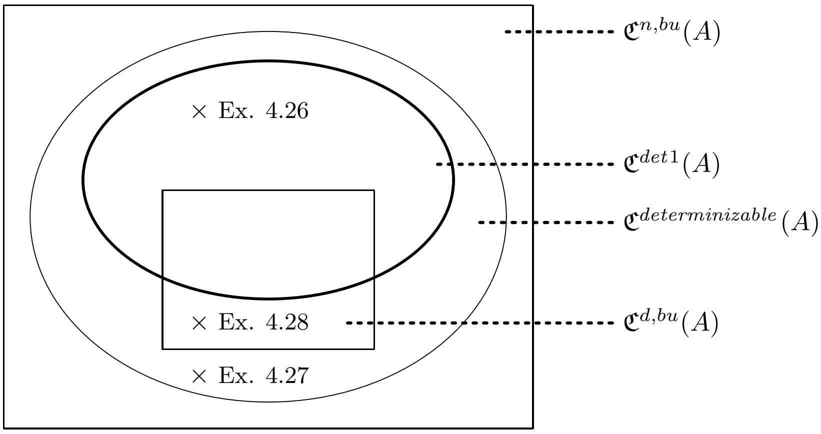

Figure 3: Classes of bu-w-fta over an arbitrary semifield A; the crosses × indicate the position of the bu-w-fta discussed in the mentioned examples

of all bu-w-fta over A. Clearly, the class $\mathfrak{C}^{d,bu}(A)$ of all deterministic bu-w-fta over A is a proper subclass of $\mathfrak{C}^{n,bu}(A)$. In Lemma 6.3 we will show that there exists an automaton which is not determinizable. (In fact, since $(\mathbb{R} \cup \{+\infty\}, +, \cdot, ^{-1}, 0, 1)$ is a semifield, this example also fits into the current environment.) Hence the class $\mathfrak{C}^{determinizable}(A)$ of all determinizable bu-w-fta over A is a proper subclass of $\mathfrak{C}^{n,bu}(A)$ which includes $\mathfrak{C}^{d,bu}(A)$. Examples 4.25, 4.26, and 4.27 show that the class $\mathfrak{C}^{det1}(A)$ of all bu-w-fta M over A such that $det(\tilde{1}, \rho_M, M)$ is a finite automaton, has to be drawn as shown in Figure 3.

Now we will avoid the second disadvantage by considering slightly more complicated φ and ρ under the additional assumption that the underlying semiring is a commutative semifield. Let $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta over a commutative semifield A and $\sqsubseteq$ be an arbitrary but fixed total order on Q, until stated otherwise. Let $\varphi'_M \in A\langle\langle T_\Sigma \rangle\rangle$, such that for every $t \in T_\Sigma$,

$$(\varphi'_M, t) = \begin{cases} h_\mu(t)_q & \text{if } h_\mu(t)_q \neq 0 \text{ and } h_\mu(t)_{q'} = 0 \text{ for every } q' \sqsubset q, \\ 1 & \text{otherwise.} \end{cases}$$

Note that, since $\sqsubseteq$ is a total order, (φ'_M, t) is well defined.

Observation 4.28 *For every $t \in T_\Sigma$, if $\hat{\mu}(t) \neq \emptyset$, then $(\varphi'_M, t) \neq 0$. Hence φ'_M has Property* (P1).

Then the tree series $S_M^{(\varphi'_M)}$ has the form

$$(S_M^{(\varphi'_M)}, t) = \begin{cases} (S_M, t) & \text{if } t \in supp(S_M), \\ h_\mu(t)_q & \text{if } t \notin supp(S_M) \text{ and } h_\mu(t)_q \neq 0 = h_\mu(t)_{q'} \text{ for every} \\ & q' \sqsubset q, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, we define the binary relation ρ'_M on T_Σ as follows:

$$\rho'_M = \{(t,t') \in T_\Sigma \times T_\Sigma \mid \exists a \in A \setminus \{0\} : \ h_\mu(t) = a \odot h_\mu(t')\}.$$

Clearly, ρ'_M is an equivalence relation. We now prove that ρ'_M is also a congruence relation with respect to the term algebra (T_Σ, Σ). This is done in Lemma 4.29 and Lemma 4.30.

Lemma 4.29 *Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, t'_1, \ldots, t_k, t'_k \in T_\Sigma$ such that there exist $a_1, \ldots, a_k \in A \setminus \{0\}$ with $h_\mu(t_1) = a_1 \odot h_\mu(t'_1), \ldots, h_\mu(t_k) = a_k \odot h_\mu(t'_k)$. Then $h_\mu(\sigma(t_1, \ldots, t_k)) = a_1 \odot \cdots \odot a_k \odot h_\mu(\sigma(t'_1, \ldots, t'_k))$.*

Proof. For every $q \in Q$,

$$\begin{aligned}
&h_\mu(\sigma(t_1, \ldots, t_k))_q \\
&\quad = \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&\quad = \sum_{(q_1,\ldots,q_k) \in Q^k} (a_1 \odot h_\mu(t'_1)_{q_1}) \odot \cdots \odot (a_k \odot h_\mu(t'_k)_{q_k}) \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&\qquad\qquad \text{(by assumption)} \\
&\quad = a_1 \odot \cdots \odot a_k \odot \sum_{(q_1,\ldots,q_k) \in Q^k} h_\mu(t'_1)_{q_1} \odot \cdots \odot h_\mu(t'_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&\qquad\qquad \text{(distributivity and commutativity)} \\
&\quad = a_1 \odot \cdots \odot a_k \odot h_\mu(\sigma(t'_1, \ldots, t'_k))_q .
\end{aligned}$$

Hence $h_\mu(\sigma(t_1, \ldots, t_k)) = a_1 \odot \cdots \odot a_k \odot h_\mu(\sigma(t'_1, \ldots, t'_k))$. □

Lemma 4.30 *ρ'_M is a congruence relation with respect to the term algebra (T_Σ, Σ).*

Proof. Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, t'_1, \ldots, t_k, t'_k \in T_\Sigma$ such that $(t_1, t'_1), \ldots, (t_k, t'_k) \in \rho'_M$. By the definition of ρ'_M there exist elements $a_1, \ldots, a_k \in A \setminus \{0\}$ with $h_\mu(t_1) = a_1 \odot h_\mu(t'_1), \ldots, h_\mu(t_k) = a_k \odot h_\mu(t'_k)$. By Lemma 4.29 then $h_\mu(\sigma(t_1, \ldots, t_k)) = a_1 \odot \cdots \odot a_k \odot h_\mu(\sigma(t_1, \ldots, t_k))$. Since $a_1, \ldots, a_k \in A \setminus \{0\}$ and A is zero divisor free (cf. Lemma 2.6), also $a_1 \odot \cdots \odot a_k \in A \setminus \{0\}$. Thus $(\sigma(t_1, \ldots, t_k), \sigma(t'_1, \ldots, t'_k)) \in \rho'_M$. □

Up to now we have proven that φ'_M has Property (P1) and that ρ'_M is a congruence relation on T_Σ. It remains to show that ρ'_M has Properties (P2), (P3), and (P4). This is done in Lemmas 4.31, 4.33, and 4.35, respectively.

Lemma 4.31 *ρ'_M has Property* (P2).

Proof. Let $t, t' \in T_\Sigma$ such that $(t, t') \in \rho'_M$, i. e., there exists an $a \in A \setminus \{0\}$ such that $h_\mu(t) = a \odot h_\mu(t')$. First let $\hat{\mu}(t) \neq \emptyset$, i. e., there exists a state $q \in Q$ such that $h_\mu(t)_q = a \odot h_\mu(t')_q \neq 0$. But then $h_\mu(t')_q \neq 0$ and hence $\hat{\mu}(t') \neq \emptyset$. Second, let $\hat{\mu}(t') \neq \emptyset$. Since also $(t, t') \in \rho'_M$, by the previous reasoning, $\hat{\mu}(t) \neq \emptyset$ follows. □

We start the proofs of Properties (P3) and (P4) with an observation which is a consequence of the distributivity of the underlying semiring.

Observation 4.32 *Let $t, t' \in T_\Sigma$. Then, for every $a \in A\setminus\{0\}$ with $h_\mu(t) = a \odot h_\mu(t')$,*

$$(S_M, t) = \sum_{q \in Q_d} h_\mu(t)_q = \sum_{q \in Q_d} a \odot h_\mu(t')_q = a \odot \sum_{q \in Q_d} h_\mu(t')_q = a \odot (S_M, t').$$

Lemma 4.33 *ρ'_M has Property* (P3).

Proof. Let $t, t' \in T_\Sigma$ such that $(t, t') \in \rho'_M$, i.e., there exists an $a \in A \setminus \{0\}$ such that $h_\mu(t) = a \odot h_\mu(t')$. By Observation 4.32 then $(S_M, t) = a \odot (S_M, t')$. Since, by Lemma 2.6, the semifield A is zero divisor free, we deduce that $t \in supp(S_M)$ iff $(S_M, t) \neq 0$ iff $(S_M, t') = a \odot (S_M, t) \neq 0$ iff $t' \in supp(S_M)$. □

To prove property (P4) we need the following lemma.

Lemma 4.34 *Let $t, t' \in T_\Sigma$ such that $\hat{\mu}(t) \neq \emptyset$. Then, for every $a \in A \setminus \{0\}$, $h_\mu(t) = a \odot h_\mu(t')$ implies $(S_M^{(\varphi'_M)}, t) = a \odot (S_M^{(\varphi'_M)}, t')$.*

Proof. First let $t \in supp(S_M)$. By Lemma 4.33, also $t' \in supp(S_M)$. Hence, by Observation 4.32, $(S_M^{(\varphi'_M)}, t) = (S_M, t) = a \odot (S_M, t') = a \odot (S_M^{(\varphi'_M)}, t')$. Now let $t \notin supp(S_M)$. By assumption we have $\hat{\mu}(t) \neq \emptyset$. By Lemma 4.31 and 4.33, also $\hat{\mu}(t') \neq \emptyset$ and $t' \notin supp(S_M)$. Hence

$$\begin{aligned} (S_M^{(\varphi'_M)}, t) &= h_\mu(t)_{\bar{q}} && \text{if } h_\mu(t)_{\bar{q}} \neq 0 \text{ and } h_\mu(t)_q = 0 \text{ for every } q \sqsubset \bar{q}, \\ (S_M^{(\varphi'_M)}, t') &= h_\mu(t')_{\hat{q}} && \text{if } h_\mu(t')_{\hat{q}} \neq 0 \text{ and } h_\mu(t')_q = 0 \text{ for every } q \sqsubset \hat{q}. \end{aligned}$$

Since $h_\mu(t)_q = a \odot h_\mu(t')_q$ for every $q \in Q$ and A is a zero divisor free semiring (cf. Lemma 2.6), $h_\mu(t)_q = a \odot h_\mu(t')_q = 0$ is equivalent to $h_\mu(t')_q = 0$. Hence $\bar{q} = \hat{q}$, so $(S_M^{(\varphi'_M)}, t) = h_\mu(t)_{\bar{q}} = a \odot h_\mu(t')_{\bar{q}} = a \odot h_\mu(t')_{\hat{q}} = a \odot (S_M^{(\varphi'_M)}, t')$. □

Lemma 4.35 *ρ'_M has Property* (P4).

Proof. Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $t_1, t'_1, \ldots, t_k, t'_k \in T_\Sigma$ such that $(t_1, t'_1), \ldots, (t_k, t'_k) \in \rho'_M$ and $\hat{\mu}(t_1) \neq \emptyset, \ldots, \hat{\mu}(t_k) \neq \emptyset$, $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$. Clearly there exist elements $a_1, \ldots, a_k \in A \setminus \{0\}$ with $h_\mu(t_1) = a_1 \odot h_\mu(t'_1), \ldots, h_\mu(t_k) = a_k \odot h_\mu(t'_k)$. From Lemma 4.29 then $h_\mu(\sigma(t_1, \ldots, t_k)) = a_1 \odot \cdots \odot a_k \odot h_\mu(\sigma(t'_1, \ldots, t'_k))$ follows. Moreover, we required that $\hat{\mu}(t_1) \neq \emptyset, \ldots, \hat{\mu}(t_k) \neq \emptyset$, $\hat{\mu}(\sigma(t_1, \ldots, t_k)) \neq \emptyset$. Hence, applying Lemma 4.34, we obtain

$$\begin{aligned} (S_M^{(\varphi'_M)}, t_1) &= a_1 \odot (S_M^{(\varphi'_M)}, t'_1), \\ &\vdots \\ (S_M^{(\varphi'_M)}, t_k) &= a_k \odot (S_M^{(\varphi'_M)}, t'_k), \\ (S_M^{(\varphi'_M)}, \sigma(t_1, \ldots, t_k)) &= a_1 \odot \cdots \odot a_k \odot (S_M^{(\varphi'_M)}, \sigma(t'_1, \ldots, t'_k)). \end{aligned}$$

From the commutativity of A now follows that

$$\begin{aligned}
&(S_M^{(\varphi'_M)}, t_k)^{-1} \odot \cdots \odot (S_M^{(\varphi'_M)}, t_1)^{-1} \odot (S_M^{(\varphi'_M)}, \sigma(t_1, \ldots, t_k)) \\
&\quad = (a_k \odot (S_M^{(\varphi'_M)}, t'_k))^{-1} \odot \cdots \odot (a_1 \odot (S_M^{(\varphi'_M)}, t'_1))^{-1} \\
&\qquad \odot\ (a_1 \odot \cdots \odot a_k \odot (S_M^{(\varphi'_M)}, \sigma(t'_1, \ldots, t'_k))) \\
&\quad = (S_M^{(\varphi'_M)}, t'_k)^{-1} \odot a_k^{-1} \odot \cdots \odot (S_M^{(\varphi'_M)}, t'_1)^{-1} \odot a_1^{-1} \\
&\qquad \odot\ a_1 \odot \cdots \odot a_k \odot (S_M^{(\varphi'_M)}, \sigma(t'_1, \ldots, t'_k)) \\
&\quad = (S_M^{(\varphi'_M)}, t'_k)^{-1} \odot \cdots \odot (S_M^{(\varphi'_M)}, t'_1)^{-1} \odot (S_M^{(\varphi'_M)}, \sigma(t'_1, \ldots, t'_k))
\end{aligned}$$

which completes the proof. □

Hence φ'_M and ρ'_M satisfy the requirements of the determinization introduced in Definition 4.2.

Theorem 4.36 *If the set of states of $det(\varphi'_M, \rho'_M, M)$ is finite, then $det(\varphi'_M, \rho'_M, M)$ is a deterministic bu-w-fta being equivalent to M.*

Proof. By Observation 4.28, and Lemmas 4.31, 4.33, and 4.35 φ'_M and ρ'_M have Properties (P1)–(P4). Hence the claimed statement is a corollary of Theorem 4.8. □

Now we show that $det(\varphi'_M, \rho'_M, M)$ improves the results concerning the finiteness of the partial determinization with respect to $\tilde{1}$ and ρ_M.

Lemma 4.37 *If $det(\tilde{1}, \rho_M, M)$ is a finite bu-w-fta, then $det(\varphi'_M, \rho'_M, M)$ is a finite bu-w-fta.*

Proof. Let Q' and Q'' denote the sets of states of $det(\tilde{1}, \rho_M, M)$ and $det(\varphi'_M, \rho'_M, M)$, respectively. Clearly, for every $t, t' \in T_\Sigma$, whenever $(t, t') \in \rho_M$, then $(t, t') \in \rho'_M$. Thus $card(Q') = card(\{[t]_{\rho_M} \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}) \geq card(\{[t]_{\rho'_M} \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}) = card(Q'')$. Hence, if Q' is a finite set, then Q'' is also a finite set. Thus if $det(\tilde{1}, \rho_M, M)$ is a finite bu-w-fta, then also $det(\varphi'_M, \rho'_M, M)$ is a finite bu-w-fta. □

Contrary to $det(\tilde{1}, \rho_M, M)$ the determinization with respect to φ'_M and ρ'_M has the nice property that for a given deterministic bu-w-fta M the resulting bu-w-ta $det(\varphi'_M, \rho'_M, M)$ is a finite and deterministic bu-w-fta. This finally is shown in Corollary 4.40. We prepare this corollary by two lemmas.

Lemma 4.38 *Let $t, t,' \in T_\Sigma$. If M is deterministic, then $\hat{\mu}(t) = \hat{\mu}(t')$ iff $(t, t') \in \rho'_M$.*

Proof. Let $\hat{\mu}(t) = \hat{\mu}(t')$. We have to prove that $(t, t') \in \rho'_M$. From Lemma 3.6 we deduce that $card\,(\hat{\mu}(t)) \leq 1$ for every tree $t \in T_\Sigma$. First let $card\,(\hat{\mu}(t)) = 1$, in other words $\hat{\mu}(t) = \hat{\mu}(t') \in Q$. We define $a = h_\mu(t)_{\hat{\mu}(t)} \odot (h_\mu(t')_{\hat{\mu}(t')})^{-1} \in A \setminus \{0\}$. Clearly $h_\mu(t)_{\hat{\mu}(t)} = a \odot h_\mu(t')_{\hat{\mu}(t')}$. Also, for every $q \in Q \setminus \{\hat{\mu}(t)\} = Q \setminus \{\hat{\mu}(t')\}$, $h_\mu(t)_q = 0 = a \odot 0 = a \odot h_\mu(t')_q$. Hence $h_\mu(t) = a \odot h_\mu(t')$ and so $(t, t') \in \rho'_M$. On the other hand, if $card\,(\hat{\mu}(t)) = 0$, that is $\hat{\mu}(t) = \hat{\mu}(t') = \emptyset$, then, for $a = 1 \in A \setminus \{0\}$ and $q \in Q$, we have $h_\mu(t)_q = 0 = 1 \odot 0 = a \odot h_\mu(t')_q$. Thus $(t, t') \in \rho'_M$.

Now let $(t, t') \in \rho'_M$. We have to show that $\hat{\mu}(t) = \hat{\mu}(t')$. Since $(t, t') \in \rho'_M$, there exists an $a \in A \setminus \{0\}$ with $h_\mu(t) = a \odot h_\mu(t')$. By Lemma 2.6 A is zero divisor free. Hence, for every $q \in Q$, $h_\mu(t)_q \neq 0$ iff $h_\mu(t')_q = a \odot h_\mu(t)_q \neq 0$. Thus $\hat{\mu}(t) = \hat{\mu}(t')$. $\square$

We note that $\hat{\mu}(t) = \hat{\mu}(t')$ neither implies $(t, t') \in \rho_M$ nor $(t, t') \in \rho'_M$ which shows the difference between the two instantiations of the partial determinization.

Lemma 4.39 *If M is deterministic, then the set of states of $det(\varphi'_M, \rho'_M, M)$ is finite.*

Proof. Let $det(\varphi'_M, \rho'_M, M) = (Q', \Sigma, Q'_d, A, \mu')$ and recall that $Q' = \{[t]_{\rho'_M} \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}$. Thus $card(Q') \leq index(\rho'_M)$. Moreover, by Lemma 4.38, $index(\rho'_M) = card(\{\hat{\mu}(t) \mid t \in T_\Sigma\})$ which is finite. In fact, since M is deterministic, by Lemma 3.6, either $\hat{\mu}(t) \in Q$ or $\hat{\mu}(t) = \emptyset$, hence $card(\{\hat{\mu}(t) \mid t \in T_\Sigma\}) \leq card(Q) + 1$. $\square$

Corollary 4.40 *If M is deterministic, then $det(\varphi'_M, \rho'_M, M)$ is a finite automaton.*

Example 4.41 We again consider the bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$ from Example 4.26, given by $Q = Q_d = \{q_1, q_2\}$, $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$, $A = Trop_{sf}$, and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}, \qquad \mu_2(\sigma) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{array}{c} (q_1, q_1) \\ (q_1, q_2) \\ (q_2, q_1) \\ (q_2, q_2) \end{array} \end{array}.$$

In Example 4.26 we have shown that $det(\tilde{1}, \rho_M, M)$ is an infinite bu-w-ta. But $A = Trop_{sf}$ is a commutative semifield. Hence we can apply the determinization with respect to φ'_M and ρ'_M which generates the bu-w-ta $det(\varphi'_M, \rho'_M, M) = (Q'', \Sigma, Q''_d, A, \mu'')$. To do so we have to calculate the equivalence classes of $\rho'_M = \{(t, t') \in T_\Sigma \times T_\Sigma \mid \exists a \in A \setminus \{0\} : \ h_\mu(t) = a + h_\mu(t')\}$.

We claim that $\rho'_M = T_\Sigma \times T_\Sigma$. Let $t, t' \in T_\Sigma$. Recall that, by Equation (†) of Example 4.26, for every $q \in Q$, $h_\mu(t)_q = size(t)$ and $h_\mu(t')_q = size(t')$. Then

$$\begin{aligned} &\forall q \in Q \ h_\mu(t)_q = size(t) - size(t') + h_\mu(t')_q \\ &\quad \Rightarrow h_\mu(t) = [size(t) - size(t')] + h_\mu(t') \\ &\quad \Rightarrow (t, t') \in \rho'_M. \end{aligned}$$

Hence $\rho'_M = T_\Sigma \times T_\Sigma$ and Q'' consists of one state $[\alpha]_{\rho'_M}$ which clearly is a final one. Then $Q'' = Q''_d = \{[\alpha]_{\rho'_M}\}$, $\mu''_0(\alpha) = (1)$, and $\mu''_2(\sigma) = (1)$. Thus $det(\varphi'_M, \rho'_M, M)$ is a finite bu-w-fta.

It seems that, for every M, $\rho'_M = T_\Sigma \times T_\Sigma$ by arguing that, for every $q \in Q$, the equation $h_\mu(t)_q = \underbrace{h_\mu(t)_q \odot h_\mu(t')_q^{-1}}_{a} \odot h_\mu(t')_q$ holds. However, then $a = h_\mu(t)_q \odot h_\mu(t')_q^{-1}$ depends on q, whereas it should be independent from q (cf. definition of ρ'_M). Thus in general $\rho'_M \neq T_\Sigma \times T_\Sigma$.

Now one could expect that a deterministic bu-w-fta M and its determinization $det(\varphi'_M, \rho'_M, M)$ with respect to φ'_M and ρ'_M are isomorphic. Unfortunately this does not hold in general.

Example 4.42 Consider the bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$, where $Q = Q' = \{q_1, q_2\}$, $A = \mathbb{B}_{sf}$, $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$, and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}, \qquad \mu_2(\sigma) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} & \begin{array}{c} (q_1, q_1) \\ (q_1, q_2) \\ (q_2, q_1) \\ (q_2, q_2) \end{array} \end{array}.$$

Clearly, M is a deterministic bu-w-fta such that, for every tree $t \in T_\Sigma$, we have $\hat{\mu}(t) = q_1$. Hence $\rho'_M = T_\Sigma \times T_\Sigma$ and the set of states of $det(\varphi'_M, \rho'_M, M) = (Q', \Sigma, Q'_d, A, \mu')$ is given by $Q' = \{[\alpha]_{\rho'_M}\}$. Moreover, $Q' = Q'_d = \{[\alpha]_{\rho'_M}\}$, $\mu'_0(\alpha) = (1)$, and $\mu_2(\sigma) = (1)$. Since M has two states and $det(\varphi'_M, \rho'_M, M)$ has only one state, they are not isomorphic.

The reason, why M and $det(\varphi'_M, \rho'_M, M)$ of Example 4.42 are not isomorphic, is the fact that M has a state $q_2 \in Q$ which is redundant, i.e., for every tree $t \in T_\Sigma$, we have $h_\mu(t)_{q_2} = 0$. If we delete all those states, we obtain a deterministic bu-w-fta which is isomorphic to $det(\rho'_M, \varphi'_M, M)$. Since the proof of this statement is long and would lead us too far away from our main subject, we here only state a necessary definition and the formal result. The proof can be found in the appendix.

Definition 4.43 *Let $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta and $\tilde{Q} \subseteq Q$. The* restriction of M with respect to $\tilde{Q}$ *is the bu-w-fta $rest_{\tilde{Q}}(M) = (\tilde{Q}, \Sigma, Q_d \cap \tilde{Q}, A, \tilde{\mu})$, where $\tilde{\mu} = (\tilde{\mu}_k \mid k \geq 0)$ and, for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$, $\tilde{\mu}_k : \Sigma^{(k)} \to A^{\tilde{Q}^k \times \tilde{Q}}$ such that, for every $q_1, \ldots, q_k, q \in \tilde{Q}$, the equation $\tilde{\mu}_k(\sigma)_{(q_1,\ldots,q_k),q} = \mu_k(\sigma)_{(q_1,\ldots,q_k),q}$ holds.*

Lemma 4.44 *Let M be deterministic and $\tilde{Q} = \{\hat{\mu}(t) \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}$. Then $det(\varphi'_M, \rho'_M, M)$ and $rest_{\tilde{Q}}(M)$ are isomorphic via $\psi : Q' \to \tilde{Q}$ such that $\psi([t]_{\rho'_M}) = \hat{\mu}(t)$, where Q' denotes the set of states of $det(\varphi'_M, \rho'_M, M)$.*

Now let us compare the classes $\mathfrak{C}^{n,bu}(A)$, $\mathfrak{C}^{d,bu}(A)$, $\mathfrak{C}^{determinizable}(A)$, and $\mathfrak{C}^{det2}(A)$ of bu-w-fta, where A is a commutative semifield (cf. Figure 4) and $\mathfrak{C}^{det2}(A)$ denotes the class of all bu-w-fta M over A such that $det(\varphi'_M, \rho'_M, M)$ is a finite automaton. Recall that $\mathfrak{C}^{n,bu}(A)$, $\mathfrak{C}^{d,bu}(A)$, and $\mathfrak{C}^{determinizable}(A)$ are the classes of all, deterministic, and determinizable bu-w-fta over A, respectively. By Corollary 4.40, $\mathfrak{C}^{d,bu}(A)$ is a subclass of $\mathfrak{C}^{det2}(A)$.

5. Comparison of Determinization for $\mathbb{B}$ and Power Set Construction

In Section 3.1 we have shown that bu-w-fta over the Boolean semiring and bu-fta coincide (cf. Lemma 3.9). Now we investigate the relationship between the deter-

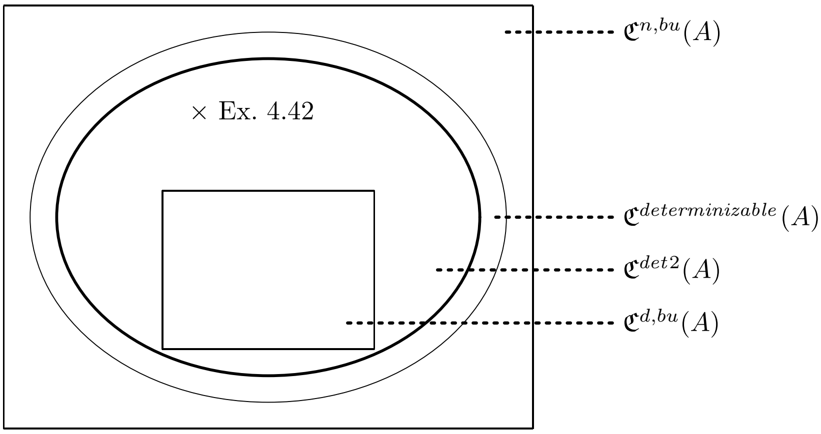

Figure 4: Classes of bu-w-fta over a commutative semifield A

minization which we have introduced in Section 4.2, and the power set construction known from the theory of bu-fta (cf. Section 2.2).

We first show that for a bu-w-fta $M = (Q, \Sigma, Q_d, \mathbb{B}, \mu)$ its determinization with respect to $\tilde{1}$ and ρ_M equals its determinization with respect to φ'_M and ρ'_M. Obviously, since the underlying algebra is the Boolean semiring, we have $\tilde{1} = \varphi'_M$ and $\rho_M = \rho'_M$ which implies the following observation.

Observation 5.1 *If M is a bu-w-fta over $\mathbb{B}$, then $det(\tilde{1}, \rho_M, M) = det(\varphi'_M, \rho'_M, M)$.*

Now we investigate the relationship between the determinization of bu-w-fta over the Boolean semiring and the power set construction of bu-fta. We proceed as follows: We start with an arbitrary bu-w-fta M over the Boolean semiring and construct its determinization $det(\tilde{1}, \rho_M, M)$. Then we consider its isomorphic image $\psi(det(\tilde{1}, \rho_M, M))$ with respect to the bijection $\psi : Q' \to \{\hat{\mu}(t) \mid t \in T_\Sigma, \hat{\mu}(t) \neq \emptyset\}$, defined by $\psi([t]_{\rho_M}) = \hat{\mu}(t)$ for every $[t]_{\rho'_M} \in Q'$ (cf. Lemma 5.3). On the other hand, we consider the bu-fta $\tilde{M}$ which is related to M in the sense of page 431. Then we apply the power set construction which generates $det(\tilde{M})$. This automaton we restrict to the set $P'' = \{\hat{\delta}(t) \mid t \in T_\Sigma, \hat{\delta}(t) \neq \emptyset\}$ of all reachable states which we denote by $rest_{P''}(det(\tilde{M}))$ (cf. Definition 5.2). In the rest of this section we show that $\psi(det(\tilde{1}, \rho_M, M))$ and $rest_{P''}(det(\tilde{M}))$ are related in the sense of page 431 (cf. Figure 5).

Definition 5.2 *Let $\tilde{M} = (Q, \Sigma, \delta, Q_d)$ with $\delta = \{\delta_\sigma\}_{k\geq 0, \sigma\in\Sigma^{(k)}}$ be a bu-fta and $\bar{Q} \subseteq Q$. We call the bu-fta $rest_{\bar{Q}}(\tilde{M}) = (\bar{Q}, \Sigma, \delta_{\bar{Q}}, Q_d \cap \bar{Q})$* restriction of $\tilde{M}$ with respect to $\bar{Q}$ *if $\delta_{\bar{Q}} = \{(\delta_{\bar{Q}})_\sigma\}_{k\geq 0, \sigma\in\Sigma^{(k)}}$ such that, for every $k \geq 0$ and $\sigma \in \Sigma^{(k)}$, we have $(\delta_{\bar{Q}})_\sigma = (\delta_\sigma)_{|\bar{Q}\times\cdots\times\bar{Q}}$.*

$$
\begin{tikzcd}
M := (Q, \Sigma, Q_d, \mathbb{B}, \mu) \arrow[r, "\text{related}"] \arrow[d, "\text{determin.}"'] & \tilde{M} := (Q, \Sigma, \delta, Q_d) \arrow[d, "\substack{\text{power set}\\ \text{construction}}"] \\
\begin{array}{c} det(\tilde{1}, \rho_M, M) \\ := (Q', \Sigma, Q'_d, \mathbb{B}, \mu') \end{array} \arrow[d, "\psi"'] & \begin{array}{c} det(\tilde{M}) \\ := (P', \Sigma, \delta', P'_d) \end{array} \arrow[d, "\text{restriction}"] \\
\begin{array}{c} \psi(det(\tilde{1}, \rho_M, M)) \\ := (Q'', \Sigma, Q''_d, \mathbb{B}, \mu'') \end{array} \arrow[r, "\text{related *}"] & \begin{array}{c} rest_{P''}(det(\tilde{M})) \\ := (P'', \Sigma, \delta'', P''_d) \end{array}
\end{tikzcd}
$$

Figure 5: Relationship of determinization over the Boolean semiring and power set construction

Lemma 5.3 *The mapping ψ is well defined and bijective.*

Proof. First we prove that ψ is well defined, i. e., that $\psi([t]_{\rho_M})$ is independent from the choice of the representant of $[t]_{\rho_M}$. Recall that $Q' = \{[t]_{\rho_M} \mid t \in T_\Sigma,\ \hat{\mu}(t) \neq \emptyset\}$. Then, for every $t, t' \in T_\Sigma$, $(t, t') \in \rho_M$ iff $h_\mu(t) = h_\mu(t')$ iff $\hat{\mu}(t) = \hat{\mu}(t')$ iff $\psi([t]_{\rho_M}) = \psi([t']_{\rho_M})$, where the second equivalence follows from the fact that the underlying semiring is $\mathbb{B}$. Hence ψ is well defined. We also deduce from the above consideration that ψ is one to one. Clearly, ψ is also onto, hence a bijection. □

Observation 5.4 *$\psi(det(\tilde{1}, \rho_M, M))$ is a deterministic bu-w-fta. Moreover, $\hat{\mu''}(t) = \psi(\hat{\mu'}(t))$ for every $t \in T_\Sigma$ with $\hat{\mu}(t) \neq \emptyset$.*

Now we investigate the sets Q'' and P'' of states of $\psi(det(\tilde{1}, \rho_M, M))$ and $rest_{P''}(det(\tilde{M}))$. Note that $Q'' = \{\hat{\mu}(t) \mid t \in T_\Sigma,\ \hat{\mu}(t) \neq \emptyset\}$.

Lemma 5.5 *$Q'' = P''$.*

Proof. From Lemma 3.9 we deduce that $\hat{\mu}(t) = \hat{\delta}(t)$ for every $t \in T_\Sigma$. The proof of $Q'' = P''$ then becomes trivial: $Q'' = \{\hat{\mu}(t) \mid t \in T_\Sigma,\ \hat{\mu}(t) \neq \emptyset\} = \{\hat{\delta}(t) \mid t \in T_\Sigma,\ \hat{\delta}(t) \neq \emptyset\} = P''$. □

Thus the sets of states of $\psi(det(\tilde{1}, \rho_M, M))$ and $rest_{P''}(det(\tilde{M}))$ are equal. How about the sets of designated states?

Lemma 5.6 *$Q''_d = P''_d$.*

Proof. In Lemma 5.5 we have proven $Q'' = P''$. Hence, for every state $q'' \in P''$, there exists a tree $t \in T_\Sigma$ such that $q'' = \hat{\mu}(t) \neq \emptyset$. Then

$$
\begin{aligned}
& q'' = \hat{\mu}(t) \in P''_d \\
& \quad \text{iff } \hat{\mu}(t) \in P'_d \cap P'' \text{ iff } \hat{\mu}(t) \in P'_d \text{ and } \hat{\mu}(t) \in P'' \text{ iff } \hat{\mu}(t) \in P'_d \text{ and } \hat{\mu}(t) \in Q'' \\
& \qquad \text{(by Lemma 5.5)} \\
& \quad \text{iff } \hat{\mu}(t) \in P'_d \text{ and } \hat{\mu}(t) \neq \emptyset \text{ iff } \hat{\mu}(t) \cap Q_d \neq \emptyset \text{ and } \hat{\mu}(t) \neq \emptyset \text{ iff } \hat{\mu}(t) \cap Q_d \neq \emptyset \\
& \quad \text{iff } \exists q \in Q_d \cap \hat{\mu}(t) \text{ iff } \exists q \in Q_d : h_\mu(t)_q = 1 \text{ iff } \bigvee_{q \in Q_d} h_\mu(t)_q = 1 \\
& \qquad \text{(calculating in } \mathbb{B}) \\
& \quad \text{iff } t \in supp(S_M) \text{ iff } [t]_{\rho_M} \in Q'_d \\
& \quad \text{iff } q'' = \psi\left([t]_{\rho_M}\right) \in Q''_d.
\end{aligned}
$$

Hence $Q''_d = P''_d$. □

Finally we show that the tree representation μ'' of $\psi(det(\tilde{1}, \rho_M, M))$ and the family δ'' of transition functions of $rest_{P''}(det(\tilde{1}, \rho_M, M))$ coincide. For this purpose we first prove two lemmas.

Lemma 5.7 *Let $t \in T_\Sigma$ such that $\hat{\mu}(t) \neq \emptyset$. Then $\hat{\mu''}(t) = \hat{\mu}(t)$.*

Proof. By Observation 5.4, Lemma 4.6 (i), and the definition of ψ we have $\hat{\mu''}(t) = \psi(\hat{\mu'}(t)) = \psi([t]_{\rho_M}) = \hat{\mu}(t)$. □

Lemma 5.8 *Let $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$ such that $\hat{\mu}(t_1) \neq \emptyset$, ..., $\hat{\mu}(t_k) \neq \emptyset$. Then $\left\{q'' \in Q'' \,\middle|\, \mu''_k(\sigma)_{\hat{\mu}(t_1),\ldots,\hat{\mu}(t_k),q''} \neq 0\right\} = \delta''_\sigma(\hat{\mu}(t_1), \ldots, \hat{\mu}(t_k))$.*

Proof. The following straightforward calculation proves the claim.

$$
\begin{aligned}
& \{q'' \in Q'' \mid \mu''_k(\sigma)_{\hat{\mu}(t_1),\ldots,\hat{\mu}(t_k),q''} \neq 0\} \\
& = \{q'' \in Q'' \mid \mu''_k(\sigma)_{\hat{\mu''}(t_1),\ldots,\hat{\mu''}(t_k),q''} \neq 0\} \qquad \text{(by Lemma 5.7)} \\
& = \bigcup_{q''_i \in \hat{\mu''}(t_i)} \{q'' \in Q'' \mid \mu''_k(\sigma)_{(q''_1,\ldots,q''_k),q''} \neq 0\} \\
& \qquad\qquad (card(\hat{\mu}(t_i)) = 1 \text{ by assumption and Lemma 3.6)} \\
& = \hat{\mu''}(\sigma(t_1, \ldots, t_k)) \qquad (\psi(det(\tilde{1}, \rho_m, M)) \text{ is deterministic, then Lemma 3.5)} \\
& = \hat{\mu}(\sigma(t_1, \ldots, t_k)) \qquad \text{(by Lemma 5.7)} \\
& = \hat{\delta}(\sigma(t_1, \ldots, t_k)) \qquad \text{(by Lemma 3.9)} \\
& = \bigcup \{\delta_\sigma(p_1, \ldots, p_k) \mid p_1 \in \hat{\delta}(t_1), \ldots, p_k \in \hat{\delta}(t_k)\} \\
& = \bigcup \{\delta_\sigma(p_1, \ldots, p_k) \mid p_1 \in \hat{\mu}(t_1), \ldots, p_k \in \hat{\mu}(t_k)\} \qquad \text{(by Lemma 3.9)} \\
& = \delta'_\sigma(\hat{\mu}(t_1), \ldots, \hat{\mu}(t_k)) \\
& = \delta''_\sigma(\hat{\mu}(t_1), \ldots, \hat{\mu}(t_k)). \qquad (q'' \in Q'' = P'' \text{ by assumption and Lemma 5.5)}
\end{aligned}
$$
□

Lemma 5.9 *For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q''_1, \ldots, q''_k, q'' \in Q''$, the equivalence $\mu''_k(\sigma)_{(q''_1,\ldots,q''_k),q''} = 1$ iff $q'' \in \delta''_\sigma(q''_1, \ldots, q''_k)$ holds.*

Proof. Since $q_1'', \ldots, q_k'' \in Q''$, there are trees $t_1, \ldots, t_k \in T_\Sigma$ with $q_1'' = \psi([t_1]_{\rho_M}) = \hat{\mu}(t_1) \neq \emptyset$, $\ldots$, $q_k'' = \psi([t_k]_{\rho_M}) = \hat{\mu}(t_k) \neq \emptyset$. Hence, by Lemma 5.8, $q'' \in \delta''_\sigma(\hat{\mu}(t_1), \ldots, \hat{\mu}(t_k)) = \{q'' \in Q'' \mid \mu_k''(\sigma)_{\hat{\mu}(t_1),\ldots,\hat{\mu}(t_k),q''} \neq 0$ iff $\mu_k''(\sigma)_{(q_1'',\ldots,q_k''),q''} = 1$ (calculating in $\mathbb{B}$). □

Now we show the relationship between $\psi(det(\tilde{1}, \rho_M, M))$ and $rest_{P''}(det(\tilde{M}))$.

Theorem 5.10 *$\psi(det(\tilde{1}, \rho_M, M))$ and $rest_{P''}(det(\tilde{M}))$ are related.*

Proof. The claim is a corollary of Lemmas 5.5, 5.6, and 5.9. □

Recall that, by Observation 5.1, $det(\tilde{1}, \rho_M, M) = det(\varphi_M', \rho_M', M)$. Thus also $det(\varphi_M', \rho_M', M)$ and $rest_{P''}(det(\tilde{M}))$ are related.

6. Comparison of Classes of Recognizable Tree Series

In this section we compare the classes $A^{n,bu}\langle\langle T_\Sigma \rangle\rangle$, $A^{d,bu}\langle\langle T_\Sigma \rangle\rangle$, $A^{n,td}\langle\langle T_\Sigma \rangle\rangle$, and $A^{d,td}\langle\langle T_\Sigma \rangle\rangle$ for classes of semirings.

First let us compare $A^{n,bu}\langle\langle T_\Sigma \rangle\rangle$ and $A^{d,bu}\langle\langle T_\Sigma \rangle\rangle$. We present a condition on the semifield which ensures that the determinization with respect to $\tilde{1}$ and ρ_M is a finite automaton, i. e., we have to ensure that $det(\tilde{1}, \rho_M, M)$ has a finite set of states. A sufficient such condition is that the semifield is finitely generated. For the rest of this section let $M = (Q, \Sigma, Q_d, A, \mu)$ be a bu-w-fta over an arbitrary semifield A if not stated otherwise.

Lemma 6.1 *Let M be a bu-w-fta over a locally finite semifield. Then the set of states of $det(\tilde{1}, \rho_M, M)$ is a finite set. Thus $det(\tilde{1}, \rho_M, M)$ is a finite and deterministic bu-w-fta being equivalent to M.*

Proof. Let $det(\tilde{1}, \rho_M, M) = (Q', \Sigma, Q_d', A, \mu')$ and consider the set

$$\hat{A} = \{\mu_k(\sigma)_{(q_1,\ldots,q_k),q} \mid k \geq 0,\ \sigma \in \Sigma^{(k)},\ q_1, \ldots, q_k, q \in Q\}.$$

Both, Σ and Q are finite, hence $\hat{A}$ is finite. Since A is a locally finite semiring, $\langle \hat{A} \rangle$ is finite. It is an easy exercise to prove by structural induction that $h_\mu(t)_q \in \langle \hat{A} \rangle$ for every $t \in T_\Sigma$ and $q \in Q$. Thus $h_\mu(t) \in \langle \hat{A} \rangle^Q$ for every $t \in T_\Sigma$ and hence $\{h_\mu(t) \mid t \in T_\Sigma\} \subseteq \langle \hat{A} \rangle^Q$. Since $Q' = \{[t]_{\rho_M} \mid t \in T_\Sigma,\ \hat{\mu}(t) \neq \emptyset\}$, we deduce that $card(Q') \leq card((T_\Sigma)/_{\rho_M}) = card(\{h_\mu(t) \mid t \in T_\Sigma\}) \leq card(\langle \hat{A} \rangle^Q) = card(\langle \hat{A} \rangle)^{card(Q)}$. Both, $\langle \hat{A} \rangle$ and Q are finite. Thus Q' is finite. □

Theorem 6.2 *If A is a locally finite semifield, then the following equality holds: $A^{n,bu}\langle\langle T_\Sigma \rangle\rangle = A^{d,bu}\langle\langle T_\Sigma \rangle\rangle$.*

Proof. The inclusion $A^{n,bu}\langle\langle T_\Sigma \rangle\rangle \supseteq A^{d,bu}\langle\langle T_\Sigma \rangle\rangle$ is trivially true. Let M be a bu-w-fta over a locally finite semifield A. Then, by Lemma 6.1, $det(\tilde{1}, \rho_M, M)$ is a finite automaton. Hence, by Theorem 4.36, $det(\tilde{1}, \rho_M, M)$ is a deterministic bu-w-fta which is equivalent to M. Thus $A^{n,bu}\langle\langle T_\Sigma \rangle\rangle \subseteq A^{d,bu}\langle\langle T_\Sigma \rangle\rangle$. □

Does this theorem also hold for arbitrary semifields? The answer is no.

Lemma 6.3 *There is a semifield A such that $A^{n,bu}\langle\langle T_\Sigma\rangle\rangle \not\subseteq A^{d,bu}\langle\langle T_\Sigma\rangle\rangle$.*

Proof. Let M be the bu-w-fta given by $Q = Q_d = \{q_1, q_2\}$, $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}\}$, $A = (\mathbb{R} \cup \{+\infty\}, +, \cdot, ^{-1}, 0, 1)$, and

$$\mu_0(\alpha) = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}, \qquad \mu_1(\gamma) = \begin{array}{cc} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & \begin{array}{c} (q_1) \\ (q_2) \end{array} \end{array}.$$

One can easily show that $h_\mu(\gamma^n(\alpha)) = \begin{pmatrix} 1 \\ 2^n \end{pmatrix}$ for every $n \geq 0$ and thus $(S_M, \gamma^n(\alpha)) = 1 + 2^n$.

We prove by contradiction that $S_M \notin A^{d,bu}\langle\langle T_\Sigma\rangle\rangle$. Assume that there exists a deterministic finite bu-w-fta $M' = (Q', \Sigma, Q'_d, A, \mu')$ such that $S_M = S_{M'}$. By Lemma 3.6 we have $card(\hat{\mu}'(\gamma^n(\alpha))) \leq 1$ for every $n \in \mathbb{N}$. If $card(\hat{\mu}'(\gamma^n(\alpha))) = 0$ or $\hat{\mu}'(\gamma^n(\alpha)) \notin Q'_d$, then $0 = \sum_{q' \in Q'_d} h_{\mu'}(\gamma^n(\alpha))_{q'} = (S_{M'}, \gamma^n(\alpha)) = (S_M, \gamma^n(\alpha)) = 2^n + 1$ which is a contradiction. Hence $\hat{\mu}'(\gamma^n(\alpha)) \in Q'_d$. Let us denote the state $\hat{\mu}'(\gamma^n(\alpha))$ by q'_n. Also, $h_{\mu'}(\gamma^n(\alpha))_{q'_n} = \sum_{q' \in Q'_d} h_{\mu'}(\gamma^n(\alpha))_{q'} = (S_{M'}, \gamma^n(\alpha)) = (S_M, \gamma^n(\alpha)) = 2^n + 1$. Let us denote this last equation by $(*)$. It is an easy exercise to show that, for every $q' \in Q'$ and $n \geq 1$,

$$\mu'_0(\alpha)_{(),q'} = \begin{cases} 2 & \text{if } q' = \hat{\mu}'(\alpha), \\ 0 & \text{otherwise} \end{cases} \qquad \text{and}$$

$$\mu'_1(\gamma)_{q'_{n-1},q'} = \begin{cases} (2^{n-1}+1)^{-1} \cdot (2^n + 1) & \text{if } q' = q'_n, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $n \geq 0$ and assume that there exists an i with $0 \leq i < n$ such that $q'_i = q'_n$. Then

$$\begin{aligned} & h_{\mu'}(\gamma^{n+1}(\alpha))_{q'_{n+1}} \\ & \quad = \sum_{q' \in Q'} h_{\mu'}(\gamma^n(\alpha))_{q'} \cdot \mu'_1(\gamma)_{q',q'_{n+1}} \\ & \quad = h_{\mu'}(\gamma^n(\alpha))_{q'_n} \odot \mu'_1(\gamma)_{q'_n,q'_{n+1}} \qquad \text{(by Lemma 3.6)} \\ & \quad = (2^n + 1) \cdot \mu'_1(\gamma)_{q'_n,q'_{n+1}} \qquad \text{(by } (*)) \\ & \quad = (2^n + 1) \cdot \mu'_1(\gamma)_{q'_i,q'_{i+1}} \qquad \text{(by assumption, } M' \text{ is deterministic)} \\ & \quad = (2^n + 1) \cdot (2^i + 1)^{-1} \cdot (2^{i+1} + 1) \\ & \quad \neq 2^{n+1} + 1, \qquad \text{(straightforward calculation)} \end{aligned}$$

which contradicts $(*)$. Thus $q'_i \neq q'_j$ whenever $i \neq j$. Hence Q' is an infinite set and thus M' is not a finite automaton. □

Now let us compare the subclasses $\mathfrak{C}^{det1}(A)$, $\mathfrak{C}^{determinizable}(A)$, and $\mathfrak{C}^{d,bu}(A)$ of $\mathfrak{C}^{n,bu}(A)$, where A is a locally finite semifield. From Theorem 6.2 we deduce that every bu-w-fta over A is determinizable. Moreover, by Lemma 4.39, for every bu-w-fta M over A, $det(\tilde{1}, \rho_M, M)$ is a finite automaton. Hence

$$\mathfrak{C}^{n,bu}(A) = \mathfrak{C}^{det1}(A) = \mathfrak{C}^{determinizable}(A).$$

Now we prove that the classes $A^{n,bu}\langle\langle T_\Sigma\rangle\rangle$ and $A^{n,td}\langle\langle T_\Sigma\rangle\rangle$ are equal assuming that A is commutative. The proof is the straightforward generalization of the tree case.

Theorem 6.4 *Let A be a commutative semiring. Then $A^{n,bu}\langle\langle T_\Sigma\rangle\rangle = A^{n,td}\langle\langle T_\Sigma\rangle\rangle$.*

Proof. We term the bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$ and the td-w-fta $N = (Q, \Sigma, Q_d, A, \mu')$ related if $\mu'_k(\sigma)_{q,(q_1,\ldots,q_k)} = \mu_k(\sigma)_{(q_1,\ldots,q_k),q}$ for every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, q \in Q$. Note that, for every bu-w-fta, there exists a related td-w-fta and vice versa. Let M and N be related bu-w-fta and td-w-fta, respectively. We claim that, for every $t \in T_\Sigma$ and $q \in Q$, the equation $h_{\mu'}(t)_q = h_\mu(t)_q$ holds and prove this by induction on the structure of $t \in T_\Sigma$: Let $k \geq 0$, $\sigma \in \Sigma^{(0)}$, $t_1, \ldots, t_k \in T_\Sigma$, and $t = \sigma(t_1, \ldots, t_k)$. Then, for every $q \in Q$,

$$\begin{aligned}
h_\mu(t)_q &= \overline{\mu_k(\sigma)}\left(h_\mu(t_1), \ldots, h_\mu(t_k)\right)_q \\
&= \sum_{(q_1,\ldots,q_k)\in Q^k} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \\
&= \sum_{(q_1,\ldots,q_k)\in Q^k} h_{\mu'}(t_1)_{q_1} \odot \cdots \odot h_{\mu'}(t_k)_{q_k} \odot \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
&= \sum_{(q_1,\ldots,q_k)\in Q^k} h_{\mu'}(t_1)_{q_1} \odot \cdots \odot h_{\mu'}(t_k)_{q_k} \odot \mu'_k(\sigma)_{q,(q_1,\ldots,q_k)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(M and N are related)} \\
&= \overline{\mu'_k(\sigma)}\left(h_{\mu'}(t_1), \ldots, h_{\mu'}(t_k)\right)_q \qquad \text{(A is commutative)} \\
&= h_{\mu'}(t)_q.
\end{aligned}$$

Note that the induction base is given by $k = 0$. Now

$$\begin{aligned}
& S \in A^{n,bu}\langle\langle T_\Sigma\rangle\rangle \\
& \Leftrightarrow \exists M = (Q, \Sigma, Q_d, A, \mu) \text{ bu-w-fta with } (S, t) = (S_M, t) = \sum_{q\in Q_d} h_\mu(t)_q \\
& \Leftrightarrow \exists N = (Q, \Sigma, Q_d, A, \mu') \text{ td-w-fta with } (S, t) = (S_N, t) = \sum_{q\in Q_d} h_{\mu'}(t)_q \\
& \qquad \text{(M, N are related)} \\
& \Leftrightarrow S \in A^{n,td}\langle\langle T_\Sigma\rangle\rangle
\end{aligned}$$

□

Now let us investigate the relationship between the classes $A^{n,td}\langle\langle T_\Sigma\rangle\rangle$ and $A^{d,td}\langle\langle T_\Sigma\rangle\rangle$. From the theory of tree languages we know that there exist nondeterministic top-down tree automata which are not determinizable. We now extend this result.

Theorem 6.5 *Let A be a commutative or zero divisor free semiring or a semifield. Also, let Σ be a ranked alphabet with card $\left(\Sigma^{(0)}\right) \geq 2$ and $\bigcup_{k\geq 2}\Sigma^{(k)} \neq \emptyset$. Then $A^{d,td}\langle\langle T_\Sigma\rangle\rangle \subsetneq A^{n,td}\langle\langle T_\Sigma\rangle\rangle$.*

Proof. Let $\alpha, \beta \in \Sigma^{(0)}$ and $\sigma \in \Sigma^{(k)}$ for some $k \geq 2$. Consider the tree series

$$S = 1\cdot\sigma(\alpha,\dots,\alpha,\beta) + 1\cdot\sigma(\beta,\alpha,\dots,\alpha).$$

First we show $S \in A^{n,td}\langle\langle T_\Sigma\rangle\rangle$ by constructing an appropriate td-w-fta $M = (Q,\Sigma,Q_d,A,\mu)$. Let $Q = \{q_\alpha, q_\beta, q_d\}, Q_d = \{q_d\}$, and

$$\mu_0(\alpha)_{q,()} = \begin{cases} 1 & \text{if } q = q_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_0(\beta)_{q,()} = \begin{cases} 1 & \text{if } q = q_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_k(\sigma)_{q,(q_1,\dots,q_k)} = \begin{cases} 1 & \text{if } (q = q_d \text{ and } q_1 = \cdots = q_{k-1} = q_\alpha \text{ and } q_k = q_\beta) \text{ or} \\ & \quad (q = q_d \text{ and } q_1 = q_\beta \text{ and } q_2 = \cdots = q_k = q_\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $S_M = 1\cdot\sigma(\alpha,\dots,\alpha,\beta) + 1\cdot\sigma(\beta,\alpha,\dots,\alpha) = S$, hence $S \in A^{n,td}\langle\langle T_\Sigma\rangle\rangle$.

Now we prove $S \notin A^{d,td}\langle\langle T_\Sigma\rangle\rangle$ by contradiction. Assume the converse, i. e., there exists a deterministic td-w-fta $M' = (Q',\Sigma,\{q'_d\},A,\mu')$ with $S_{M'} = S = 1\cdot\sigma(\alpha,\dots,\alpha,\beta) + 1\cdot\sigma(\beta,\alpha,\dots,\alpha)$. By the definition of deterministic td-w-fta there exists at most one k-tuple $(q'_1,\dots,q'_k) \in Q'^k$ such that $\mu'_k(\sigma)_{q'_d,(q'_1,\dots,q'_k)} \neq 0$. Moreover,

$$\begin{aligned} 1 &= (S_{M'}, \sigma(\alpha,\dots,\alpha,\beta)) \\ &= h_{\mu'}(\sigma(\alpha,\dots,\alpha,\beta))_{q'_d} \\ &= \mu'_k(\sigma)_{q'_d,(q'_1,\dots,q'_k)} \odot \mu'_0(\alpha)_{q'_1,()} \odot \cdots \odot \mu'_0(\alpha)_{q'_{k-1},()} \odot \mu'_0(\beta)_{q'_k,()} \end{aligned} \tag{1}$$

$$\begin{aligned} 1 &= (S_{M'}, \sigma(\beta,\alpha,\dots,\alpha)) \\ &= h_{\mu'}(\sigma(\beta,\alpha,\dots,\alpha))_{q'_d} \\ &= \mu'_k(\sigma)_{q'_d,(q'_1,\dots,q'_k)} \odot \mu'_0(\beta)_{q'_1,()} \odot {\mu'}_0(\alpha)_{q'_2,()} \odot \cdots \odot \mu'_0(\alpha)_{q'_k,()} \end{aligned} \tag{2}$$

$$\begin{aligned} 0 &= (S_{M'}, \sigma(\alpha,\dots,\alpha)) \\ &= h_{\mu'}(\sigma(\alpha,\dots,\alpha))_{q'_d} \\ &= \mu'_k(\sigma)_{q'_d,(q'_1,\dots,q'_k)} \odot \mu'_0(\alpha)_{q'_1,()} \odot \cdots \odot \mu'_0(\alpha)_{q'_k,()} \end{aligned} \tag{3}$$

Clearly, by (1) and (2),

$$\mu'_0(\alpha)_{q'_1,()} \neq 0, \ldots, \mu'_0(\alpha)_{q'_k,()} \neq 0, \mu'_0(\beta)_{q'_1,()} \neq 0, \mu'_0(\beta)_{q'_k,()} \neq 0,$$
$$\mu'_k(\sigma)_{q'_d,(q'_1,\ldots,q'_k)} \neq 0. \tag{4}$$

- If A is a commutative semiring, then, by (1) and (3),

$$\begin{aligned}
&\mu'_0(\alpha)_{q'_k,()}\\
&= 1 \odot \mu'_0(\alpha)_{q'_k,()}\\
&= \Big(\mu'_k(\sigma)_{q'_d,(q'_1,\ldots,q'_k)} \odot \mu'_0(\alpha)_{q'_1,()} \odot \cdots \odot \mu'_0(\alpha)_{q'_{k-1},()} \odot \mu'_0(\beta)_{q'_k,()}\Big)\\
&\quad \odot\ \mu'_0(\alpha)_{q'_k,()}\\
&= \Big(\mu'_k(\sigma)_{q'_d,(q'_1,\ldots,q'_k)} \odot \mu'_0(\alpha)_{q'_1,()} \odot \cdots \odot \mu'_0(\alpha)_{q'_{k-1},()} \odot \mu'_0(\alpha)_{q'_k,()}\Big)\\
&\quad \odot\ \mu'_0(\beta)_{q'_k,()}\\
&= 0 \odot \mu'_0(\beta)_{q'_k,()}\\
&= 0
\end{aligned}$$

  which contradicts to $\mu'_0(\alpha)_{q'_k,()} \neq 0$ (cf. (4)).
- If A is a zero divisor free semiring, then

$$\begin{aligned}
0 &\neq \mu'_k(\sigma)_{q'_d,(q'_1,\ldots,q'_k)} \odot \mu'_0(\alpha)_{q'_1,()} \odot \cdots \odot \mu'_0(\alpha)_{q'_k,()} && \text{(by (4))}\\
&= 0. && \text{(by (3))}
\end{aligned}$$

  Contradiction.
- If A is a semifield, then, by Lemma 2.6, A is zero divisor free. Hence the above calculation applies.

In all three cases we obtained contradictions. Hence there does not exist a deterministic td-w-fta M' with $S_{M'} = S$, i. e., $S \notin A^{d,td}\langle\langle T_\Sigma\rangle\rangle$. □

Summing up the results of this section gives the following theorem.

Theorem 6.6 *Let A be a commutative and locally finite semifield and Σ a ranked alphabet. Then $A^{d,bu}\langle\langle T_\Sigma\rangle\rangle = A^{n,bu}\langle\langle T_\Sigma\rangle\rangle = A^{n,td}\langle\langle T_\Sigma\rangle\rangle$. Moreover, if Σ satisfies card $\left(\Sigma^{(0)}\right) \geq 2$ and $\bigcup_{k\geq 2} \Sigma^{(k)} \neq \emptyset$, then $A^{d,td}\langle\langle T_\Sigma\rangle\rangle \subsetneq A^{n,td}\langle\langle T_\Sigma\rangle\rangle$.*

Proof. The claim is a corollary of Theorems 6.2, 6.4, and 6.5. □

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A. The Behaviour of $det(\varphi', \rho'_M, M)$ on Deterministic bu-w-fta

In Example 4.42 we have shown that a deterministic bu-w-fta M and its determinization $det(\varphi'_M, \rho'_M, M)$ need not to be isomorphic. The reason for this was the existence of redundant states. Now we restrict M to all those states which occur in a nonzero derivation of any tree. We show that this restricted bu-w-fta is isomorphic to $det(\varphi'_M, \rho'_M, M)$. Thus the determinization with respect to φ'_M and ρ'_M performs this restriction. Since we decrease the number of states, we obtain as a corollary that the determinization with respect to φ' and ρ'_M calculates for every deterministic bu-w-fta M a finite bu-w-fta.

For the rest of this section let M, $det(\varphi'_M, \rho'_M, M)$, $\tilde{Q}$, $rest_{\tilde{Q}}(M)$, and ψ stand for

- a deterministic bu-w-fta $M = (Q, \Sigma, Q_d, A, \mu)$ over a commutative semifield A,
- the determinization $det(\varphi'_M, \rho'_M, M) = (Q', \Sigma, Q'_d, A, \mu')$ of M with respect to φ'_M and ρ'_M,
- $\tilde{Q} = \{\hat{\mu}(t) \mid t \in T_\Sigma,\ \hat{\mu}(t) \neq \emptyset\} \subseteq Q$,
- the restriction $rest_{\tilde{Q}}(M) = (\tilde{Q}, \Sigma, Q_d \cap \tilde{Q}, A, \tilde{\mu})$ or M' with respect to $\tilde{Q}$, and
- $\psi : Q' \to \tilde{Q} : [t]_{\rho'_M} \mapsto \hat{\mu}(t)$.

Observation A.1 *For every $t \in T_\Sigma$ and $q \in \tilde{Q}$, $h_\mu(t)_q = h_{\tilde{\mu}}(t)_q$. Hence M and $rest_{\tilde{Q}}(M)$ are equivalent. Moreover, $rest_{\tilde{Q}}(M)$ is a deterministic bu-w-fta.*

We now show that $det(\varphi'_M, \rho'_M, M)$ and $rest_{\tilde{Q}}(M)$ are isomorphic via ψ.

Lemma A.2 *ψ is a bijection.*

Proof. In Lemma 4.38 we have proven that, for every $t, t' \in T_\Sigma$, $(t, t') \in \rho'_M$ iff $\hat{\mu}(t) = \hat{\mu}(t')$. Hence $\psi([t]_{\rho'_M})$ is independent from the choice of the representant $t' \in [t]_{\rho'_M}$ and ψ is injective. Clearly, ψ is also surjective, hence bijective. □

It remains to prove the equality of the tree representations of $det(\varphi'_M, \rho'_M, M)$ and $rest_{\tilde{Q}}(M)$. We start with a lemma.

Lemma A.3 *Let $t \in T_\Sigma$, and $q' \in Q'$. Then $h_{\mu'}(t)_{q'} = h_{\tilde{\mu}}(t)_{\psi(q')}$.*

Proof. First let $t \in supp(S_M)$, i. e., $\hat{\mu}(t) \neq \emptyset$. (a) Let $q' = [t]_{\rho'_M}$. Then

$$\begin{aligned} h_{\mu'}(t)_{[t]_{\rho'_M}} &= (S_M^{(\varphi'_M)}, t) \qquad \text{(by Lemma 4.6(ii))} \\ &= (S_M, t) \qquad (t \in supp(S_M)) \\ &= \sum_{q \in Q_d} h_\mu(t)_q \\ &= h_\mu(t)_{\hat{\mu}(t)} \qquad \text{(by Lemma 3.6)} \\ &= h_{\tilde{\mu}}(t)_{\hat{\mu}(t)} \qquad \text{(by Observation A.1)} \\ &= h_{\tilde{\mu}}(t)_{\psi([t]_{\rho'_M})}. \end{aligned}$$

(b) Now let $q' \in Q' \setminus \{[t]_{\rho'_M}\}$. We will show $h_{\mu'}(t)_{q'} = 0 = h_{\tilde{\mu}}(t)_{\psi(q')}$. From Lemma 4.6 (i) we deduce that $[t]_{\rho'_M} = \hat{\mu}'(t)$. Hence $q' \notin \hat{\mu}'(t)$, i. e., $h_{\mu'}(t)_{q'} = 0$. On the other hand, from $[t]_{\rho'_M} = \hat{\mu}'(t)$ and the above calculation follows $0 \neq h_{\mu'}(t)_{[t]_{\rho'_M}} = h_{\tilde{\mu}}(t)_{\psi([t]_{\rho'_M})}$, i. e., $\psi([t]_{\rho'_M}) \in \hat{\tilde{\mu}}(t)$. Since M and hence $rest_{\tilde{Q}}(M)$ is deterministic (cf. Observation A.1), $\psi([t]_{\rho'_M}) = \hat{\tilde{\mu}}(t)$ by Lemma 3.6. By Lemma A.2 ψ is a bijection, thus $\psi(q') \neq \psi([t]_{\rho'_M}) = \hat{\tilde{\mu}}(t)$, i. e., $h_{\tilde{\mu}}(t)_{\psi(q')} = 0$.

Now let $t \notin supp(S_M)$). (a) Let $\hat{\mu}(t) \neq \emptyset$. First assume $q' = [t]_{\rho'_M}$. Then

$$\begin{aligned} h_{\mu'}(t)_{q'} &= (S_M^{(\varphi'_M)}, t) \qquad \text{(by Lemma 4.6(ii))} \\ &= h_\mu(t)_{\hat{\mu}(t)} \\ &\qquad (t \notin supp(S_M),\ \hat{\mu}(t) \neq \emptyset, \text{ by Lemma 3.6 then } card\,(\hat{\mu}(t)) = 1) \\ &= h_{\tilde{\mu}}(t)_{\psi(q')}. \end{aligned}$$

The proof of $h_{\tilde{\mu}}(t)_{q'} = 0 = h_{\mu'}(t)_{q'}$ for $q' \in Q' \setminus \{[t]_{\rho'_M}\}$ is similar to the case $t \in supp(S_M)$. (b) Let $\hat{\mu}(t) = \emptyset$. We again show $h_{\tilde{\mu}}(t)_{q'} = 0 = h_{\mu'}(t)_{q'}$. From Lemma 4.6 (i) we deduce that $\hat{\mu}'(t) = \emptyset$, hence $h_{\mu'}(t)_{q'} = 0$. On the other hand $\hat{\mu}(t) = \emptyset$ implies $h_\mu(t)_q = 0$ for every $q \in \tilde{Q} \subseteq Q$. By Observation A.1 we have $h_{\tilde{\mu}}(t)_q = 0$. From $\psi(q') \in \tilde{Q}$ we deduce that $h_{\tilde{\mu}}(t)_{\psi(q')} = 0$ which completes the proof. □

Corollary A.4 *For every $t \in T_\Sigma$, $\hat{\tilde{\mu}}(t) = \psi(\hat{\mu'}(t))$.*

Lemma A.5 *For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $q'_1, \ldots, q'_k, q' \in Q'$,*

$$\mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'} = \tilde{\mu}_k(\sigma)_{(\psi(q'_1),\ldots,\psi(q'_k)),\psi(q')}.$$

Proof. The proof is based on the equation $h_{\mu'}(t)_{q'} = h_{\tilde{\mu}}(t)_{\psi(q')}$ for every $t \in T_\Sigma$ and $q' \in Q'$ which we have proven in Lemma A.3. First we will calculate both sides. We start with $h_{\mu'}(t)_{q'}$. By the definition of Q' there exists trees $t_1, \ldots, t_k \in T_\Sigma$ such that $q'_1 = [t_1]_{\rho'_M}, \ldots, q'_k = [t_k]_{\rho'_M}$. Then

$$\begin{aligned}
&h_{\mu'}(t)_{q'}\\
&\quad= \sum_{(p'_1,\ldots,p'_k)\in Q'^k} h_{\mu'}(t_1)_{p'_1} \odot \cdots \odot h_{\mu'}(t_k)_{p'_k} \odot \mu'_k(\sigma)_{(p'_1,\ldots,p'_k),q'}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(\hat{\mu'}(t_1),\ldots,\hat{\mu'}(t_k)),q'}\\
&\qquad\quad \text{(by Lemma 3.6)}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{([t_1]_{\rho'_M},\ldots,[t_k]_{\rho'_M}),q'}\\
&\qquad\quad \text{(by Lemma 4.6 (i))}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&h_{\tilde{\mu}}(t)_{\psi(q')}\\
&\quad= \sum_{(p'_1,\ldots,p'_k)\in Q'^k} h_{\tilde{\mu}}(t_1)_{p'_1} \odot \cdots \odot h_{\tilde{\mu}}(t_k)_{p'_k} \odot \mu'_k(\sigma)_{(p'_1,\ldots,p'_k),\psi(q')}\\
&\quad= h_{\tilde{\mu}}(t_1)_{\hat{\tilde{\mu}}(t_1)} \odot \cdots \odot h_{\tilde{\mu}}(t_k)_{\hat{\tilde{\mu}}(t_k)} \odot \mu'_k(\sigma)_{(\hat{\tilde{\mu}}(t_1),\ldots,\hat{\tilde{\mu}}(t_k)),\psi(q')}\\
&\qquad\quad \text{(by Lemma 3.6)}\\
&\quad= h_{\tilde{\mu}}(t_1)_{\psi(\hat{\mu'}(t_1))} \odot \cdots \odot h_{\tilde{\mu}}(t_k)_{\psi(\hat{\mu'}(t_k))} \odot \mu'_k(\sigma)_{(\psi(\hat{\mu'}(t_1)),\ldots,\psi(\hat{\mu'}(t_k))),\psi(q')}\\
&\qquad\quad \text{(by Corollary A.4)}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(\psi(\hat{\mu'}(t_1)),\ldots,\psi(\hat{\mu'}(t_k))),\psi(q')}\\
&\qquad\quad \text{(by Lemma A.3)}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(\psi([t_1]_{\rho'_M}),\ldots,\psi([t_k]_{\rho'_M})),\psi(q')}\\
&\qquad\quad \text{(by Lemma 4.6 (i))}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(\psi(q'_1),\ldots,\psi(q'_k)),\psi(q')}\\
&\qquad\quad \text{(by Lemma A.3).}
\end{aligned}$$

Since, by Lemma A.3, $h_{\mu'}(t)_{q'} = h_{\tilde{\mu}}(t)_{\psi(q')}$, we deduce that

$$\begin{aligned}
&h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'}\\
&\quad= h_{\mu'}(t_1)_{\hat{\mu'}(t_1)} \odot \cdots \odot h_{\mu'}(t_k)_{\hat{\mu'}(t_k)} \odot \mu'_k(\sigma)_{(\psi(q'_1),\ldots,\psi(q'_k)),\psi(q')}. \qquad (\nabla)
\end{aligned}$$

By the definition of $\hat{\mu}(t_i)$ we have $h_{\mu'}(t_i)_{\hat{\mu}'(t_i)} \neq 0$ for every i with $1 \leq i \leq k$. Since A is a semifield, $\big(h_{\mu'}(t_i)_{\hat{\mu}'(t_1)}\big)^{-1}, \ldots, \big(h_{\mu'}(t_i)_{\hat{\mu}'(t_k)}\big)^{-1}$ exist. To complete the prove we multiply (∇) from the left with $\big(h_{\mu'}(t_i)_{\hat{\mu}'(t_k)}\big)^{-1} \odot \cdots \odot \big(h_{\mu'}(t_i)_{\hat{\mu}'(t_1)}\big)^{-1}$ and obtain $\mu'_k(\sigma)_{(q'_1,\ldots,q'_k),q'} = \mu'_k(\sigma)_{(\psi(q'_1),\ldots,\psi(q'_k)),\psi(q')}$. □

Theorem A.6 *$det(\varphi'_M, \rho'_M, M)$ and $rest_{\tilde{Q}}(M)$ are isomorphic via ψ.*

Proof. The claim is a corollary of Lemma A.2 and Lemma A.5. □

Finally we show that the application of the partial determinization with respect to φ'_M and ρ'_M is idempotent in the sense that for an arbitrary bu-w-fta N, $M = det(\varphi'_N, \rho'_N, N)$ and $det(\varphi'_M, \rho'_M, M)$ are isomorphic.

Theorem A.7 *Let $N = (P, \Sigma, P_d, A, \nu)$ be an arbitrary bu-w-fta over a commutative semifield and $M = det(\varphi'_N, \rho'_N, N)$. If M is a finite bu-w-fta, then M and $det(\varphi'_M, \rho'_M, M)$ are isomorphic.*

Proof. By Lemma 4.6 (i) we have $Q = \{[t]_{\rho'_M} \mid t \in T_\Sigma,\ \hat{\nu}(t) \neq \emptyset\} = \{\hat{\mu}(t) \mid t \in T_\Sigma,\ \hat{\nu}(t) \neq \emptyset\} = \{\hat{\mu}(t) \mid t \in T_\Sigma,\ \hat{\mu}(t) \neq \emptyset\} = \tilde{Q}$. But then $M = rest_{\tilde{Q}}(M)$. By Theorem A.6 M and $det(\varphi'_M, \rho'_M, M)$ are isomorphic. □

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