Bachelor's Thesis

Context-free grammars over the free magmoid

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Aufgabenstellung für die Bachelorarbeit

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Thema: Kontextfreie Grammatiken über dem freien Magmoid

Hintergrund: Ein aktueller Trend in den Forschungsgebieten des *Natural Language Processing (NLP)* und der *Machine Translation (MT)* ist der zunehmende Bezug auf die syntaktische, d.h. grammatikalische, Struktur der zu verarbeitenden Sätze, welche in Form eines *Parsebaums* des Satzes vorliegt [vgl. 8, 9]. Als formale Modelle, um die Sprache der erlaubten Parsebäume zu beschreiben, dienen u.a. reguläre Baumgrammatiken (RTG) [4], aber auch (eingeschränkte Varianten von) kontextfreien Baumgrammatiken (CFTG) [11].

Die regulären Baumsprachen bilden eine robuste Klasse mit zahlreichen Abschlusseigenschaften, und verhalten sich größtenteils analog zur klassischen Theorie über Wörtern. Anders stellt sich die Lage bei den kontextfreien Baumsprachen dar: unter anderem sind sie nicht unter inversen linearen Baumhomomorphismen abgeschlossen [1]. Überraschenderweise gilt dies selbst für die Einschränkung auf *lineare* CFTG (l-CFTG) [10]. Dieses Negativresultat ist nicht nur theoretisch unbefriedigend, es erschwert auch das kompositionelle Design von Übersetzungssystemen auf Grundlage von CFTG.

Das Problem kann jedoch gelöst werden, indem eine stärkere zugrundeliegende algebraische Struktur gewählt wird, die der *Magmoiden* [2, 3]. Die Trägermenge des freien (projektiven) Magmoid enthält *Tupel* von Bäumen, und die grundlegenden Operationen erlauben "vertikale" Substitution und "horizontale" Konkatenation solcher Tupel. Die kontextfreien Sprachen über dem freien Magmoid werden erzeugt von *kontextfreien Magmoidgrammatiken (CFMG)*. Sie enthalten echt die kontextfreien Baumsprachen, und zudem sind sie unter inversen Homomorphismen abgeschlossen. In [14] wird Erkennbarkeit und Rationalität für Magmoide behandelt und ein Kleene-Theorem für den freien Magmoid bewiesen. Zudem existiert ein enger Zusammenhang zwischen Magmoiden und dem *multi bottom-up tree transducer* [6, 7].

In dieser Arbeit sollen die folgenden Fragestellungen zu kontextfreien Sprachen über Magmoiden bearbeitet werden.

- Die kontextfreien Baumsprachen sind unter Schnitt mit erkennbaren Baumsprachen abgeschlossen [12, 13]. Lässt sich diese Konstruktion auf CFMG verallgemeinern?

- Es soll ein Dekompositionsresultat $\mathcal{L}(\mathrm{CFMG}) = \mathcal{T}(\mathcal{L}(\mathrm{CFTG}))$ für eine geeignete Klasse von Baumtransformationen $\mathcal{T}$ gezeigt werden, analog für l-CFMG und l-CFTG. Dabei bezeichnet $\mathcal{L}(\mathrm{CF}X\mathrm{G})$ die Klasse der Sprachen die von CFXG, $X \in \{\mathrm{M}, \mathrm{T}\}$, erzeugt werden. Vermutung: $\mathcal{T}$ ist die Klasse der Transformationen von *deterministischen top-down tree transducer (d-TOP)*, oder sogar von *single-use* d-TOP [vgl. 5, 6].

Wünschenswert, im Rahmen dieser Arbeit aber optional, ist die Bearbeitung einiger der folgenden Fragestellungen:

- Es sollen *lineare* CFMG (l-CFMG) untersucht werden, analog zu den linearen CFTG. Gilt der Abschluss unter Schnitt mit erkennbaren Sprachen? Sind die Sprachen von l-CFMG unter inversen linearen Homomorphismen abgeschlossen?

- Sind die Sprachen von l-CFMG unter Anwendung von linearen multi bottom-up tree transducer abgeschlossen? Wenn ja, soll eine direkte Konstruktion angegeben werden.

- Wie verhalten sich die Sprachen von linearen *monadischen* Grammatiken über dem Magmoid (also mit maximal einer Variable) zu denen von linearen und monadischen über Bäumen? Vermutung: sie sind identisch. Das würde erklären warum die Sprachen von monadischen l-CFTG bessere Abschlusseigenschaften aufweisen.

Die Arbeit muss den üblichen Standards wie folgt genügen. Die Arbeit muss in sich abgeschlossen sein und alle nötigen Definitionen und Referenzen enthalten. Die Struktur der Arbeit muss klar erkenntlich sein, und der Leser soll gut durch die Arbeit geführt werden. Die Darstellung aller Begriffe und Verfahren soll mathematisch formal fundiert sein. Für jeden wichtigen Begriff sollen Beispiele angegeben werden, ebenso für die Abläufe der beschriebenen Verfahren. Wo es angemessen ist, sollten Illustrationen die Darstellung vervollständigen. Schließlich sollen alle Lemmata und Sätze möglichst lückenlos bewiesen werden. Die Beweise sollen leicht nachvollziehbar dokumentiert sein.

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Selbstständigkeitserklärung

Hiermit versichere ich, die vorliegende Arbeit selbstständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der von mir angegebenen Quellen angefertigt zu haben. Sämtliche aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche gekennzeichnet.

Die Arbeit wurde noch keiner Prüfungsbehörde in gleicher oder ähnlicher Form vorgelegt.

Dresden, den 18. Juli 2016

Contents

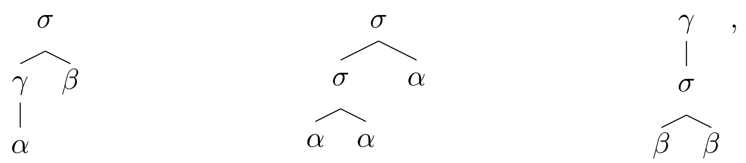
Chapter 1: Introduction

1.1 Motivation

The field of *Natural Language Processing (NLP)* is dedicated to the formal representation, structuring and arithmetic properties of natural languages. Not only is this relevant for a scientific purpose, but also in our everyday life. *Machine Learning*, *Machine Translation* and *Human-Computer Interaction* profit vastly from research in NLP, since interparticipant communication in the modern world relies on the integrability of communication devices into translation processes.

As an example, picture the following scenery: A group of people whose native languages are pairwise different meet at a workshop and want to communicate their ideas. Since all of them share basic knowledge about the English language, they try to speak in English. As additional assistance, they use smartphones to translate unknown phrases. The use of NLP in this setting is at least twofold: On the one hand, whenever someone translates a grammatically correct phrase with a smartphone, the resulting translation should be both substantially and grammatically correct. On the other hand, it should be possible for the translating devices to understand incorrect sentences. That is, if someone misunderstood a word and enters the erroneous text into his smartphone, it should still dispense a correct translation.

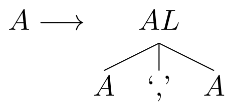
In contrast to "basic" formal language theory that is affine to *words*, i.e. sequences of symbols ($a, bab, abaacdb, \dots$) from an alphabet Σ, NLP motivates the use of a structure called *trees* to achieve these goals. In the context of this thesis, a tree will always be a tree over a ranked alphabet. That is, we will give every symbol in an alphabet a number of successors and a tree is made up of these symbols, such that each position in the tree respects the corresponding number of successors. Trees over a ranked alphabet are usually drawn as a graph. Examples are



where $\sigma, \gamma, \beta, \alpha$ are symbols from the ranked alphabet. One very intuitive reason to introduce this structure is given by the syntactical structure of natural language. In the English language, a sentence is a compound grammatical object consisting of subphrases, nouns, verbs, adjectives, adverbs et cetera. Whilst the actual sentence is stored as the *yield* (i.e. the sequence of leaves) of a tree, the inner positions are used to store information about the syntactical structure of the text. This makes a grammatically correct translation possible.

A set of trees is often referred to as a *tree language* and using these concepts, NLP introduces a vast scale of machinery that for example generates, translates or accepts tree languages. In Chapter 2, we will gather all the elementary definitions and lemmas that are necessary to understand the following chapters of this thesis. If the reader is not familiar with these instruments, it is strongly recommended to read the preliminaries.

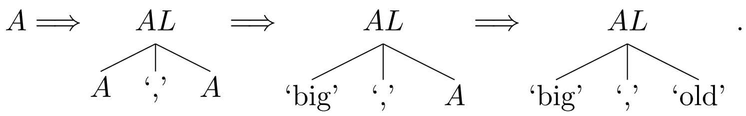
One very important class of such machinery is represented by the *context-free tree grammars* or short *CFTGs*. The main idea is to describe syntactically "well-formed" components of a tree and successively derive a tree into a (possibly more complex) new tree that might reuse parts from earlier steps in the derivation. A conceivable task is to describe the structure of an enumeration of adjectives, which might be done as follows:



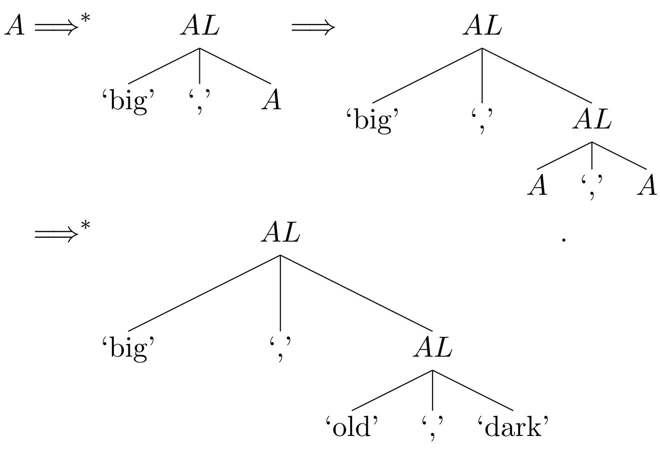
$$A \longrightarrow \text{'big'}$$
$$A \longrightarrow \text{'old'}$$
$$A \longrightarrow \text{'dark'}.$$

We think of A as "representing adjectives" and AL as "representing an enumeration (or list) of adjectives". Using the above *productions* in a corresponding CFTG results in derivations of the following form:

$$A \Longrightarrow \text{'big'},$$



The occurrences of A within a derivation may again be decomposed to a list of adjectives as in the derivation



The yields of the derived trees can now be concatenated and are enumerations of adjectives.

A CFTG distinguishes between symbols in a tree that it may further derive, so called *nonterminals*, and those who are final, the *terminals*. Moreover, a CFTG has a unique *initial nonterminal* which is usually denoted by Z and is the starting point for any derivation. Thus a CFTG is made up of a terminal ranked alphabet, a nonterminal ranked alphabet, an initial nonterminal and a (finite) set of productions.

One important limitation to the expressiveness of CFTGs is a property called "non-determinism followed by copying", where a part of a tree is always derived completely, before being copied several times. The problem is, that the grammar may also copy parts of a tree before deriving them, so CFTGs can not model this property. We will see an example for this in later Chapters (e.g. Examples 27 and 29).

Furthermore, *context-free tree languages* (that is, the class of languages generated by CFTGs) are somewhat theoretically unsatisfying, since they are neither closed under application of tree-homomorphisms ([11], there: Example 6.7.), inverse tree-homomorphisms nor linear inverse tree-homomorphisms, both proven in [3] (there: Théorème 3.1.).

Therefore, Arnold and Dauchet introduced in [4] and [5] the structure of *magmoids*. This structure works with tuples of trees, which we will call *lavas* and allows "horizontal concatenation" and "vertical substitution" of these lavas. Similar to the case with trees, *context-free magmoid grammars* (or short: *CFMGs*) are defined. In contrast to CFTGs, a CFMG has lavas as terminals and the generated language projects the derived lavas on their first components. Context-free magmoid languages show a more algebraic nature than context-free tree languages. As we will show, they are for example closed under tree-homomorphisms and intersection with regular tree languages, yet the emptiness and membership problems are decidable for them.

A main goal of this thesis is to portray magmoids and context-free magmoid grammars. The entirety of Chapter 3 is dedicated to their construction and examples, where we will put a lot of effort into formalizing a structure over which a CFMG operates. We will call this structure $T_k(\Sigma, V)$ and it corresponds to $T(\Sigma \cup V)$ found in [5]. We thought of this elaboration as necessary, since we were not able to find an adequate definition of the latter structure in the given article by Arnold and Dauchet.

Chapters 4 and 5 solve the tasks given for this thesis. First we will show that the language classes of CFMGs and CFTGs are connected by an algebraic nexus. Namely the class of languages generated by CFMGs is equal to the image of the class of languages generated by CFTGs under total and deterministic top-down tree transducers, or in symbols

$$\mathscr{L}(\text{CFMG}) = \text{td-TOP}(\mathscr{L}(\text{CFTG})).$$

We will deduce that this also holds for the class of deterministic top-down tree transducers. Next we will show that for any language generated by a CFMG and any recognizable tree language, the intersection of both languages can be generated by a CFMG. All given proofs will be constructive and we will include proof intuition at the beginning of each chapter.

In the last chapter, we will reflect on our work and summarize the results.

Chapter 2: Preliminaries

2.1 Notations and Basic Definitions

We use the conventional set-theoretic approach to mathematics after Zermelo and Fraenkel with the axiom of choice. For a set M, we denote the **cardinality of** M by $\#M$.

The **set of positive integers** is denoted by $\mathbb{N}_+ := \{1, 2, 3, ...\}$. The **set of nonnegative integers** is denoted by $\mathbb{N}_0 := \mathbb{N}_+ \cup \{0\}$. If not stated differently we write $\mathbb{N}$ for $\mathbb{N}_+$. We abbreviate $[n, m] := \{n, n+1, ..., m\}$ and $[n] := [1, n]$ for any $n, m \in \mathbb{N}$ with $n \leq m$. Moreover we denote $[0] := \emptyset$.

Furthermore $\mathbb{Z}$ denotes the **set of all integers**, $\mathbb{Q}$ the **set of rational numbers** and $\mathbb{R}$ the **set of real numbers**.

A **relation** between sets A and B is a subset $r \subseteq A \times B$. For sets A, B, C and relations $\rho \subseteq A \times B$ and $\pi \subseteq B \times C$, we define the **(contravariant) composition of** ρ **and** π as $\pi \circ \rho := \{(r, p) \in A \times C \mid \exists s \in B : (r, s) \in \rho \wedge (s, p) \in \pi\}$. The dual covariant notation is $\rho \star \pi := \pi \circ \rho$.

For a set A, the **identity relation on** A is $id_A := \{(a, a) \mid a \in A\}$ and for a relation $r \subseteq A \times A$, define r^n for $n \in \mathbb{N}_0$ inductively by $r^0 := id_A$ and $r^{k+1} := r \circ r^k,\ k \in \mathbb{N}_0$ which is called the **n-fold composition of** r. Moreover the **reflexive, transitive closure of** r is $r^* := \bigcup_{n \geq 0} r^n$ and the **symmetric closure of** r is $r \cup \{(x, y) \mid (y, x) \in r\}$.

A **mapping** between sets A and B is a triplet (A, f, B) where $f \subseteq A \times B$ is a relation, denoted $f : A \longrightarrow B$ such that for all $a \in A$ there exists exactly one $b \in B$ such that $(a, b) \in f$. We then write $f(a) = b$. If for a relation $f \subseteq A \times B$ it only holds that for all $a \in A$ there exists at most one $b \in B$ such that $(a, b) \in f$, we call (A, f, B) a **partial mapping**. It is customary to omit the sets A and B when talking about mappings.

Let A, B be sets, $C \subseteq A$, $f : A \longrightarrow B$. The **restriction of** f **to** C is the mapping $f|_C : C \longrightarrow B$ such that $f|_C(x) = f(x)$ for every $x \in C$.

For $k \in \mathbb{N}_0$ and a set A, a k**-ary operation on** A is a mapping $\phi : A^k \longrightarrow A$. We write **nonary** instead of 0-ary, **unary** instead of 1-ary and **binary** instead of 2-ary. For a nonary operation $\varphi : A^0 \to A$ on a set A we identify φ with $\varphi(())$.

An **alphabet** is a set Σ, such that $\#\Sigma \in \mathbb{N}$, i.e. Σ is a finite and nonempty set. Σ^* denotes the **set of words over** Σ, i.e. finite ordered sequences of elements from Σ. The **length of** $\omega \in \Sigma^*$ is denoted $|\omega|$. Let $\Sigma^n := \{\omega \in \Sigma^* \mid |\omega| = n\}$ for $n \in \mathbb{N}_0$ and ε be the unique element of Σ^0. For $\sigma \in \Sigma, \omega \in \Sigma^*$, we denote the **number of occurrences of** σ **in** ω as $|\omega|_\sigma$.

Let Σ be an alphabet and $r : \Sigma \longrightarrow \mathbb{N}_0$ a mapping. We call the pair (Σ, r) a **ranked alphabet** and for any $a \in \Sigma$, $r(a)$ is the **rank of** a. If the context is clear, we will withhold r and simply write Σ instead of (Σ, r).

Let Σ be a ranked alphabet and $k \in \mathbb{N}_0$. We define $\Sigma^{(k)} := r^{-1}(\{k\}) = \{a \in \Sigma \mid r(a) = k\}$. Since Σ is finite and nonempty, there exists $\mathrm{maxrk}(\Sigma) := \max\ r(\Sigma)$ called the **maximal rank of** Σ.

We fix the sets $X := \{x_i \mid i \in \mathbb{N}\}$ and for any $n \in \mathbb{N}$, $X_n := \{x_i \mid i \in [n]\}$.

Let Σ be a ranked alphabet and A a set. Then the **set of trees over Σ indexed by** A, abbreviated by $T_\Sigma(A)$, is the smallest set $T \subseteq (\Sigma \cup A \cup C)^*$ (where C consists of open and closed round brackets and a comma), that satisfies the following conditions:

1. $A \subseteq T$
2. $\sigma(t_1, ..., t_k) \in T$ for any $k \in \mathbb{N}_0, \sigma \in \Sigma^{(k)}, t_1, ...t_k \in T$.

Moreover $T_\Sigma := T_\Sigma(\emptyset)$ and for each $\sigma \in \Sigma^{(k)}$ we identify σ with $\sigma(x_1, \ldots, x_k)$.

A **tree language** is a set of trees $L \subseteq T_\Sigma$.

A tree $t \in T_\Sigma(X)$ is called **linear** if for every $i \in \mathbb{N}$ x_i occurs at most once in t.

Let $l \in \mathbb{N}_0, \xi \in T_\Sigma(X_l)$ and $t_1, ..., t_l \in T_\Sigma(A)$. Define the **substitution of ξ with** $t_1, ..., t_l$ inductively by:

$$\xi[t_1, ..., t_l] := t_i$$

whenever $\xi = x_i,\ i \in [l]$ and

$$\xi[t_1, ..., t_l] := \sigma(\zeta_1[t_1, ..., t_l], ..., \zeta_k[t_1, ..., t_l])$$

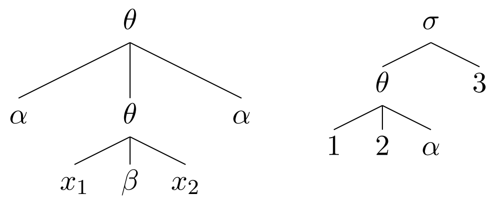
for $k \in \mathbb{N}_0, \sigma \in \Sigma^{(k)}, \zeta_1, ..., \zeta_k \in T_\Sigma, \xi = \sigma(\zeta_1, ..., \zeta_k)$.

Define $\tilde{T}_\Sigma(X_q)$ as the set of trees $u \in T_\Sigma(X_q)$ such that the left-to-right sequence of variables in u is $x_1 \ldots x_q$ (where $q \in \mathbb{N}_0$).

Example 1. Consider the set $\Sigma := \{\theta, \sigma, \beta, \alpha\}$ with ranks $r(\theta) = 3, r(\sigma) = 2, r(\beta) = r(\alpha) = 0$. That is, (Σ, r) is a ranked alphabet where for example

$$\theta(\alpha, \theta(x_1, \beta, x_2), \alpha) \in T_\Sigma(X_2) \text{ and}$$
$$\sigma(\theta(1, 2, \alpha), 3) \in T_\Sigma(\mathbb{N})$$

hold. The trees can be visualized as follows:



Note that the first tree is also in $\tilde{T}_\Sigma(X_2)$. ■

2.2 Tree Transformations and Tree Grammars

The following definitions for tree transducers are strongly oriented on [24] and [9]. Since it is not the task of this thesis to explain these concepts in more depth, we waive examples.

Let Σ, Δ be ranked alphabets. A relation $\tau \subseteq T_\Sigma \times T_\Delta$ is called **tree transformation**. For a subset $L \subseteq T_\Sigma(X)$ and a set Q define

$$Q\langle L\rangle := \{q(\xi) \mid q \in Q, \xi \in L\}.$$

If $\#Q = k \in \mathbb{N}$ and $Q = \{q_1, \ldots, q_k\}$, we moreover introduce a notational convention for substitution in trees from $T_\Sigma(Q\langle X\rangle)$: Let $(\xi_{i,j})_{i\in[k],j\in\mathbb{N}}$ be a double-indexed family of trees from $T_\Sigma(X)$ and $\zeta \in T_\Sigma(Q\langle X\rangle)$. Define the expression $\zeta[q_i(x_j)/\xi_{i,j}]$ inductively by

$$(q_\iota(x_\kappa))[q_i(x_j)/\xi_{i,j}] := \xi_{\iota,\kappa}$$

for some $\iota \in [k], \kappa \in \mathbb{N}$ and

$$(\sigma(t_1, \ldots, t_n))[q_i(x_j)/\xi_{i,j}] := \sigma(t_1[q_i(x_j)/\xi_{i,j}], \ldots, t_n[q_i(x_j)/\xi_{i,j}])$$

for some $n \geq 0$, $\sigma \in \Sigma^{(n)}$, $t_1, \ldots, t_n \in T_\Sigma(Q\langle X\rangle)$. Thus $\zeta[q_i(x_j)/\xi_{i,j}]$ is the element of $T_\Sigma(X)$ derived from ζ by replacing any occurrence of $q_i(x_j)$ with $\xi_{i,j}$.

A **top-down tree transducer (td-tt)** is a tuple $T = (Q, \Sigma, \Delta, I, R)$ where Q is a finite set (of **states**), $I \subseteq Q$ and R is a finite set of **rules** of the form

$$q(\sigma(x_1, \ldots, x_k)) \longrightarrow t$$

where $k \geq 0, \sigma \in \Sigma^{(k)}, q \in Q, t \in T_\Delta(Q\langle X_k\rangle)$.

The **derivation relation of** T is a relation $\Longrightarrow_T \subseteq T_\Delta(Q\langle T_\Sigma\rangle)^2$ such that $\forall \phi, \psi \in T_\Delta(Q\langle T_\Sigma\rangle)$:

$$\begin{aligned}
&\phi \Longrightarrow_T \psi :\Longleftrightarrow \\
&\qquad \exists u_1, u_2 \in (Q \cup \Sigma \cup \Delta \cup \{(,,,)\})^*, (q(\sigma(x_1, \ldots, x_k)) \longrightarrow t) \in R, s_1, \ldots, s_k \in T_\Sigma : \\
&\qquad \phi = u_1 \cdot q(\sigma(s_1, \ldots, s_k)) \cdot u_2, \psi = u_1 \cdot t[s_1, \ldots, s_k] \cdot u_2
\end{aligned}$$

The **tree transformation induced by** T is

$$\tau(T) := \{(s, t) \in T_\Sigma \times T_\Delta \mid \exists q \in I : q(s) \Longrightarrow_T^* t\}.$$

Let $T = (Q, \Sigma, \Delta, I, R)$ be a td-tt, $q \in Q$. We define $T_q := (Q, \Sigma, \Delta, \{q\}, R)$, the td-tt with initial state q.

T is called **deterministic** if $\#I = 1$ and for every $q \in Q, k \geq 0, \sigma \in \Sigma^{(k)}, q(\sigma(x_1, \ldots, x_k))$ is the left hand side of at most one rule in R.

T is called **total** if for every $q \in Q, k \geq 0, \sigma \in \Sigma^{(k)}, q(\sigma(x_1, \ldots, x_k))$ is the left hand side of at least one rule in R.

Note that a total and deterministic td-tt (td-td-tt) induces a tree mapping $\tau(T) : T_\Sigma \longrightarrow T_\Delta$. Thus the expression $\tau(T)(\xi)$ for some $\xi \in T_\Sigma$ is the unique $\zeta \in T_\Delta$ such that $(\xi, \zeta) \in \tau(T)$. In this case we can extend the tree transformation to trees containing variables:

$$\overline{\tau(T)} : T_\Sigma(X) \longrightarrow T_\Delta(Q(X))$$

where $\overline{\tau(T)}(\xi)$ is the unique element in $T_\Delta(Q(X))$ that can be derived from $\xi \in T_\Sigma(X)$ in T. Since obviously $\overline{\tau(T)}|_{T_\Sigma} = \tau(T)$ holds, we can write $\tau(T)$ instead of $\overline{\tau(T)}$. Using this definition, we can express for some $\sigma \in \Sigma$, $q \in Q$ the right hand side of the unique rule $(q(\sigma) \longrightarrow t) \in R$ by $\tau(T_q)(\sigma)$.

Lemma 2. Let Σ be a ranked alphabet, $T = (Q, \Sigma, \Delta, I, R)$ a td-td-tt.
For any $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k \in T_\Sigma$, $q \in Q$ it holds that

$$\tau(T_q)(\sigma(t_1, \ldots, t_k)) = \tau(T_q)(\sigma)[q_i(x_j)/\tau(T_{q_i})(t_j)].$$

Proof. This is a well-known result and was proven by Fülöp and Vogler in [12] (there: Theorem 3.25.). □

T is called **top-down tree homomorphism** (or simply **tree homomorphism**) if T is total, deterministic and $\#Q = 1$.
T is called **top-down relabeling** (or simply **relabeling**) if $\#Q = 1$ and for every $q(\sigma(x_1, \ldots, x_k)) \longrightarrow t \in R$ there exists $\delta \in \Delta^{(k)}$ such that $t = \delta(q(x_1), \ldots, q(x_k))$.
We denote the class of tree transformations induced by deterministic (total and deterministic) tree transducers with d-TOP (td-TOP). The class of tree transformations induced by homomorphisms is denoted h-TOP or HOM.
For a class of tree transformations Ξ and a class of tree languages Λ, we define

$$\Xi(\Lambda) := \{\tau(L) \mid \tau \in \Xi, L \in \Lambda, \tau \subseteq (T_\Sigma \times T_\Delta), L \subseteq T_\Sigma, \Sigma, \Delta \text{ r.a.}\}.$$

A **bottom-up tree transducer (bu-tt)** is a tuple $B = (Q, \Sigma, \Delta, F, R)$ where Q is a finite set (of **states**), $F \subseteq Q$ and R is a finite set of **rules** of the form

$$\sigma(q_1(x_1), \ldots, q_k(x_k)) \longrightarrow q(t)$$

where $k \geq 0, \sigma \in \Sigma^{(k)}, q_1, \ldots, q_k, q \in Q, t \in T_\Delta(X_k)$.
The **derivation relation of** B is the relation $\Longrightarrow_T \subseteq T_\Sigma(Q\langle T_\Delta\rangle)^2$ such that $\forall \phi, \psi \in T_\Sigma(Q\langle T_\Delta\rangle)$:

$$\begin{aligned} &\phi \Longrightarrow_B \psi :\Longleftrightarrow \\ &\qquad \exists b \in \tilde{T}_{Q\cup\Sigma\cup\Delta}(X_1), (\sigma(q_1(x_1), \ldots, q_k(x_k)) \longrightarrow q(t)) \in R, s_1, \ldots, s_k \in T_\Sigma : \\ &\qquad \phi = b[\sigma(q_1(s_1), \ldots, q_k(s_k))], \psi = b[q(t[s_1, \ldots, s_k])] \end{aligned}$$

The **tree transformation induced by** B is

$$\tau(B) := \{(s,t) \in T_\Sigma \times T_\Delta \mid \exists q \in F : s \Longrightarrow_B^* q(t)\}.$$

B is called **deterministic** if for every $k \geq 0, \sigma \in \Sigma^{(k)}, q_1 \ldots, q_k \in Q$ there is at most one rule in R with left hand side $\sigma(q_1(x_1), \ldots, q_k(x_k))$.

B is called **total** if for every $k \geq 0, \sigma \in \Sigma^{(k)}, q_1 \ldots, q_k \in Q$ there is at least one rule in R with left hand side $\sigma(q_1(x_1), \ldots, q_k(x_k))$.

B is called **finite-state tree automaton (fta)** if $\Sigma = \Delta$ and for every rule

$$(\sigma(q_1(x_1), \ldots, q_k(x_k)) \longrightarrow q(t)) \in R,$$

t has the form $\sigma(x_1, \ldots, x_k)$.

B is called **bottom-up deterministic finite-state tree automaton (budet-fta)** if B is deterministic and total fta.

Let $L \subseteq T_\Sigma$. L is called **recognizable tree language** if there exists a budet-fta B such that $\tau(B) = \{(t, t) \mid t \in L\}$.

The class of all recognizable tree languages over Σ is denoted $\mathrm{REC}(\Sigma)$.

A **context-free tree grammar (CFTG)** is a tuple $G = (V, \Sigma, Z, P)$ where

- V, Σ are ranked alphabets, elements of V are called **nonterminals**, elements of Σ are called **terminals** respectively,
- $Z \in V^{(0)}$ and
- P is a finite set of **productions** of the form $A(x_1, \ldots, x_q) \longrightarrow \zeta$, with $A \in V^{(q)}, \zeta \in T_{\Sigma \cup V}(X_q), q \in \mathbb{N}_0$.

The **derivation relation induced by** G is the relation $\Longrightarrow_G \subseteq T_{\Sigma \cup V} \times T_{\Sigma \cup V}$ that has for $\phi, \psi \in T_{\Sigma \cup V}$

$$\begin{aligned} \phi \Longrightarrow_G \psi :\Longleftrightarrow \exists (A(x_1, \ldots, x_q) \longrightarrow \zeta) \in P, \xi \in \tilde{T}_{\Sigma \cup V}(X_1), \zeta_1, \ldots, \zeta_q \in T_{\Sigma \cup V} : \\ \phi = \xi[A[\zeta_1, \ldots, \zeta_q]] \wedge \psi = \xi[\zeta[\zeta_1, \ldots, \zeta_q]]. \end{aligned}$$

The **language generated by** G is $L(G) := \{\zeta \mid Z \Longrightarrow_G^* \zeta, \zeta \in T_\Sigma\}$.

A CFTG $G = (V, \Sigma, Z, P)$ is called **linear** or **l-CFTG** $:\Longleftrightarrow$ for every production $(A \longrightarrow \zeta) \in P$, ζ is linear as an element of $T_{\Sigma \cup V}(X)$.

The **class of tree-languages generated by context-free tree grammars** is denoted $\mathscr{L}(\mathrm{CFTG})$. The **class of tree-languages generated by linear context-free tree grammars** is denoted $\mathscr{L}(\text{l-CFTG})$.

Let $G = (V, \Sigma, Z, P)$ be a CFTG. We say G is in **normal form** or **nf-CFTG** if for any rule $(A \longrightarrow \zeta) \in P$, ζ is of one of the following forms:

$$\zeta \in T_V(X) \qquad , \text{ or } \qquad \sigma(x_1, \ldots, x_q) \text{ , for some } \sigma \in \Sigma^{(q)}, q \in \mathbb{N}_0.$$

This normal form will allow us to have perspicuous constructions for the following results.

Lemma 3. Let $G = (V, \Sigma, Z, P)$ be a CFTG. There exists $G' = (V', \Sigma, Z', P')$ CFTG in normal form (nf-CFTG) such that $L(G) = L(G')$.

Proof. This is a well-known result for CFTGs and can be found in [23]. $\square$

Let $G = (V, \Sigma, Z, P)$ be a CFTG. The **OI derivation relation induced by** G is the relation $\overset{\circ}{\Longrightarrow}_G \subseteq \Longrightarrow_G$ such that for any $(A(x_1, \ldots, x_q) \longrightarrow \zeta) \in P$, $\xi \in \tilde{T}_{\Sigma \cup V}(X_1)$ and $\zeta_1, \ldots, \zeta_q \in T_{\Sigma \cup V}$, it holds that $(\xi[A[\zeta_1, \ldots, \zeta_q]], \xi[\zeta[\zeta_1, \ldots, \zeta_q]]) \in \overset{\circ}{\Longrightarrow}_G$ is true if and only if the path from the root of ξ to the single occurrence of x_1 in ξ only consists of elements of Σ (or x_1).

The **OI language generated by** G is $L_\circ(G) := \{\zeta \mid Z \overset{\circ}{\Longrightarrow}_G{}^* \zeta, \zeta \in T_\Sigma\}$.

Lemma 4. Let $G = (V, \Sigma, Z, P)$ be a CFTG. It holds that

$$L(G) = L_\circ(G).$$

Proof. This Lemma is well known and a proof can be found in [10] (there: Theorem 3.4. and the remark after Theorem 3.4.). $\square$

2.3 Algebraic Structures

Let $\Omega \neq \emptyset$ be a set and $\circ$ a binary operation on Ω. We call $(\Omega, \circ)$ a

semigroup $:\Longleftrightarrow \forall x, y, z \in \Omega : x \circ (y \circ z) = (x \circ y) \circ z$.

monoid $:\Longleftrightarrow (\Omega, \circ)$ is a semigroup and $\exists 1 \in \Omega : \forall x \in \Omega : x \circ 1 = 1 \circ x = x$.

Now let $\Omega \neq \emptyset$ be a set, $+$ and $*$ binary operations on Ω and $0 \in \Omega$. The tuple $(\Omega, +, *, 0)$ is called a **semiring** $:\Longleftrightarrow$ (i) – (iv) hold, where

(i) $(\Omega, +, 0)$ is a commutative monoid,
(ii) $(\Omega, *)$ is a semigroup,
(iii) $\forall x, y, z \in \Omega : x * (y + z) = x * y + x * z \wedge (y + z) * x = y * x + z * x$,
(iv) $\forall x \in \Omega : x * 0 = 0 * x = 0$.

A detailed study of group theory can be found in [7] and [16] contains a vast exploration of the theory of semirings.

Chapter 3: Magmoids and Context-Free Magmoid Grammars

3.1 Magmoids

We first introduce the definition of an algebraic structure – the *magmoid* – as seen in [4] and proceed with giving examples of this structure.

Definition 5. Let $\mathbb{M}$ be a set, $\cdot : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{M}$ a partial binary operation on $\mathbb{M}$, $\oplus : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{M}$ a binary operation on $\mathbb{M}$ and o and $\imath$ two nonary operations on $\mathbb{M}$.

We call the tuple $(\mathbb{M}, \cdot, \oplus, \mathrm{o}, \imath)$ a **magmoid** $:\Longleftrightarrow$
the conditions (M1), (M2), (M2'), (M3), (M3'), (M4) and (M5) hold, where

(M1) $\forall p, q \in \mathbb{N}_0 \exists \mathbb{M}^p_q \subseteq \mathbb{M}$ such that $\mathbb{M} = \bigcup_{p,q\in\mathbb{N}_0} \mathbb{M}^p_q$ and $\forall p, p', q, q' \in \mathbb{N}_0$ with $(p, q) \neq (p', q') : \mathbb{M}^p_q \cap \mathbb{M}^{p'}_{q'} = \emptyset$. ($\mathbb{M}$ is the union of disjoint sets).

(M2) $\forall p, p', q, q' \in \mathbb{N}_0, m \in \mathbb{M}^p_q, m' \in \mathbb{M}^{p'}_{q'} : m \cdot m'$ is defined $\Leftrightarrow q = p'$ and if the composition is defined we have $m \cdot m' \in \mathbb{M}^p_{q'}$.

(M2') $\forall m, m', m'' \in \mathbb{M}$ where $(m \cdot m') \cdot m''$ and $m \cdot (m' \cdot m'')$ are defined, $(m \cdot m') \cdot m'' = m \cdot (m' \cdot m'')$ (associativity on domain).

(M3) $\forall p, p', q, q' \in \mathbb{N}_0, m \in \mathbb{M}^p_q, m' \in \mathbb{M}^{p'}_{q'} : m \oplus m' \in \mathbb{M}^{p+p'}_{q+q'}$.

(M3') $\oplus$ is associative.

(M4) $\forall m_1, m_2, n_1, n_2 \in \mathbb{M} : (m_1 \cdot m_2) \oplus (n_1 \cdot n_2)$ is defined $\implies (m_1 \oplus n_1) \cdot (m_2 \oplus n_2)$ is defined and both terms are equal.

(M5) $\imath \in \mathbb{M}^1_1, \mathrm{o} \in \mathbb{M}^0_0$ and with $\imath_p := \bigoplus_{i=1}^p \imath \in \mathbb{M}^p_p$ for $p \in \mathbb{N}$ it holds that
$\forall p, q \in \mathbb{N}_0, m \in \mathbb{M}^p_q : \imath_p \cdot m = m \cdot \imath_q = m$ and $m \oplus \mathrm{o} = \mathrm{o} \oplus m = m$.

Note that because of (M1), we call $\mathbb{M}$ biranked. For any $p, q \in \mathbb{N}_0$ we call p the **superrank of** $\mathbb{M}^p_q$ and q the **subrank of** $\mathbb{M}^p_q$, whereas $\mathbb{M}^p_q$ is called the **fibre of** $\mathbb{M}$ **for** p, q or p**-**q**-fibre of** $\mathbb{M}$. Moreover we call the partial operation $\cdot$ the **product of composition** and the operation $\oplus$ is called the **tensor product**. Since the tensor product is associative, we use the big operator $\bigoplus$ as the natural extension of $\oplus$ to arbitrarily many operands. It is customary to refer to a magmoid by its support. Thus we will write $\mathbb{M}$ instead of the tuple. An element of $\mathbb{M}$ can be referred to as **lava**.

If $\mathbb{M}^0_1$ contains a unique element o_1, then we define two helpful lavas: $\mathrm{o}_p := \bigoplus_{i=1}^p \mathrm{o}_1$ for $p \in \mathbb{N}_0$ and for any $p \in \mathbb{N}, i \in [p]$ the p, i**-projection in** $\mathbb{M}$ is $\pi^i_p := \mathrm{o}_{i-1} \oplus \imath \oplus \mathrm{o}_{p-i} \in \mathbb{M}^1_p$ ■

Corollary 6. Let $(\mathbb{M}, \cdot, \oplus, \mathrm{o}, \imath)$ be a magmoid. In $\mathbb{M}$ it holds that

$$(a_1 \cdot b_1) \oplus \cdots \oplus (a_n \cdot b_n) = (a_1 \oplus \cdots \oplus a_n) \cdot (b_1 \oplus \cdots \oplus b_n) \qquad (1)$$

for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{M}$ such that the left hand side of the equation is defined.

Proof. We use induction over n. For $n = 1$, the equation holds trivially, for $n = 2$ the equation follows directly from axiom **(M4)**.

Now let $n \in \mathbb{N}, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{M}$ such that equation (1) holds (and its left hand side is defined). Let $a, b \in \mathbb{M}$ such that $a \cdot b$ is defined. Then

$$\begin{aligned}(a \cdot b) \oplus (a_1 \cdot b_1) \oplus \cdots \oplus (a_n \cdot b_n) &= (a \cdot b) \oplus ((a_1 \cdot b_1) \oplus \cdots \oplus (a_n \cdot b_n)) \\ &\overset{IH}{=} (a \cdot b) \oplus ((a_1 \oplus \cdots \oplus a_n) \cdot (b_1 \oplus \cdots \oplus b_n)) \\ &\overset{\textbf{(M4)}}{=} (a \oplus (a_1 \oplus \cdots \oplus a_n)) \cdot (b \oplus (b_1 \oplus \cdots \oplus b_n)) \\ &= (a \oplus a_1 \oplus \cdots \oplus a_n) \cdot (b \oplus b_1 \oplus \cdots \oplus b_n)\end{aligned}$$

□

Example 7. As a first simple example consider the (p, q)-matrices over the real numbers $\mathbb{R}$ together with matrix multiplication and disjoint matrix union. For the latter (binary) operation, the two matrices are just written diagonally next to each other and all other positions in the resulting matrix are filled up with zeros. The p-th unit matrix becomes the unit for multiplication and the empty matrix with 0 rows and 0 columns the unit for disjoint matrix union.

This idea can be generalized to matrix magmoids over semirings with a unit element:

Let $(R, +, *, 0_R, 1_R)$ be a semiring with unit element 1_R for $*$ and define $\mathbb{M}^p_q := R^{p,q}$ (the set of p, q-matrices with values in R) and $\mathbb{M} := \bigcup_{p,q \in \mathbb{N}_0} \mathbb{M}^p_q$. The tuple $\mathbb{M}(R) := (\mathbb{M}, *, \oplus, \text{o}, ((1_R)))$, where

- $*$ is the usual matrix multiplication,
- for any $p, p', q, q' \in \mathbb{N}_0, m \in \mathbb{M}^p_q, m' \in \mathbb{M}^{p'}_{q'}$ we define $m \oplus m' := \begin{pmatrix} m & 0_{p,q'} \\ 0_{p',q} & m' \end{pmatrix}$ with zero-matrices $0_{p,q'}$ and $0_{p',q}$, and
- o is the empty matrix with 0 rows and 0 columns

is called **matrix magmoid over R**.

Note that the p-th unit $\imath_p$ becomes the p-dimensional unit matrix. Furthermore all $\mathbb{M}^p_0$ and $\mathbb{M}^0_q$ contain only one single element that is an empty matrix, but still the $\mathbb{M}^p_q$ are pairwise disjoint since two empty matrices with different numbers of rows or columns are posited to be different.

To prove that $\mathbb{M}(R)$ is in fact a magmoid, we show that the axioms from Definition 2 hold:

(M1) follows directly from the definition of $\mathbb{M}$,

(M2) and **(M2')** are usual arithmetics for matrix multiplication and

(M3) and **(M3')** follow directly from the definition of $\oplus$.

The last part of axiom **(M5)** can be seen as follows: since $m \oplus \text{o}$ (and $\text{o} \oplus m$) doesn't add any rows or columns to m (recall that $\text{o} \in \mathbb{M}^0_0$) and m itself isn't changed by the operation, using the tensorproduct on m and o gives m. The rest of **(M5)** follows from the definitions of $\mathbb{M}$ and matrix multiplication.

To prove that axiom **(M4)** holds, we evaluate both sides of the equation. Let $m, m', n, n' \in \mathbb{M}$ such that $\lambda := (m*m') \oplus (n*n')$ is defined. Now $\exists p, q, r, p', q', r' \in \mathbb{N}_0$ such that $m \in \mathbb{M}^p_q, m' \in \mathbb{M}^q_r, n \in \mathbb{M}^{p'}_{q'}$ and $n' \in \mathbb{M}^{q'}_{r'}$. Thus $m \oplus n \in \mathbb{M}^{p+p'}_{q+q'}$ and $m' \oplus n' \in \mathbb{M}^{q+q'}_{r+r'}$, which gives us that $\rho := (m \oplus n) * (m' \oplus n')$ is defined (and $\in \mathbb{M}^{p+p'}_{r+r'}$). Because of

$$\lambda = (m * m') \oplus (n * n') = \begin{pmatrix} m * m' & 0_{p,r'} \\ 0_{p',r} & n * n' \end{pmatrix},$$
$$\rho = (m \oplus n) * (m' \oplus n') = \begin{pmatrix} m & 0_{p,q'} \\ 0_{p',q} & n \end{pmatrix} * \begin{pmatrix} m' & 0_{q,r'} \\ 0_{q',r} & n' \end{pmatrix}$$

it is then fairly obvious that $\lambda = \rho$.
Thus all axioms for magmoids hold and the name "matrix magmoid" is appropriate. ∎

For our purposes, the magmoid of tree-tuples over a ranked alphabet is the most important example. It is crucial to gain intuition about this structure, which is why we give a detailed presentation of this so called *free projectable magmoid* in the rest of this section.

Example 8.
Recall that the set $T_\Sigma(X_k)$ for some $k \in \mathbb{N}_0$ and ranked alphabet Σ is the set of all trees over Σ indexed by variables (denoted $x_1, ..., x_k$). For $p, q \in \mathbb{N}_0$ we will denote the $(p+1)$-tuples in $\mathbb{M}^p_q := \{q\} \times T_\Sigma(X_q)^p$ using angle brackets, for this notation is easier to read. Defining $\mathbb{M} := \bigcup_{p,q \in \mathbb{N}_0} \mathbb{M}^p_q$ raises the question whether the fibres of $\mathbb{M}$ are disjoint. Since the first component of each element in $\mathbb{M}^p_q$ is q, we only need to show that for any $p, p', q \in \mathbb{N}_0$ with $p \neq p'$, $\mathbb{M}^p_q$ and $\mathbb{M}^{p'}_q$ are disjoint. But since p and p' are the numbers of trees in the tuples, we have that the fibres of $\mathbb{M}$ are disjoint. Let $\Delta := \{\sigma^{(2)}, \gamma^{(1)}, \delta^{(1)}, \beta^{(0)}, \alpha^{(0)}\}$.
Define

$$\begin{aligned} t_1 :=& \sigma(\gamma(x_1), x_2) && \in T_\Delta(X_2) \\ t_2 :=& \beta && \in T_\Delta(X_2) \\ t_3 :=& \delta(x_2) && \in T_\Delta(X_2) \\ t_4 :=& x_3 && \in T_\Delta(X_4) \\ t_5 :=& \sigma(x_1, x_4) && \in T_\Delta(X_4) \end{aligned}$$

We then for example have that

$$\begin{aligned} \tau_1 :=& \langle 2, t_1, t_2, t_3 \rangle && \in T_\Delta(X_2)^3 = \mathbb{M}^3_2 \\ \tau_2 :=& \langle 4, t_4, t_5 \rangle && \in T_\Delta(X_4)^2 = \mathbb{M}^2_4 \end{aligned}$$

It is important to note that for $p = 0$, $\mathbb{M}^p_q$ contains the single element $\langle q \rangle$.

Now define $\cdot$ as the partial operation on $\mathbb{M}$ that for $\langle q, u_1, ..., u_p \rangle \in \mathbb{M}^p_q$ and $\langle r, v_1, ..., v_q \rangle \in \mathbb{M}^q_r$ outputs $\langle r, u'_1, ..., u'_p \rangle \in \mathbb{M}^p_r$ where $u'_i := u_i[v_1, ..., v_q]$ for $i \in [p]$.

Moreover define $\oplus$ as the operation on $\mathbb{M}$ that for $\langle q, u_1, ..., u_p \rangle \in \mathbb{M}^p_q$ and $\langle q', v_1, ..., v_{p'} \rangle \in \mathbb{M}^{p'}_{q'}$ outputs $\langle q + q', w_1, ..., w_{p+p'} \rangle \in \mathbb{M}^{p+p'}_{q+q'}$ where $w_i := u_i$ for $i \in [p]$ and $w_i := v_{i-p}[x_{q+1}, ..., x_{q+q'}]$ for $i \in [p+p'] \setminus [p]$.

Furthermore for $n \geq 1$, $i \in [n]$, $p_i, q_i \in \mathbb{N}_0$, $u_i := \langle q_i, u_{i,1}, ..., u_{i,p_i} \rangle \in \mathbb{M}_{q_i}^{p_i}$ define $\langle\!\langle u_1, \ldots, u_n \rangle\!\rangle := \langle q, u_{1,1}, ..., u_{1,p_1}, \ldots, u_{n,1}, ..., u_{n,p_n} \rangle \in \mathbb{M}_q^p$, where $p = \sum_{i=1}^n p_i$ and $q = \max_{i \in [n]} q_i$.

It is now

$$\begin{aligned} \tau_1 \cdot \tau_2 =& \langle 4, \sigma(\gamma(x_3), \sigma(x_1, x_4)), \beta, \delta(\sigma(x_1, x_4)) \rangle && \in T_\Delta(X_4)^3 = \mathbb{M}_4^3 \\ \tau_1 \oplus \tau_2 =& \langle 6, t_1, t_2, t_3, x_5, \sigma(x_3, x_6) \rangle && \in T_\Delta(X_6)^5 = \mathbb{M}_6^5 \end{aligned}$$

Defining $\mathtt{0} := \langle 0 \rangle$ and $\mathtt{1} := \langle 1, x_1 \rangle$ with $\mathtt{1}_p := \bigoplus_{i=1}^p \mathtt{1}$, we see that for any $m \in \mathbb{M}$ it holds that $m \oplus \mathtt{0} = \mathtt{0} \oplus m = m$ and for any $m \in \mathbb{M}_q^p$ $(p, q \in \mathbb{N}_0)$ we have $m \cdot \mathtt{1}_q = \mathtt{1}_p \cdot m = m$.

Note that for $p \in \mathbb{N}$,$i \in [p]$, the p,i-th projection is $\pi_p^i = \langle p, x_i \rangle$.

We continue the example for trees over Δ with a portrayal of one representative for the equation in axiom **(M4)** from Definition 2:
Let

$$\begin{aligned} s_1 :=& \sigma(x_1, x_1) && \in T_\Delta(X_1) \\ s_2 :=& \gamma(\alpha) && \in T_\Delta(X_3), \end{aligned}$$

thus

$$\begin{aligned} \chi_1 :=& \langle 1, s_1 \rangle && \in T_\Delta(X_1)^1 = \mathbb{M}_1^1 \\ \chi_2 :=& \langle 3, s_2 \rangle && \in T_\Delta(X_3)^1 = \mathbb{M}_3^1 \\ \chi_1 \cdot \chi_2 =& \langle 3, \sigma(\gamma(\alpha), \gamma(\alpha)) \rangle && \in T_\Delta(X_3)^1 = \mathbb{M}_3^1 \end{aligned}$$

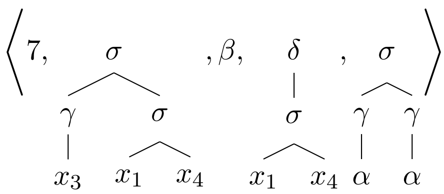
and since

$$\begin{aligned} \tau_1 \oplus \chi_1 =& \langle 3, t_1, t_2, t_3, \sigma(x_3, x_3) \rangle && \in T_\Delta(X_3)^4 = \mathbb{M}_3^4 \\ \tau_2 \oplus \chi_2 =& \langle 7, t_4, t_5, s_2 \rangle && \in T_\Delta(X_7)^3 = \mathbb{M}_7^3 \end{aligned}$$

we get

$$\begin{aligned} (\tau_1 \cdot \tau_2) \oplus (\chi_1 \cdot \chi_2) =& \langle 7, \sigma(\gamma(x_3), \sigma(x_1, x_4)), \beta, \delta(\sigma(x_1, x_4)), \sigma(\gamma(\alpha), \gamma(\alpha)) \rangle \\ =& (\tau_1 \oplus \chi_1) \cdot (\tau_2 \oplus \chi_2) \end{aligned}$$

which can be visualized as



Note that in this example we used $T_\Delta(X_k) \subseteq T_\Delta(X_l)$ for $k \leq l$ when we stated $s_2 \in T_\Delta(X_3)$ to adjust the ranks of the lavas to fit for later operations.

The assumption that Σ is a finite ranked alphabet can be dropped and instead $\mathbb{M}$ can be defined for possibly infinite ranked alphabets analogous to the finite case. ■

Lemma 9.
Let Σ be a ranked alphabet. Defining $\mathbb{M}_q^p$ for $p, q \in \mathbb{N}_0$, $\mathbb{M}$, $\cdot$, $\oplus$, o and $\imath$ as in Example 4, the tuple $(\mathbb{M}, \cdot, \oplus, o, \imath)$ is a magmoid – the so called **free projectable magmoid over** Σ. The common notation for $\mathbb{M}$ is $T(\Sigma)$. Moreover it is customary to identify $T_\Sigma(X_k) \ni t$ with $\langle k, t\rangle \in T(\Sigma)_k^1$.

Proof. Since the axioms **(M1)**, **(M2)**, **(M3)**, **(M5)** from Definition 5 are proven in Example 8, we focus on axioms **(M2')**, **(M3')** and **(M4)**.

While the associativity of $\cdot$ has been proven in [15] (there: Proposition 2.4) and the associativity of $\oplus$ is easy to verify, axiom **(M4)** can be seen as follows: For $m, m', n, n' \in \mathbb{M}$ such that $\lambda := (m \cdot m') \oplus (n \cdot n')$ is defined, we find $p, p', q, q', r, r' \in \mathbb{N}_0$ such that $m \in \mathbb{M}_q^p, m' \in \mathbb{M}_r^q, n \in \mathbb{M}_{q'}^{p'}$ and $n' \in \mathbb{M}_{r'}^{q'}$. Thus $(m \oplus n) \cdot (m' \oplus n')$ is defined.

Let the operands be decomposed as follows: $m = \langle q, u_1, \ldots, u_p\rangle, m' = \langle r, v_1, \ldots, v_q\rangle, n = \langle q', u_1', \ldots, u_{p'}'\rangle, n' = \langle r', v_1', \ldots, v_{q'}'\rangle$. It then holds that

$$\begin{aligned} m \cdot m' &= \langle r, u_1[v_1, \ldots, v_q], \ldots, u_p[v_1, \ldots, v_q]\rangle \\ n \cdot n' &= \langle r', u_1'[v_1', \ldots, v_{q'}'], \ldots, u_{p'}'[v_1', \ldots, v_{q'}']\rangle \\ (m \cdot m') \oplus (n \cdot n') &= \langle r + r', w_1, \ldots, w_p, z_1[x_{r+1}, \ldots, x_{r+r'}], \ldots, z_{p'}[x_{r+1}, \ldots, x_{r+r'}]\rangle \end{aligned}$$

where $w_i := u_i[v_1, \ldots, v_q], z_j := u_j'[v_1', \ldots, v_{q'}']$ for $i \in [p], j \in [p']$. Whereas

$$\begin{aligned} m \oplus n &= \langle q + q', u_1, \ldots, u_p, u_1'[x_{q+1}, \ldots, x_{q+q'}], \ldots, u_{p'}'[x_{q+1}, \ldots, x_{q+q'}]\rangle \\ m' \oplus n' &= \langle r + r', v_1, \ldots, v_q, v_1'[x_{r+1}, \ldots, x_{r+r'}], \ldots, v_{q'}'[x_{r+1}, \ldots, x_{r+r'}]\rangle. \end{aligned}$$

Taking into account that $u_1, \ldots, u_p$ and $u_1', \ldots, u_{p'}'$ use disjoint variables we get the required equality in **(M4)**. $\square$

Definition 10. Let Σ be a ranked alphabet. For notational purposes we define the **torsion-free subset of** $T(\Sigma)$, $\tilde{T}(\Sigma) := \bigcup_{p,q\in\mathbb{N}_0} \tilde{T}(\Sigma)_q^p$, where $\tilde{T}(\Sigma)_q^p$ is the set of lavas u from $T(\Sigma)_q^p$ such that the left-to-right sequence of variables in u is $x_1 \ldots x_q$. In fact, $\tilde{T}(\Sigma)$ is a magmoid, as found in [4] (there: Théorème 1). ■

Definition 11. Let $(\mathbb{M}, \cdot_\mathbb{M}, \oplus_\mathbb{M}, o_\mathbb{M}, \imath_\mathbb{M})$ and $(\mathbb{O}, \cdot_\mathbb{O}, \oplus_\mathbb{O}, o_\mathbb{O}, \imath_\mathbb{O})$ be magmoids. A mapping $\varphi : \mathbb{M} \longrightarrow \mathbb{O}$ is called **magmoid homomorphism** if the following axioms hold:

(1) $\varphi(\mathbb{M}_q^p) \subseteq \mathbb{O}_q^p$ for any $p, q \in \mathbb{N}_0$,
(2) $\varphi(m \cdot_\mathbb{M} n) = \varphi(m) \cdot_\mathbb{O} \varphi(n)$ for any $m, n \in \mathbb{M}$ such that $m \cdot_\mathbb{M} n$ is defined,
(3) $\varphi(m \oplus_\mathbb{M} n) = \varphi(m) \oplus_\mathbb{O} \varphi(n)$ for any $m, n \in \mathbb{M}$.

This is equivalent to the definition in [4] (there: first definition in Chapter 4.1.) since there, axiom A3 implies axiom A2. That is, our axioms (2) and (3) imply that neutral elements are preserved.

Due to the fact that this is a very common definition, we shall not give an example of magmoid homomorphisms. ■

3.2 The Construction of $T_k(\Sigma, V)$

Since our goal is to formulate results for context-free magmoid grammars (CFMGs), we need an appropriate structure on which their derivation relation can be defined. In this subchapter we give one very important characterization of this structure to complete the theoretic picture of CFMGs given in [5].

Definition 12. Let $(\mathbb{M}, \cdot, \oplus, \mathrm{o}, \mathrm{1})$ be a magmoid and $k \in \mathbb{N}$. Define the magmoid $(k\text{-dil}(\mathbb{M}), \cdot|_{k\text{-dil}(\mathbb{M})}, \oplus|_{k\text{-dil}(\mathbb{M})}, \mathrm{o}, \mathrm{1}_k)$ where $k\text{-dil}(\mathbb{M})^p_q = \mathbb{M}^{kp}_{kq}$ $(\forall p, q \in \mathbb{N}_0)$. Since $\mathbb{M}$ is a magmoid and the restricted operations preserve fibres of k-dil$(\mathbb{M})$, all axioms for magmoids hold immediately for k-dil$(\mathbb{M})$. Thus, k-dil$(\mathbb{M})$ is well-defined. We call k-dil$(\mathbb{M})$ the **k-dilatation of** $\mathbb{M}$. ■

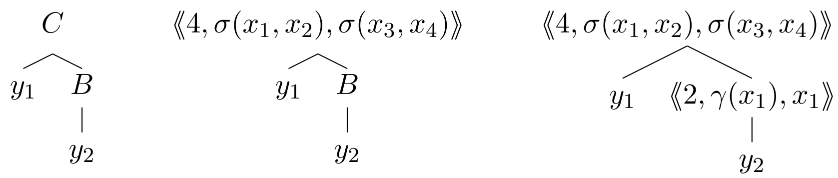
Example 13. Let $\Sigma := \{\sigma^{(2)}, \gamma^{(1)}, \delta^{(1)}, \beta^{(0)}, \alpha^{(0)}\}$ and $k := 2$. Then we e.g. have that

$$\begin{aligned}
\tau_1 &:= \langle 6, \sigma(x_1, \sigma(\beta, x_4)), \gamma(x_1)\rangle && \in k\text{-dil}(T(\Sigma))^1_3\\
\tau_2 &:= \langle 0, \sigma(\alpha, \beta), \delta(\gamma(\alpha))\rangle && \in k\text{-dil}(T(\Sigma))^1_0\\
\tau_3 &:= \langle 6, \delta(x_2), \alpha, \delta(x_5), \alpha\rangle && \in k\text{-dil}(T(\Sigma))^2_3\\
\tau_4 &:= \langle 4, x_1, \sigma(x_1, x_2), \gamma(x_3), \gamma(x_3)\rangle && \in k\text{-dil}(T(\Sigma))^2_2.
\end{aligned}$$

The next step in constructing $T_k(\Sigma, V)$ is to allow occurrences of elements of a second ranked alphabet V, which will be used as nonterminals for magmoid grammars. ■

Example 14. Let Σ, V be ranked alphabets, $k \in \mathbb{N}$. The elements of k-dil$(T(\Sigma))^1$ are together with their subranks a *possibly infinite* ranked alphabet. Define $\hat{T}_k(\Sigma, V) := T(k\text{-dil}(T(\Sigma))^1 \cup V)$. That is, $\hat{T}_k(\Sigma, V)$ is the magmoid of trees over k-dil$(T(\Sigma))^1 \cup V$ – meaning any vertex in a tree in a lava is either an element of V or a k-tuple of trees over Σ. We continue by giving several graphical examples of elements of $\hat{T}_k(\Sigma, V)$. Moreover to distinguish between lavas in $\hat{T}_k(\Sigma, V)$ and lavas in k-dil$(T(\Sigma))^1$, we denote the latter ones using double angle brackets.

From now on let $\Sigma := \{\sigma^{(2)}, \gamma^{(1)}, \beta^{(0)}, \alpha^{(0)}\}$, $V := \{A^{(0)}, B^{(1)}, C^{(2)}\}$, $k := 2$. To construct elements of $\hat{T}_k(\Sigma, V)$, we need some trees over k-dil$(T(\Sigma))^1 \cup V$. The trees



can be formally written as

$$\begin{aligned}
\chi_1 &= C(y_1, B(y_2))\\
\chi_2 &= \langle\!\langle 4, \sigma(x_1, x_2), \sigma(x_3, x_4)\rangle\!\rangle(y_1, B(y_2))
\end{aligned}$$

$$\chi_3 = \langle\!\langle 4, \sigma(x_1, x_2), \sigma(x_3, x_4)\rangle\!\rangle(y_1, \langle\!\langle 2, \gamma(x_1), x_1\rangle\!\rangle(y_2)).$$

To be able to distinguish between variables in $\hat{T}_k(\Sigma, V)$ and those in $T(\Sigma)$, we denoted the variables of the trees χ_1, χ_2 and χ_3 with y_1 and y_2.

Finally, we now have

$$\langle 2, \chi_1, \chi_2, \chi_3\rangle \in \hat{T}_k(\Sigma, V)^3_2.$$

The idea of replacing exactly one element of V (a nonterminal) in a tree over $k\text{-dil}(T(\Sigma))^1 \cup V$ will be heavily used by the derivation relation of CFMGs, which is why in this example χ_2 results from χ_1 by replacing C and respectively χ_3 results from χ_2 by replacing B. In CFMGs however it is possible to derive any tree in $\hat{T}_k(\Sigma, V)^1$ from a nonterminal. ■

Definition 15. The components of an element in $\hat{T}_k(\Sigma, V)$ are from a CFMG point of view composited terminals and nonterminals. Derivations in a CFMG will eventually produce adjacent terminals in a lava which we want to be viewed as a single terminal. To achieve this, we construct a congruence relation on $\hat{T}_k(\Sigma, V)$ that allows us to execute substitutions within a derived tree. This has the advantage of a much easier handling of example derivations and permits the use of $k\text{-dil}(T(\Sigma))^1$ to express the terminals produced by a CFMG. This construction does not affect the language generated by a CFMG as we will prove in Corollary 17 and Lemma 19.

We call lavas $\omega_1, \omega_2 \in \hat{T}_k(\Sigma, V)^p_q$ **sub-congruent** if there exists an $i \in [p]$ and $a, b, c \in \mathbb{N}_0$, $l_1, \ldots, l_a, m_1, \ldots, m_b, r_1, \ldots, r_c \in T_{k\text{-dil}(T(\Sigma))^1 \cup V}(X)$, $v \in \tilde{T}_{k\text{-dil}(T(\Sigma))^1 \cup V}(X_1)$, $u \in k\text{-dil}(T(\Sigma))^1_{a+1+c}$, $m \in k\text{-dil}(T(\Sigma))^1_b$ such that for every $j \in [p] \setminus \{i\}$ we have $\pi^j_p \cdot \omega_1 = \pi^j_p \cdot \omega_2$ and $\pi^i_p \cdot \omega_1 = v[\phi]$ and $\pi^i_p \cdot \omega_2 = v[\psi]$ where

$$\begin{aligned}
\phi &:= u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)\\
\psi &:= (u \cdot (\imath_a \oplus m \oplus \imath_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1, \ldots, r_c).
\end{aligned}$$

The calculation of $u \cdot (\imath_a \oplus m \oplus \imath_c)$ takes place in $k\text{-dil}(T(\Sigma))$. Define the relation $\asymp \subseteq \hat{T}_k(\Sigma, V) \times \hat{T}_k(\Sigma, V)$ as the smallest equivalence relation that includes sub-congruence. That is, $\asymp$ is the symmetric, reflexive, transitive closure of sub-congruence.

We observe that two lavas are equivalent if and only if they can be derived from each other by merging and decomposing terminal-parts. It especially holds that two equivalent lavas contain the same nonterminals.

The fact that $\asymp$ is a congruence relation with respect to the tensor product in $\hat{T}_k(\Sigma, V)$ is obvious.

We prove the fact that $\asymp$ is a congruence relation with respect to the product of composition in $\hat{T}_k(\Sigma, V)$.

Let $p, q, r \in \mathbb{N}_0$, $u_1, u_2 \in \hat{T}_k(\Sigma, V)^p_q$, $w \in \hat{T}_k(\Sigma, V)^q_r$ such that u_1,u_2 are sub-congruent. Moreover let u_1 and u_2 be decomposed as in the definition of sub-congruence:

$$\pi^i_p \cdot u_1 = v[u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)]$$

$$\pi_p^i \cdot u_2 = v[(u \cdot (1_a \oplus m \oplus 1_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1, \ldots, r_c)].$$

Now we have by associativity of the product of composition that

$$\begin{aligned}
\pi_p^i \cdot (u_1 \cdot w) &= v \cdot (u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c) \cdot w) \\
&= v \cdot u(l_1 \cdot w, \ldots, l_a \cdot w, m(m_1 \cdot w, \ldots, m_b \cdot w), r_1 \cdot w, \ldots, r_c \cdot w) \\
&\asymp v \cdot (u \cdot (1_a \oplus m \oplus 1_c))(l_1 \cdot w, \ldots, l_a \cdot w, m_1 \cdot w, \ldots, m_b \cdot w, r_1 \cdot w, \ldots, r_c \cdot w) \\
&= v \cdot ((u \cdot (1_a \oplus m \oplus 1_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1, \ldots, r_c) \cdot w) \\
&= \pi_p^i \cdot (u_2 \cdot w).
\end{aligned}$$

For any $j \in [p] \setminus \{i\}$ it trivially holds that $\pi_p^j \cdot (u_1 \cdot w) = \pi_p^j \cdot (u_2 \cdot w)$. Thus $u_1 \cdot w$, $u_2 \cdot w$ are sub-congruent. We can prove analogously that if u_2, u_1 are sub-congruent, so are $u_2 \cdot w$ and $u_1 \cdot w$.

Now let $u_1, u_2 \in \hat{T}_k(\Sigma, V)_q^p, w \in \hat{T}_k(\Sigma, V)_r^q$ such that $u_1 \asymp u_2$. Since $\asymp$ is the symmetric, reflexive, transitive closure of sub-congruence, there exist $n \in \mathbb{N}$, $w_1, \ldots, w_n \in \hat{T}_k(\Sigma, V)_q^p$ such that $u_1 = w_1$, $u_2 = w_n$ and for any $i \in [n-1]$ either w_i, w_{i+1} or w_{i+1}, w_i are sub-congruent. Thus we have

$$u_1 \cdot w = w_1 \cdot w \asymp \cdots \asymp w_n \cdot w = u_2 \cdot w$$

which proves the congruence property of $\asymp$ for left multiplication.

Let $p, q, r \in \mathbb{N}_0$, $w \in \hat{T}_k(\Sigma, V)_q^p, u_1, u_2 \in \hat{T}_k(\Sigma, V)_r^q$ such that u_1,u_2 are sub-congruent. Moreover let u_1 and u_2 be decomposed as in the definition of sub-congruence:

$$\begin{aligned}
\pi_q^i \cdot u_1 &= v[u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)] \\
\pi_q^i \cdot u_2 &= v[(u \cdot (1_a \oplus m \oplus 1_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1, \ldots, r_c)].
\end{aligned}$$

Let $j \in [p]$ and let m be the number of occurrences of x_i in $\pi_p^j \cdot w$. Straightforward induction on m yields that $\pi_p^j \cdot w \cdot u_1 \asymp \pi_p^j \cdot w \cdot u_2$. Thus

$$\begin{aligned}
w \cdot u_1 &= \langle r, \pi_p^1 \cdot w \cdot u_1, \ldots, \pi_p^p \cdot w \cdot u_1 \rangle \\
&\asymp \langle r, \pi_p^1 \cdot w \cdot u_2, \ldots, \pi_p^p \cdot w \cdot u_1 \rangle \\
&\ldots \\
&\asymp \langle r, \pi_p^1 \cdot w \cdot u_2, \ldots, \pi_p^p \cdot w \cdot u_2 \rangle \\
&= w \cdot u_2.
\end{aligned}$$

Analogous to the case of left multiplication it is true that for any $w \in \hat{T}_k(\Sigma, V)_q^p, u_1, u_2 \in \hat{T}_k(\Sigma, V)_r^q$ such that $u_1 \asymp u_2$ we have $w \cdot u_1 \asymp w \cdot u_2$.

Therefore $\asymp$ is a congruence relation with respect to all magmoid operations.

Finally we define $T_k(\Sigma, V) := \hat{T}_k(\Sigma, V)/_{\asymp}$ and $\tilde{T}_k(\Sigma, V)$ denotes the torsion-free subset of $T_k(\Sigma, V)$ (i.e., the equivalence classes of torsion-free lavas).

Note that in [5], the structure $T_k(\Sigma, V)$ is denoted by $T(\Sigma \cup V)$. ■

Corollary 16. Let Σ be a ranked alphabet, $k \geq 1$, $\zeta \in T_k(\Sigma, \emptyset)_0^1$. There exists a unique $\xi \in k\text{-dil}(T(\Sigma))_0^1$ such that $\xi \in [\zeta]_{\asymp}$. We will denote ξ by $\mu(\zeta)$.

Proof. We use the fact that $\asymp$ is the symmetric, reflexive, transitive closure of sub-congruence.

Define the mapping $\mu : T_k(\Sigma, \emptyset)_0^1 \longrightarrow k\text{-dil}(T(\Sigma))_0^1$ recursively by

$$\mu(\sigma(s_1, \ldots, s_n)) = \sigma \cdot (\mu(s_1) \oplus \cdots \oplus \mu(s_n))$$

for $n \in \mathbb{N}_0, \sigma \in k\text{-dil}(T(\Sigma))_n^1, s_1, \ldots, s_n \in T_k(\Sigma, \emptyset)_0^1$.

Note that $\mu(\sigma) = \sigma$ for every $\sigma \in k\text{-dil}(T(\Sigma))_0^1$.

Let $u_1, u_2 \in T_k(\Sigma, \emptyset)_0^1$ such that u_1, u_2 are sub-congruent. We show that $\mu(u_1) = \mu(u_2)$. Let therefore u_1 and u_2 be decomposed as in the definition of sub-congruence:

$$\begin{aligned} u_1 &= v[u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)] \\ u_2 &= v[(u \cdot (\mathbf{1}_a \oplus m \oplus \mathbf{1}_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1 \ldots, r_c)]. \end{aligned}$$

Using the recursive definition of μ we have for some $\nu \in k\text{-dil}(T(\Sigma))_1^1$ (that depends on v) that

$$\begin{aligned} \mu(u_1) = \quad & \nu \cdot \Big(\mu\big(u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)\big)\Big) \\ = \quad & \nu \cdot \Big(u \cdot \big(\mu(l_1) \oplus \cdots \oplus \mu(l_a) \oplus \mu(m(m_1, \ldots, m_b)) \oplus \mu(r_1) \oplus \cdots \oplus \mu(r_c)\big)\Big) \\ = \quad & \nu \cdot \Big(\big(u \cdot (\mathbf{1}_a \oplus \mathbf{1} \oplus \mathbf{1}_c)\big) \cdot \big(\mu(l_1) \oplus \cdots \oplus \mu(l_a) \oplus \\ & (m \cdot (\mu(m_1) \oplus \cdots \oplus \mu(m_b))) \oplus \mu(r_1) \oplus \cdots \oplus \mu(r_c)\big)\Big) \\ = \quad & \nu \cdot \Big(u \cdot \big((\mathbf{1}_a \oplus m \oplus \mathbf{1}_c) \cdot (\mu(l_1) \oplus \cdots \oplus \mu(l_a) \oplus \\ & \mu(m_1) \oplus \cdots \oplus \mu(m_b) \oplus \mu(r_1) \oplus \cdots \oplus \mu(r_c))\big)\Big) \\ = \quad & \nu \cdot \Big(\mu\big((u \cdot (\mathbf{1}_a \oplus m \oplus \mathbf{1}_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1, \ldots, r_c)\big)\Big) \\ = \quad & \mu(u_2). \end{aligned}$$

Analogously it is true that if u_2, u_1 are sub-congruent, then $\mu(u_2) = \mu(u_1)$.

Let $u, w \in T_k(\Sigma, \emptyset)_0^1$ such that $u \asymp w$. Thus there exists $n \in \mathbb{N}, u_1, \ldots, u_n \in T_k(\Sigma, \emptyset)_0^1$ such that $u = u_1, w = u_n$ and for every $i \in [n-1]$ either u_i, u_{i+1} or u_{i+1}, u_i are sub-congruent. Since it holds that

$$\mu(u_i) = \mu(u_{i+1}) \text{ for every } i \in [n-1],$$

we deduce that $\mu(u) = \mu(u_1) = \mu(u_n) = \mu(w)$.

It is obvious that $\mu(\zeta) \in k\text{-dil}(T(\Sigma))_0^1, \mu(\zeta) \in [\zeta]_{\asymp}$. Moreover let $\alpha, \beta \in k\text{-dil}(T(\Sigma))_0^1$ such that $\alpha, \beta \in [\zeta]_{\asymp}$. Using

$$\alpha = \mu(\alpha) = \mu(\beta) = \beta,$$

we have that $\mu(\zeta)$ is the only element of $k\text{-dil}(T(\Sigma))_0^1$ that is equivalent to ζ. $\square$

Corollary 17. Let Σ, V be ranked alphabets, $k \geq 0$, $B \in V$, $\xi, \xi' \in T_k(\Sigma, V)^1_0$ such that $\xi \asymp \xi'$. Let moreover $\xi_1 \in \tilde{T}_k(\Sigma, V), \xi_2 \in T_k(\Sigma, V)$ such that $\xi = \xi_1 \cdot B \cdot \xi_2$ and the right hand side is defined (note that this forces ξ_1 to have subrank 1). Then there exist $\xi'_1 \in \tilde{T}_k(\Sigma, V), \xi'_2 \in T_k(\Sigma, V)$ such that

$$\xi' = \xi'_1 \cdot B \cdot \xi'_2 \text{ is defined and } \xi'_i \asymp \xi_i \text{ for } i \in [2].$$

Proof. We again use the fact that $\asymp$ is the symmetric, reflexive, transitive closure of sub-congruence.

Let $\xi, \xi' \in T_k(\Sigma, V)^1_0$ such that ξ and ξ' are sub-congruent and be decomposed as in the definition of sub-congruence:

$$\begin{aligned} \xi &= v[u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)], \\ \xi' &= v[(u \cdot (1_a \oplus m \oplus 1_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1 \ldots, r_c)]. \end{aligned}$$

Let moreover $B \in V$, $\xi_1 \in \tilde{T}_k(\Sigma, V)^1_1, \xi_2 \in T_k(\Sigma, V)$ such that $\xi = \xi_1 \cdot B \cdot \xi_2$. We observe that this decomposition labels a single occurrence of B in ξ. Obviously the position of this occurrence of B in ξ is equal to the position of x_1 in ξ_1. Let ω be this position.

Case 1 – $\omega \in \mathrm{pos}(v)$: There exists $\zeta \in T_k(\Sigma, V)_1$ such that $v = \xi_1 \cdot B \cdot \zeta$. Thus we have

$$\xi_2 = \zeta[u(l_1, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)]$$

and deduce that for

$$\begin{aligned} \xi'_1 &:= \xi_1, \\ \xi'_2 &:= \zeta \cdot ((u \cdot (1_a \oplus m \oplus 1_c))(l_1, \ldots, l_a, m_1, \ldots, m_b, r_1 \ldots, r_c)) \end{aligned}$$

we get $\xi'_1 \asymp \xi_1$, $\xi'_2 \asymp \xi_2$ and $\xi' = \xi'_1 \cdot B \cdot \xi'_2$.

Case 2 – $\omega \notin \mathrm{pos}(v)$: Since B is neither u nor m, w.l.o.g. the occurrence of interest of B can be assumed to be in l_1. Thus there exists $\zeta \in T_k(\Sigma, V)^1_1$ such that $l_1 = \zeta[B \cdot \xi_2]$. Therefore

$$\xi_1 = v[u(\zeta, \ldots, l_a, m(m_1, \ldots, m_b), r_1, \ldots, r_c)]$$

and we deduce that for

$$\begin{aligned} \xi'_1 &:= v[(u \cdot (1_a \oplus m \oplus 1_c))(\zeta, \ldots, l_a, m_1, \ldots, m_b, r_1 \ldots, r_c)], \\ \xi'_2 &:= \xi_2 \end{aligned}$$

we get $\xi'_1 \asymp \xi_1$, $\xi'_2 \asymp \xi_2$, $\xi' = \xi'_1 \cdot B \cdot \xi'_2$ and $\xi'_1 \in \tilde{T}_k(\Sigma, V)$.

This proves the claim for sub-congruent ξ and ξ'. Analogously we can prove that for sub-congruent ξ' and ξ the claim also holds.

Now let $\xi, \xi' \in T_k(\Sigma, \emptyset)^1_0$ such that $\xi \asymp \xi'$. Thus there exist $n \in \mathbb{N}, \zeta_1, \ldots, \zeta_n \in T_k(\Sigma, \emptyset)^1_0$ such that $\xi = \zeta_1, \xi' = \zeta_n$ and for every $i \in [n-1]$ either ζ_i, ζ_{i+1} or ζ_{i+1}, ζ_i are sub-congruent. Let moreover $B \in V$, $\xi'_{(1,1)} \in \tilde{T}_k(\Sigma, V)^1_1, \xi'_{(1,2)} \in T_k(\Sigma, V)$ such that $\xi = \xi'_{(1,1)} \cdot B \cdot \xi'_{(1,2)}$.

Since it holds that for any $i \in \{2, \ldots, n\}$ there are $\xi'_{(i,1)} \in \tilde{T}_k(\Sigma, V), \xi'_{(i,2)} \in T_k(\Sigma, V)$ such that

$$\zeta_i = \xi'_{(i,1)} \cdot B \cdot \xi'_{(i,2)} \text{ and } \xi'_{(i,j)} \asymp \xi'_{(i-1,j)} \text{ for } j \in [2],$$

we have that $\xi' = \xi'_{(n,1)} \cdot B \cdot \xi'_{(n,2)}$ where $\xi'_{(1,1)} \asymp \xi'_{(n,1)}$ and $\xi'_{(1,2)} \asymp \xi'_{(n,2)}$. □

3.3 Context-Free Magmoid Grammars (CFMG)

The most important difference between the construction of $T_k(\Sigma, V)$ and context-free magmoid grammars is that the latter ones will always only use lavas from $T_k(\Sigma, V)$ with superrank 1. Thus, we can at every point express the derivations and productions of a CFMG using trees over $k\text{-dil}(T(\Sigma))^1 \cup V$ instead of the (notationally correct) corresponding 1-tuples. The constructed magmoid-machinery from the last subchapter will prove to be useful when we start proving properties and giving examples.

Definition 18. Let $k \geq 1$. A **k-dilated context-free magmoid grammar** (**k-dilated CFMG**) is a tuple $G = (V, \Sigma, Z, P)$ such that

- V, Σ are ranked alphabets, V are called **nonterminals**
- $Z \in V^{(0)}$ and
- P is a finite set of **productions** of the form
  $A \longrightarrow \zeta$, with $q \in \mathbb{N}_0, A \in V^{(q)}, \zeta \in T_k(\Sigma, V)^1_q$.

Furthermore we call the elements of $k\text{-dil}(T(\Sigma))^1$ **terminals**.

A **context-free magmoid grammar** (**CFMG**) is a tuple $G = (V, \Sigma, Z, P)$ such that G is a k-dilated CFMG for some $k \geq 1$. We will call k the **dilatation index of** G.

The **derivation relation induced by** G is the relation $\Longrightarrow_G \subseteq T_k(\Sigma, V)^1 \times T_k(\Sigma, V)^1$ that has for $\xi_1, \xi_2 \in T_k(\Sigma, V)^1$

$$\xi_1 \Longrightarrow_G \xi_2 :\Longleftrightarrow \exists \zeta_1 \in \tilde{T}_k(\Sigma, V)^1_1, \zeta_2 \in T_k(\Sigma, V), (A \longrightarrow \zeta) \in P : \\ \xi_1 = \zeta_1 \cdot A \cdot \zeta_2 \wedge \xi_2 = \zeta_1 \cdot \zeta \cdot \zeta_2.$$

The **language generated by** G is $L(G) := \{\pi^1_k \cdot \zeta \mid Z \Longrightarrow^*_G \zeta, \zeta \in k\text{-dil}(T(\Sigma))^1_0\}$. Note that formally $\langle 0, Z\rangle$ produces (some tree over $k\text{-dil}(T(\Sigma))^1$ equivalent to) $\langle 0, \zeta\rangle$. A first "layer of notation" allows us to say that Z produces ζ. But since $\zeta \in k\text{-dil}(T(\Sigma))^1_0$, we can work with ζ as an element of $T(\Sigma)^k_0$. Thus in the definition of $L(G)$, π^1_k is the projection in $T(\Sigma)$, not in $k\text{-dil}(T(\Sigma))$. Then again the result of $\pi^1_k \cdot \zeta$ is in $T(\Sigma)^1$, but is notationally identified with an element of T_Σ. Therefore we can perpend $L(G) \subseteq T_\Sigma$ and will use this identity throughout the rest of this thesis.

A CFMG $G = (V, \Sigma, Z, P)$ is called **linear** or **l-CFMG** $:\Longleftrightarrow$ for every production $(A \longrightarrow \zeta) \in P$, ζ is linear as an element of $T_{k\text{-dil}(T(\Sigma))^1 \cup V}(X)$.

Note that this is not the same definition of linear CFMGs as in [5].

The **class of tree-languages generated by context-free magmoid grammars over** Σ is denoted $\mathscr{L}(\text{CFMG}(\Sigma))$.

The **class of tree-languages generated by context-free magmoid grammars** is denoted $\mathscr{L}(\text{CFMG})$.

Analogously, the classes $\mathscr{L}(\text{l-CFMG}(\Sigma))$ and $\mathscr{L}(\text{l-CFMG})$ are defined for the case of linear context-free magmoid grammars and the classes $\mathscr{L}(\text{CFMG}(\Sigma)_k)$ ($\mathscr{L}(\text{l-CFMG}(\Sigma)_k)$) and $\mathscr{L}(\text{CFMG}_k)$ ($\mathscr{L}(\text{l-CFMG}_k)$) are defined for the case of k-dilated (and linear) context-free magmoid grammars respectively. ■

The derivation relation of CFMGs is defined on $T_k(\Sigma, V)$ which contains equivalence classes. Thus we can swap the representatives of such equivalence classes while deriving.

The following lemma proves that a change of representatives does not affect the possible derivations in a CFMG.

Lemma 19. Let $G = (V, \Sigma, Z, P)$ be a k-dilated CFMG, $n \in \mathbb{N}_0$, $\xi, \xi', \zeta \in T_k(\Sigma, V)^1$ such that $\xi \asymp \xi'$ and $\xi \Longrightarrow_G^n \zeta$. Then

$$\exists \zeta' \in [\zeta]_\asymp : \xi' \Longrightarrow_G^n \zeta'. \tag{2}$$

Proof. We use complete induction on n.

For $n = 0$ we have $\xi = \zeta$, thus $\xi' \asymp \zeta$ and moreover $\zeta' = \xi'$. This proves the induction base.

Assume that for some $n \in \mathbb{N}_0$ we have

$$\forall \xi, \xi', \zeta \in T_k(\Sigma, V)^1, \xi \asymp \xi' : (\xi \Longrightarrow_G^n \zeta) \implies (\exists \zeta' \in [\zeta]_\asymp : \xi' \Longrightarrow_G^n \zeta').$$

Let $\xi, \xi', \zeta \in T_k(\Sigma, V)^1, \xi \asymp \xi'$ such that $\xi \Longrightarrow_G^{n+1} \zeta$. By definition of $(n+1)$-fold composition of $\Longrightarrow_G$ we have

$$\exists \theta \in T_k(\Sigma, V)^1 : \xi \Longrightarrow_G^n \theta \text{ and } \theta \Longrightarrow_G^1 \zeta.$$

Thus by assumption there exists some $\theta' \in [\theta]_\asymp$ such that $\xi' \Longrightarrow_G^n \theta'$.

The definition of $\Longrightarrow_G$ implies the existence of $\theta_1 \in \tilde{T}_k(\Sigma, V), \theta_2 \in T_k(\Sigma, V)$ and $(B \longrightarrow \tilde{\theta}) \in P$ such that

$$\theta = \theta_1 \cdot B \cdot \theta_2 \text{ and } \zeta = \theta_1 \cdot \tilde{\theta} \cdot \theta_2.$$

By Corollary 17 there exist $\theta_1', \theta_2' \in T_k(\Sigma, V)$ such that

$$\theta' = \theta_1' \cdot B \cdot \theta_2' \text{ and } \theta_i \asymp \theta_i' \text{ for } i \in [2].$$

We deduce that for $\zeta' := \theta_1' \cdot \tilde{\theta} \cdot \theta_2'$ we have

$$\theta' \Longrightarrow_G^1 \zeta'$$

and furthermore since $\asymp$ is a congruence relation it holds that $\zeta' \asymp \zeta$.

We thus have $\xi' \Longrightarrow_G^{n+1} \zeta'$ for some $\zeta' \in [\zeta]_\asymp$ which by the principle of complete induction proves the lemma. □

Example 20. Let $\Sigma := \{\sigma^{(2)}, \eta^{(1)}, \gamma^{(1)}, \delta^{(1)}, \beta^{(0)}, \alpha^{(0)}\}$, $V := \{A^{(0)}, B^{(2)}, C^{(1)}, D^{(1)}\}$. We give an introductory example for CFMGs. For the sake of keeping track of the used notations we successively construct the right hand sides of rules, one by one abbreviating the mathematical objects. We shall then use these notations unmentioned throughout the rest of this thesis.

Let $G := (V, \Sigma, A, P)$ where

$$\begin{aligned} P: \quad & A \xrightarrow{r_1} \quad && \Big\langle 0; B(C(\langle\!\langle 0, \beta, \alpha \rangle\!\rangle), \langle\!\langle 0, \alpha, \beta \rangle\!\rangle) \Big\rangle \\ & B \xrightarrow{r_2} \quad && \Big\langle 2; \langle\!\langle 2, \eta(x_1), \alpha \rangle\!\rangle (B(C(x_1), D(x_2))) \Big\rangle \end{aligned}$$

$$
\begin{aligned}
B &\xrightarrow{r_3} && \Big\langle 2; \langle\!\langle 4, \sigma(x_1, x_3), \sigma(x_2, x_4)\rangle\!\rangle(x_1, x_1)\Big\rangle \\
C &\xrightarrow{r_4} && \Big\langle 1; \langle\!\langle 2, \gamma(x_1), \gamma(x_2)\rangle\!\rangle(x_1)\Big\rangle \\
C &\xrightarrow{r_5} && \Big\langle 1; \langle\!\langle 2, \delta(x_2), \delta(x_1)\rangle\!\rangle(x_1)\Big\rangle \\
D &\xrightarrow{r_6} && \Big\langle 1; \langle\!\langle 2, \delta(x_1), \gamma(x_2)\rangle\!\rangle(x_1)\Big\rangle
\end{aligned}
$$

This is a 2-dilated CFMG. The right hand sides of productions are not abbreviated and formally represent the mathematical objects in P. Note that this time we also did not differentiate between the variables in $T_2(\Sigma, V)$ and those in 2-dil$(T(\Sigma))^1$. As discussed earlier, we can reduce the right hand sides of productions to the trees that are enclosed in the tuples. Moreover using a graphical representation of those trees, we get the following

$P:$

$$
A \xrightarrow{r_1} \quad
\begin{tikzcd}
 & B \arrow[dl, dash] \arrow[dr, dash] & \\
C \arrow[d, dash] & & \langle\!\langle 0, \alpha, \beta\rangle\!\rangle \\
\langle\!\langle 0, \beta, \alpha\rangle\!\rangle & &
\end{tikzcd}
$$

$$
B \xrightarrow{r_2} \quad
\begin{tikzcd}
 & \langle\!\langle 2, \eta(x_1), \alpha\rangle\!\rangle \arrow[d, dash] & \\
 & B \arrow[dl, dash] \arrow[dr, dash] & \\
C \arrow[d, dash] & & D \arrow[d, dash] \\
x_1 & & x_2
\end{tikzcd}
$$

$$
B \xrightarrow{r_3} \quad
\begin{tikzcd}
 & \langle\!\langle 4, \sigma(x_1, x_3), \sigma(x_2, x_4)\rangle\!\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
x_1 & & x_1
\end{tikzcd}
$$

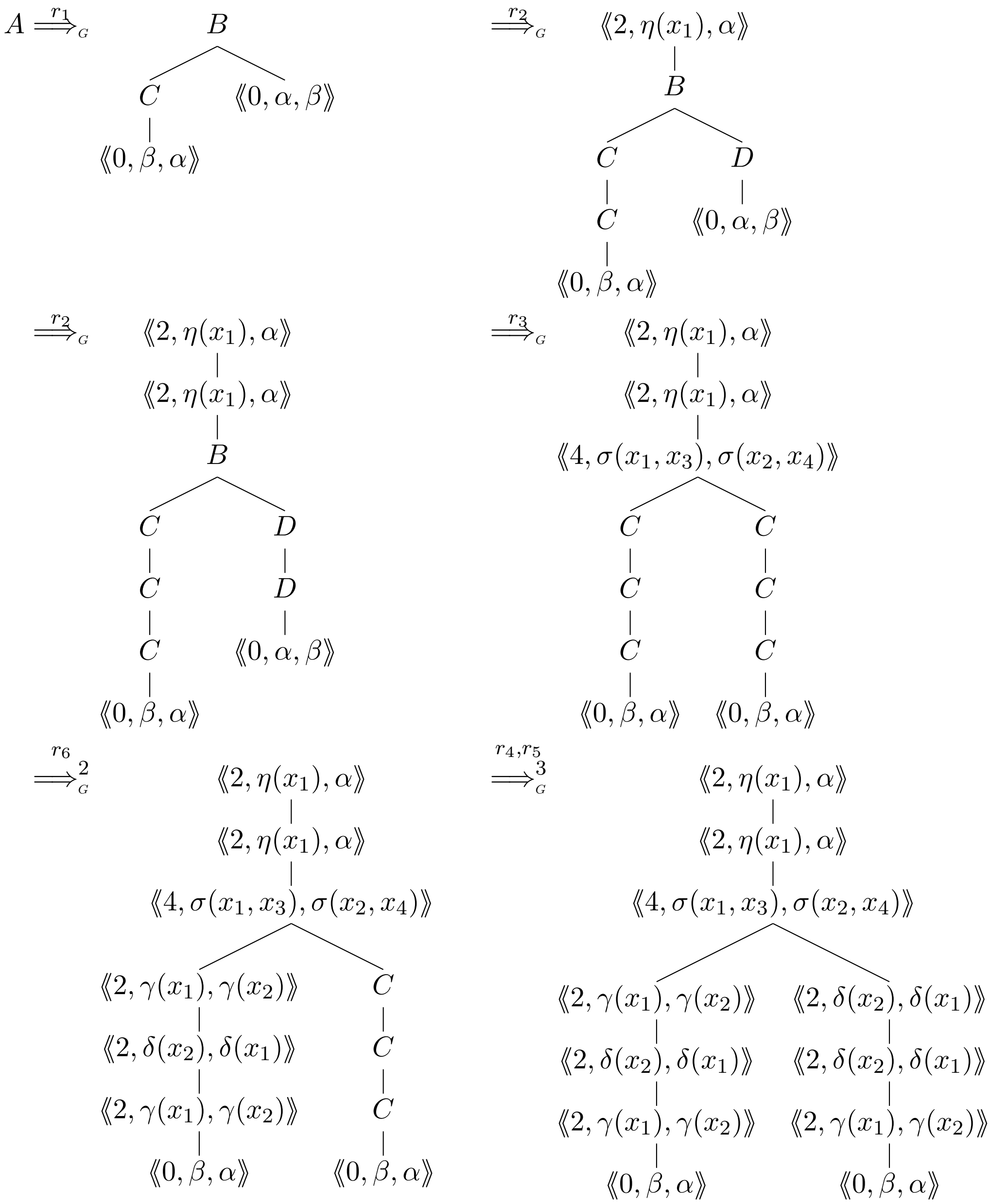
$$
C \xrightarrow{r_4} \quad
\begin{tikzcd}
\langle\!\langle 2, \gamma(x_1), \gamma(x_2)\rangle\!\rangle \arrow[d, dash] \\
x_1
\end{tikzcd}
$$

$$
C \xrightarrow{r_5} \quad
\begin{tikzcd}
\langle\!\langle 2, \delta(x_2), \delta(x_1)\rangle\!\rangle \arrow[d, dash] \\
x_1
\end{tikzcd}
$$

$$
D \xrightarrow{r_6} \quad
\begin{tikzcd}
\langle\!\langle 2, \delta(x_1), \gamma(x_2)\rangle\!\rangle. \arrow[d, dash] \\
x_1
\end{tikzcd}
$$

The right hand sides of the rules r_4 to r_6 are of the form $\tau(x_1, \ldots, x_k)$ for some $\tau \in$ 2-dil$(T(\Sigma))^1_k$. Thus we can abbreviate e.g. rule r_4 to be $C \xrightarrow{r_4} \langle\!\langle 2, \gamma(x_1), \gamma(x_2)\rangle\!\rangle$.

An example for a derivation in G is



The last tree is in $T_2(\Sigma, V)$ equal to $\langle\!\langle 0, \eta(\eta(\sigma(\gamma(\delta(\gamma(\alpha))), \delta(\delta(\gamma(\beta)))))), \alpha\rangle\!\rangle$, thus we have

$$A \Longrightarrow_G^* \langle\!\langle 0, \eta^2\sigma(\gamma\delta\gamma\alpha, \delta^2\gamma\beta), \alpha\rangle\!\rangle \in 2\text{-dil}(T(\Sigma))_0^1.$$

Moreover $\eta^2\sigma(\gamma\delta\gamma\alpha, \delta^2\gamma\beta)$ is in $L(G)$. The structure of the productions of G makes it easy to verify that $L(G) = \{\eta^n\sigma(\omega_1 a_1, \omega_2 a_2) \mid n \in \mathbb{N}_0, \omega_1, \omega_2 \in \{\gamma, \delta\}^{n+1}, a_i = o(\omega_i), i \in [2]\}$ where

$$o : \{\gamma, \delta\}^* \to \{\alpha, \beta\}$$

$$o(\omega) = \begin{cases} \alpha & , |\omega|_\delta \equiv 1 \bmod 2 \\ \beta & , |\omega|_\delta \equiv 0 \bmod 2. \end{cases}$$

■

Definition 21. Let $G = (V, \Sigma, Z, P)$ be a CFMG. The **OI derivation relation induced by** G is the relation $\overset{\circ}{\Longrightarrow}_G \subseteq \Longrightarrow_G$ such that for any $(\xi[A[\zeta_1, \ldots, \zeta_q]], \xi[\zeta[\zeta_1, \ldots, \zeta_q]]) \in \overset{\circ}{\Longrightarrow}_G$ the path from the root of ξ to the single occurrence of x_1 in ξ only consists of elements of $k\text{-dil}(T(\Sigma))^1$ (or x_1).

The **OI language generated by** G is $L_\circ(G) := \{\pi_k^1 \cdot \zeta \mid Z \overset{\circ}{\Longrightarrow}_G{}^* \zeta, \zeta \in k\text{-dil}(T(\Sigma))_0^1\}$.

Lemma 22. Let $G = (V, \Sigma, Z, P)$ be a CFMG. It holds that

$$L(G) = L_\circ(G).$$

Proof. Defining $G' := (V, \Delta, Z, P)$ where $\Delta := \{\Lambda \in k\text{-dil}(T(\Sigma))^1 \mid \Lambda \text{ occurs in } P\}$, we get a context-free tree grammar that has

$$\forall n, m \in \mathbb{N}_0, A \in V^{(m)}, \zeta \in T_k(\Sigma, V)_m^1 : (A \Longrightarrow_G^n \zeta) \Longleftrightarrow (\exists \xi \in [\zeta]_\asymp : A \Longrightarrow_{G'}^n \xi),$$

as we will prove in proposition 26, equation (5). Thus G' already expresses every possible derivation in G. Since G' is a CFTG, we can always use OI derivations in G' to derive a tree over $k\text{-dil}(T(\Sigma))^1$ which implies the claimed property for G. □

Definition 23. Let $G = (V, \Sigma, Z, P)$ be a CFMG. We say G is in **normal form** or **nf-CFMG** if for any rule $(A \longrightarrow \zeta) \in P$, ζ is of one of the following forms:

$$\zeta \in T_V(X) \qquad\qquad \zeta \in k\text{-dil}(T(\Sigma))^1.$$

Note that for CFMGs there exist normal forms that restrict right hand sides of productions even more – e.g. Chomsky-like normal forms as seen in [19] – but for our purposes the given normal form is completely adequate. ■

Lemma 24. Let $G = (V, \Sigma, Z, P)$ be a CFMG. There exists a CFMG in normal form G' such that $L(G) = L(G')$.

Proof. We give a construction of G'.

For $p = (A \longrightarrow \zeta) \in P$ define $\varkappa_p := \{\kappa \in k\text{-dil}(T(\Sigma))^1 \mid \exists \omega \in pos(\zeta) : \kappa = \zeta(\omega)\}$. Let $\varkappa := \bigcup_{p \in P} \varkappa_p$.

By defining $V' := V \dot\cup \{V_K^{(0)} \mid K \in \varkappa\}$, we add a new nonterminal to V for each lava that occurs in a right hand side of a rule in P. Moreover let τ be the relabeling that replaces any occurrence of $K \in \varkappa$ in a tree over $k\text{-dil}(T(\Sigma))^1 \cup V$ with V_K and υ the relabeling that replaces any occurrence of V_K in a tree over $k\text{-dil}(T(\Sigma))^1 \cup V'$ with K. We immediately deduce that τ and υ extended to $T_k(\Sigma, V)$ are homomorphisms with respect to the magmoid structure of $T_k(\Sigma, V)$.

The CFMG $G' = (V', \Sigma, Z, P')$ where

$$\begin{aligned} P' := \quad & \{(A \longrightarrow \xi) \mid (A \longrightarrow \zeta) \in P, \xi = \tau(\zeta)\} \\ & \cup\{(V_K \longrightarrow K) \mid K \in \kappa\} \end{aligned}$$

has the claimed properties. Obviously the right hand side of any rule in P' is either a tree over V' or an element of k-dil$(T(\Sigma))^1$.

Now we show $L(G) = L(G')$. To do that we prove the more general claim

$$\forall A \in V, \zeta \in T_k(\Sigma, V)^1_0 : (A \Longrightarrow^+_G \zeta) \Longleftrightarrow (\exists \xi \in T_k(\Sigma, V')^1_0 : A \Longrightarrow^+_{G'} \xi \wedge \upsilon(\xi) = \zeta) \quad (3)$$

by induction over the length of the derivations.

"$\Longrightarrow$": Let $A \in V, \zeta \in T_k(\Sigma, V)^1_0$ such that $A \Longrightarrow^1_G \zeta$. Thus by definition of $\Longrightarrow_G$ we have $(A \longrightarrow \zeta) \in P$, which by construction of G' implies $A \longrightarrow \tau(\zeta) \in P'$. Then again by definition we have $A \Longrightarrow^1_{G'} \tau(\zeta)$. Obviously $\upsilon(\tau(\zeta)) = \zeta$, thus the induction base is proven.

Assume for some $n \in \mathbb{N}$ we have

$$\forall A \in V, \zeta \in T_k(\Sigma, V)^1_0 : (A \Longrightarrow^n_G \zeta) \Longrightarrow (\exists \xi \in T_k(\Sigma, V')^1_0 : A \Longrightarrow^+_{G'} \xi \wedge \upsilon(\xi) = \zeta).$$

Now let $A \in V, \zeta \in T_k(\Sigma, V)^1_0$ such that $A \Longrightarrow^{n+1}_G \zeta$. Thus

$$\exists \zeta' \in T_k(\Sigma, V)^1_0 : A \Longrightarrow^n_G \zeta' \wedge \zeta' \Longrightarrow_G \zeta$$

and by induction assumption

$$\exists \xi' \in T_k(\Sigma, V')^1_0 : A \Longrightarrow^+_{G'} \xi' \wedge \upsilon(\xi') = \zeta'.$$

Moreover by definition of $\Longrightarrow_G$ we have

$$\exists \zeta_1 \in \tilde{T}_k(\Sigma, V)^1_1, \zeta_2 \in T_k(\Sigma, V), (B \longrightarrow \tilde{\zeta}) \in P : \zeta' = \zeta_1 \cdot B \cdot \zeta_2 \wedge \zeta = \zeta_1 \cdot \tilde{\zeta} \cdot \zeta_2.$$

By construction it holds that $(B \longrightarrow \tau(\tilde{\zeta})) \in P'$. Since $\upsilon(\xi') = \zeta'$ we can decompose ξ' in the same way as ζ':

$$\exists \xi_1 \in \tilde{T}_k(\Sigma, V)^1_1, \xi_2 \in T_k(\Sigma, V') : \xi' = \xi_1 \cdot B \cdot \xi_2 \wedge \upsilon(\xi_1) = \zeta_1 \wedge \upsilon(\xi_2) = \zeta_2.$$

Thus in G' we can derive $\xi' \Longrightarrow^1_{G'} (\xi_1 \cdot \tau(\tilde{\zeta}) \cdot \xi_2) =: \xi$. Finally because υ is a homomorphism we have $\upsilon(\xi) = \upsilon(\xi_1) \cdot \upsilon(\tilde{\zeta}) \cdot \upsilon(\xi_2) = \zeta_1 \cdot \tilde{\zeta} \cdot \zeta_2 = \zeta$.

All in all, there exists $\xi \in T_k(\Sigma, V')^1_0$ such that $A \Longrightarrow^+_{G'} \xi$ and $\upsilon(\xi) = \zeta$.

"$\Longleftarrow$": Let $A \in V, \zeta \in T_k(\Sigma, V)^1_0$ such that there exists $\xi \in T_k(\Sigma, V')^1_0 : A \Longrightarrow^1_{G'} \xi \wedge$ $\upsilon(\xi) = \zeta$. Thus, we have $(A \longrightarrow \xi) \in P'$ which implies by construction $(A \longrightarrow \upsilon(\xi)) \in P$ since $A \in V$. This gives the fact that $A \Longrightarrow^1_G \zeta$.

Assume for some $n \in \mathbb{N}$ we have

$$\forall A \in V, \zeta \in T_k(\Sigma, V)^1_0 : (\exists \xi \in T_k(\Sigma, V')^1_0 : A \Longrightarrow^n_{G'} \xi \wedge \upsilon(\xi) = \zeta) \Longrightarrow (A \Longrightarrow^+_G \zeta).$$

Now let $A \in V, \zeta \in T_k(\Sigma, V)_0^1$ such that there exists $\xi \in T_k(\Sigma, V')_0^1 : A \Longrightarrow_{G'}^{n+1} \xi \wedge \upsilon(\xi) = \zeta$. By definition of the n-fold composition of $\Longrightarrow_{G'}$ we have

$$\exists \xi' \in T_k(\Sigma, V')_0^1 : A \Longrightarrow_{G'}^{n} \xi' \wedge \xi' \Longrightarrow_{G'}^{1} \xi.$$

Fix $\zeta' := \upsilon(\xi')$. By induction assumption we get $A \Longrightarrow_{G}^{+} \zeta'$ and again by definition of $\Longrightarrow_{G'}$ it holds that

$$\begin{aligned} &\exists \xi_1 \in \tilde{T}_k(\Sigma, V)_1^1, \xi_2 \in T_k(\Sigma, V'), (B \longrightarrow \tilde{\xi}) \in P' : \\ &\xi' = \xi_1 \cdot B \cdot \xi_2 \wedge \xi = \xi_1 \cdot \tilde{\xi} \cdot \xi_2. \end{aligned}$$

Case 1 – $B \in V$: The production $(B \longrightarrow \upsilon(\tilde{\xi}))$ is in P and since υ is a homomorphism we have

$$\zeta' = \upsilon(\xi_1) \cdot B \cdot \upsilon(\xi_2) \wedge \zeta = \upsilon(\xi_1) \cdot \upsilon(\tilde{\xi}) \cdot \upsilon(\xi_2).$$

Thus we can derive $\zeta' \Longrightarrow_{G}^{1} \zeta$ but then $A \Longrightarrow_{G}^{+} \zeta$.
Case 2 – $B = V_K$, $K \in k\text{-dil}(T(\Sigma))^1$: Using the homomorphism property of υ we can decompose

$$\zeta' = \upsilon(\xi_1) \cdot \upsilon(V_K) \cdot \upsilon(\xi_2) \wedge \zeta = \upsilon(\xi_1) \cdot \upsilon(\tilde{\xi}) \cdot \upsilon(\xi_2).$$

Finally since $\tilde{\xi} = K = \upsilon(V_K)$ we have that $A \Longrightarrow_{G}^{+} \zeta' = \zeta$. This proves (3).

Let $A \in V$, $\zeta \in T_k(\Sigma, V)_0^1$. It obviously holds that

$$(\exists \xi \in T_k(\Sigma, V')_0^1 : A \Longrightarrow_{G'}^{+} \xi \wedge \upsilon(\xi) = \zeta) \Longleftrightarrow (A \Longrightarrow_{G'}^{+} \zeta),$$

which together with (3) results in

$$\forall A \in V, \zeta \in T_k(\Sigma, V)_0^1 : (A \Longrightarrow_{G}^{+} \zeta) \Longleftrightarrow (A \Longrightarrow_{G'}^{+} \zeta). \tag{4}$$

Therefore we deduce for $t \in T_\Sigma$

$$\begin{aligned} t \in L(G) &\overset{\text{def.}}{\Longleftrightarrow} \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1 : Z \Longrightarrow_{G}^{+} \zeta \wedge \pi_k^1 \cdot \zeta = t \\ &\overset{(4)}{\Longleftrightarrow} \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1 : Z \Longrightarrow_{G'}^{+} \zeta \wedge \pi_k^1 \cdot \zeta = t \\ &\overset{\text{def.}}{\Longleftrightarrow} t \in L(G'). \end{aligned}$$

□

Example 25. We construct the normal form of the CFMG $G = (V, \Sigma, A, P)$ from Example 20. Therefore we extract all lavas from vertices on right sides of productions in P. It is

$$\begin{aligned} \varkappa = \{ &\langle\!\langle 0, \beta, \alpha \rangle\!\rangle, \langle\!\langle 0, \alpha, \beta \rangle\!\rangle, \langle\!\langle 2, \eta(x_1), \alpha \rangle\!\rangle, \langle\!\langle 4, \sigma(x_1, x_3), \sigma(x_2, x_4) \rangle\!\rangle, \\ &\langle\!\langle 2, \gamma(x_1), \gamma(x_2) \rangle\!\rangle, \langle\!\langle 2, \gamma(x_2), \gamma(x_1) \rangle\!\rangle, \langle\!\langle 2, \delta(x_1), \delta(x_2) \rangle\!\rangle \} \end{aligned}$$

and labeling the elements of $\varkappa$ with $K_1, \ldots, K_7$ we have

$$V' = \{A^{(0)}, B^{(2)}, C^{(1)}, D^{(1)}, V_{K_1}^{(0)}, V_{K_2}^{(0)}, V_{K_3}^{(0)}, V_{K_4}^{(0)}, V_{K_5}^{(0)}, V_{K_6}^{(0)}, V_{K_7}^{(0)}, \}.$$

The productions turn out to be

$$\begin{aligned} P' : \qquad A &\xrightarrow{r'_1} && B(C(V_{K_1}), V_{K_2}) \\ B &\xrightarrow{r'_2} && V_{K_3}(B(C(x_1), D(x_2)) \\ B &\xrightarrow{r'_3} && V_{K_4} \\ C &\xrightarrow{r'_4} && V_{K_5} \\ C &\xrightarrow{r'_5} && V_{K_6} \\ D &\xrightarrow{r'_6} && V_{K_7} \\ V_{K_i} &\xrightarrow{r'_{6+i}} && K_i && (i \in [7]). \end{aligned}$$

■

This structure is very close to a context-free tree grammar (see Definition in Chapter 2.2). We thus want to examine the connection between CFTGs and CFMGs in the next chapter.

Chapter 4: Comparison of CFMG and CFTG

4.1 The Language Classes and their Connection

As we have observed earlier, a context-free magmoid grammar behaves much like a context-free tree grammar except for the terminals and the definition of its generated language. The following propositions and corollaries explore this kinship.

Our first result is a decomposition of any CFMG G into a total and deterministic td-tt T and a CFTG G'. Therefor, G' is defined to use the lavas that occur in productions of G as a ranked alphabet and have the same productions as G. Thus the only difference between the two grammars is that G is allowed to use the structure given by $T_k(\Sigma, V)$. The transducer T is constructed to "plug together" the resulting trees from G' and by that compensates the unlikeness of G and G'.

Our second result is the composition that is inverse to the first result. That is, we take a CFTG G' (in normal form) and a total and deterministic td-tt T and compose them into a CFMG G. The idea behind this construction is to let G simulate the processing of T on every output of G' for every state of T. Thus G dilates the *terminal* productions of G' to store information about T. By chosing appropriate variables in the derived lavas, we achieve the claimed composition.

Proposition 26. The following relations hold:

$$\mathscr{L}(\text{CFMG}) \subseteq \text{td-TOP}(\mathscr{L}(\text{CFMG})),$$
$$\mathscr{L}(\text{l-CFMG}) \subseteq \text{td-TOP}(\mathscr{L}(\text{l-CFMG})).$$

Proof. Let Σ be a ranked alphabet and $L \in \mathscr{L}(\text{CFMG})$ a tree language over Σ. Thus there exist $k \in \mathbb{N}$ and a k-dilated nf-CFMG $G = (V, \Sigma, Z, P)$ such that $L(G) = L$.

We construct a CFTG G' and a td-td-tt T such that $L(G) = \tau(T)(L(G'))$.

Hence let $G' = (V, \Delta, Z, P)$ and $T = (Q, \Delta, \Sigma, \{q_1\}, R)$ where

$$\Delta := \{\Lambda \in k\text{-dil}(T(\Sigma))^1 \mid \Lambda \text{ occurs in } P\},$$
$$Q := \{q_1, \dots, q_k\}$$

and R contains the rules

$$q_i(u(x_1, \dots, x_n)) \longrightarrow (\pi_k^i \cdot u)[q_1(x_1), \dots, q_k(x_1), \dots, q_1(x_n), \dots, q_k(x_n)]$$

for any $i \in [k], u \in \Delta$.

Note that R is well-defined since we identify $\pi_k^i \cdot u$ with its single component – a tree over Σ with variables in X_{kn}. Thus since Δ is finite, G' and T are well-defined and we easily verify that T is a td-td-tt. Moreover if G is linear, then so is G'.

First we prove the following claim:

$$\forall n, m \in \mathbb{N}_0, A \in V^{(m)}, \zeta \in T_k(\Sigma, V)_m^1 : (A \Longrightarrow_G^n \zeta) \Longleftrightarrow (\exists \xi \in [\zeta]_{\asymp} : A \Longrightarrow_{G'}^n \xi) \qquad (5)$$

by complete induction on n.

"$\Longrightarrow$": The induction base is trivial.

Assume that for some $n \in \mathbb{N}_0$ it holds that

$$\forall m \in \mathbb{N}_0, A \in V^{(m)}, \zeta \in T_k(\Sigma, V)^1_m : (A \Longrightarrow^n_{G} \zeta) \Longrightarrow (\exists \xi \in [\zeta]_{\asymp} : A \Longrightarrow^n_{G'} \xi).$$

Let $m \in \mathbb{N}_0, A \in V^{(m)}, \zeta \in T_k(\Sigma, V)^1_m$ such that $A \Longrightarrow^{n+1}_{G} \zeta$. Thus there exists $\zeta' \in T_k(\Sigma, V)^1_m$ such that

$$A \Longrightarrow^n_{G} \zeta' \text{ and } \zeta' \Longrightarrow^1_{G} \zeta.$$

By induction assumption there exists $\xi' \in [\zeta']_{\asymp}$ such that $A \Longrightarrow^n_{G'} \xi'$. Moreover by definition of $\Longrightarrow_{G}$ there exist $\zeta_1 \in \tilde{T}_k(\Sigma, V), \zeta_2 \in T_k(\Sigma, V)$ and $(B \longrightarrow \tilde{\zeta}) \in P$ with

$$\zeta' = \zeta_1 \cdot B \cdot \zeta_2 \text{ and } \zeta = \zeta_1 \cdot \tilde{\zeta} \cdot \zeta_2.$$

Corollary 17 implies the existence of $\xi_1, \xi_2 \in T_k(\Sigma, V)$ such that

$$\xi' = \xi_1 \cdot B \cdot \xi_2 \text{ and } \xi_i \asymp \zeta_i \text{ for } i \in [2].$$

Thus we can derive $\xi' \Longrightarrow^1_{G'} \xi_1 \cdot \tilde{\zeta} \cdot \xi_2 =: \xi$. Since $\asymp$ is a congruence relation we furthermore get $\xi \asymp \zeta$, which proves the implication.

"$\Longleftarrow$": The induction base is trivial.

Assume that for some $n \in \mathbb{N}_0$ it holds that

$$\forall m \in \mathbb{N}_0, A \in V^{(m)}, \zeta \in T_k(\Sigma, V)^1_m : (\exists \xi \in [\zeta]_{\asymp} : A \Longrightarrow^n_{G'} \xi) \Longrightarrow (A \Longrightarrow^n_{G} \zeta).$$

Let $m \in \mathbb{N}_0, A \in V^{(m)}, \zeta \in T_k(\Sigma, V)^1_m$ such that there exists $\xi \in [\zeta]_{\asymp}$ with $A \Longrightarrow^{n+1}_{G'} \xi$. We again decompose the derivation:

$$A \Longrightarrow^n_{G'} \xi' \text{ and } \xi' \Longrightarrow^1_{G'} \xi$$

for some $\xi' \in T_k(\Sigma, V)^1_m$. We get by induction assumption that $A \Longrightarrow^n_{G} \xi'$ and the definition of $\Longrightarrow_{G'}$ implies the existence of $\xi_1, \xi_2 \in T_k(\Sigma, V)$ and $(B \longrightarrow \tilde{\xi}) \in P$ such that

$$\xi' = \xi_1 \cdot B \cdot \xi_2 \text{ and } \xi = \xi_1 \cdot \tilde{\xi} \cdot \xi_2.$$

Thus we can derive $\xi' \Longrightarrow_{G} \xi$ in G which implies that $A \Longrightarrow^{n+1}_{G} \xi$. The derivation relation for G does not distinguish between ξ and ζ (since $\xi \asymp \zeta$) what gives $A \Longrightarrow^{n+1}_{G} \zeta$.

This proves claim (5).

Next we prove the following claim:

$$\forall \zeta \in T_{k\text{-dil}(T(\Sigma))^1}, i \in [k] : q_i(\zeta) \Longrightarrow^*_{T} \pi^i_k \cdot \mu(\zeta) \tag{6}$$

by structural induction over ζ (recall the definition of $\mu(\zeta)$ from Corollary 16). Note that this implies

$$\text{(I)} \qquad q_i(\zeta) \Longrightarrow^*_{T} \pi^i_k \cdot \zeta \qquad , \text{ for } \zeta \in k\text{-dil}(T(\Sigma))^1,$$

$$
\begin{array}{lll}
\text{(II)} & \tau(T)(\zeta) = \pi_k^1 \cdot \mu(\zeta) & \text{, for } \zeta \in T_{k\text{-dil}(T(\Sigma))^1} \text{ and} \\
\text{(III)} & \tau(T)(\zeta) = \tau(T)(\xi) & \text{, for } \zeta \in T_{k\text{-dil}(T(\Sigma))^1}, \xi \in [\zeta]_{\asymp}.
\end{array}
$$

Let $\zeta = \sigma(s_1, \ldots, s_m)$ for some $m \in \mathbb{N}_0$, $\sigma \in k\text{-dil}(T(\Sigma))^1$, $s_1, \ldots, s_m \in T_{k\text{-dil}(T(\Sigma))^1}$ such that for all $i \in [k], j \in [m]$ we have $q_i(s_j) \Longrightarrow_T^* \pi_k^i \cdot \mu(s_j)$. Let moreover $\iota \in [k]$. By definition of T and induction assumption it holds that

$$
\begin{aligned}
q_\iota(\zeta) &\Longrightarrow_T^1 (\pi_k^\iota \cdot \sigma)[q_1(s_1), \ldots, q_k(s_1), \ldots, q_1(s_m), \ldots, q_k(s_m)] \\
&\Longrightarrow_T^* (\pi_k^\iota \cdot \sigma)[\pi_k^1 \cdot \mu(s_1), \ldots, \pi_k^k \cdot \mu(s_1), \ldots, \pi_k^1 \cdot \mu(s_m), \ldots, \pi_k^k \cdot \mu(s_m)].
\end{aligned}
$$

Using the operations from $T(\Sigma)$ and the fact that $T(\Sigma)$ is a (projectable) magmoid we get

$$
\begin{aligned}
&(\pi_k^\iota \cdot \sigma)[\pi_k^1 \cdot \mu(s_1), \ldots, \pi_k^k \cdot \mu(s_1), \ldots, \pi_k^1 \cdot \mu(s_m), \ldots, \pi_k^k \cdot \mu(s_m)] \\
=&(\pi_k^\iota \cdot \sigma) \cdot ((\pi_k^1 \cdot \mu(s_1)) \oplus \cdots \oplus (\pi_k^k \cdot \mu(s_1)) \oplus \cdots \oplus (\pi_k^1 \cdot \mu(s_m)) \oplus \cdots \oplus (\pi_k^k \cdot \mu(s_m))) \\
=&(\pi_k^\iota \cdot \sigma) \cdot (\mu(s_1) \oplus \cdots \oplus \mu(s_m)) \\
=&\pi_k^\iota \cdot (\sigma \cdot (\mu(s_1) \oplus \cdots \oplus \mu(s_m))) \\
=&\pi_k^\iota \cdot \mu(\sigma(s_1, \ldots, s_m)) \\
=&\pi_k^\iota \cdot \mu(\zeta).
\end{aligned}
$$

Thus $q_\iota(\zeta) \Longrightarrow_T^* \pi_k^\iota \cdot \mu(\zeta)$ as claimed.

Using (5) and corollaries (I)–(III) from claim (6) we can now prove $L(G) = \tau(T)(L(G'))$.

Let $t \in T_\Sigma$. We have

$$
\begin{aligned}
t \in L(G) &\overset{\text{def.}}{\Longleftrightarrow} \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1 : Z \Longrightarrow_G^+ \zeta \wedge \pi_k^1 \cdot \zeta = t \\
&\overset{(5)}{\Longleftrightarrow} \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1, \xi \in [\zeta]_{\asymp} : Z \Longrightarrow_{G'}^+ \xi \wedge \pi_k^1 \cdot \zeta = t \\
&\overset{\text{(II)}}{\Longleftrightarrow} \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1, \xi \in [\zeta]_{\asymp} : Z \Longrightarrow_{G'}^+ \xi \wedge \tau(T)(\zeta) = t \\
&\overset{\text{(III)}}{\Longleftrightarrow} \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1, \xi \in [\zeta]_{\asymp} : Z \Longrightarrow_{G'}^+ \xi \wedge \tau(T)(\xi) = t \\
&\overset{\text{Cor.16}}{\Longleftrightarrow} \exists \xi \in T_{k\text{-dil}(T(\Sigma))^1} : Z \Longrightarrow_{G'}^+ \xi \wedge \tau(T)(\xi) = t \\
&\overset{\text{def.}}{\Longleftrightarrow} t \in \tau(T)(L(G'))
\end{aligned}
$$

This proves Proposition 26. □

Example 27. Let $\Sigma := \{\sigma^{(2)}, \gamma^{(1)}, \beta^{(0)}, \alpha^{(0)}\}$. It is known that the tree language

$$
L := \{\sigma(t, t) \mid t \in T_\Sigma\}
$$

is not in $\mathscr{L}(\text{CFTG})$ ([2], there: Theorem 4.1. and based on this also [18]). We therefore construct a CFMG G with $L(G) = L$ and apply the construction given in Proposition (26).

Consider $V := \{Z^{(0)}, A^{(0)}\}$ and the 1-dilated CFMG $G := (V, \Sigma, Z, P)$ where P consists of the following productions:

$$
\begin{aligned}
Z &\longrightarrow \langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle(A)\\
A &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(A, A)\\
A &\longrightarrow \langle\!\langle 1, \gamma(x_1)\rangle\!\rangle(A)\\
A &\longrightarrow \langle\!\langle 0, \beta\rangle\!\rangle\\
A &\longrightarrow \langle\!\langle 0, \alpha\rangle\!\rangle.
\end{aligned}
$$

It is obvious that $L(G) = L$ holds.

The constructed CFTG and td-td-tt are $G' := (V, \Delta, Z, P)$ and $T := (Q, \Delta, \Sigma, \{q_1\}, R)$ where $Q := \{q_1\}$,

$$\Delta := \{\langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle, \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle, \langle\!\langle 1, \gamma(x_1)\rangle\!\rangle, \langle\!\langle 0, \beta\rangle\!\rangle, \langle\!\langle 0, \alpha\rangle\!\rangle\}$$

and R consists of the following rules:

$$
\begin{aligned}
&q_1(\langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle(x_1)) && \longrightarrow \sigma(q_1(x_1), q_1(x_1))\\
&q_1(\langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1, x_2)) && \longrightarrow \sigma(q_1(x_1), q_1(x_2))\\
&q_1(\langle\!\langle 1, \gamma(x_1)\rangle\!\rangle(x_1)) && \longrightarrow \gamma(q_1(x_1))\\
&q_1(\langle\!\langle 0, \beta\rangle\!\rangle) && \longrightarrow \beta\\
&q_1(\langle\!\langle 0, \alpha\rangle\!\rangle) && \longrightarrow \alpha.
\end{aligned}
$$

The important difference between CFTGs and CFMGs that makes the construction of G generate the lanuage L (in contrast to the same approach for a CFTG) is that we can not copy occurrences of A into generated terminal parts as the following example derivation in G portrays:

$$
Z \Longrightarrow_G \begin{array}{c}\langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle\\ | \\ A\end{array} \Longrightarrow_G \begin{array}{c}\langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle\\ | \\ \langle\!\langle 1, \gamma(x_1)\rangle\!\rangle\\ | \\ A\end{array} \asymp \begin{array}{c}\langle\!\langle 1, \sigma(\gamma(x_1), \gamma(x_1))\rangle\!\rangle\\ | \\ A\end{array}
$$

$$
\Longrightarrow_G \begin{array}{c}\langle\!\langle 1, \sigma(\gamma(x_1), \gamma(x_1))\rangle\!\rangle\\ | \\ \langle\!\langle 0, \alpha\rangle\!\rangle\end{array} \asymp \langle\!\langle 1, \sigma(\gamma(\alpha), \gamma(\alpha))\rangle\!\rangle.
$$

Note that since $\#Q = 1$, we have that T is a homomorphism. This points out the fact that $\mathscr{L}(\text{CFTG})$ is not closed under homomorphisms – which was a major motivation to introduce the structure of magmoids.

Proposition 28. The following relations hold:

$$
\begin{aligned}
\mathscr{L}(\text{CFMG}) &\supseteq \text{td-TOP}(\mathscr{L}(\text{CFTG})),\\
\mathscr{L}(\text{l-CFMG}) &\supseteq \text{td-TOP}(\mathscr{L}(\text{l-CFTG})).
\end{aligned}
$$

Proof. Let Σ be a ranked alphabet and $L \in$ td-TOP($\mathscr{L}$(CFTG)) a tree language over Σ. Thus there exists a ranked alphabet Δ, a nf-CFTG $G = (V, \Delta, Z, P)$ and a td-td-tt $T = (Q, \Delta, \Sigma, I, R)$ such that $L = \tau(T)(L(G))$. W.l.o.g. we can assume that for some $k \in \mathbb{N}$ it is $Q = \{q_1, \ldots, q_k\}$ and $I = \{q_1\}$.

Moreover define the mapping $\eta : T_\Sigma(Q\langle X\rangle) \longrightarrow T_\Sigma(X)$ by

$$\eta(\xi) = \xi[q_i(x_j)/x_{(j-1)k+i}] \qquad , \xi \in T_\Sigma(Q\langle X\rangle).$$

We construct a CFMG G' such that $\tau(T)(L(G)) = L(G')$.

Hence let $G' = (V, \Sigma, Z, P')$ where P' contains on the one hand for every production $(A \longrightarrow \zeta) \in P$ with $\zeta \in T_V(X)$ the production

$$A \longrightarrow \zeta$$

and on the other hand for every $(A \longrightarrow \sigma(x_1, \ldots, x_n)) \in P$ the production

$$A \longrightarrow \langle\!\langle kn, \eta(t_1), \ldots, \eta(t_k)\rangle\!\rangle(x_1, \ldots, x_n)$$

where for any $i \in [k]$, $t_i = \tau(T_{q_i})(\sigma(x_1, \ldots, x_n))$.

Note that if G is linear, so is G'.

First we show the claim

$$\forall n \in \mathbb{N}, \zeta \in T_V, \xi \in T_\Delta : (\zeta \Longrightarrow_G{}^n \xi) \Longrightarrow (\zeta \Longrightarrow_{G'}{}^n \langle\!\langle 0, \tau(T_{q_1})(\xi), \ldots, \tau(T_{q_k})(\xi)\rangle\!\rangle) \qquad (7)$$

by (strong) complete induction on n.

Let $\zeta \in T_V$, $\xi \in T_\Delta$. Since G is in normal form, the following implications hold:

$$\begin{aligned} \zeta \Longrightarrow_G{}^1 \xi &\Longrightarrow \zeta = A \in V^{(0)} \wedge \xi \in \Delta^{(0)} \wedge (A \longrightarrow \xi) \in P \\ &\Longrightarrow \zeta \Longrightarrow_{G'}{}^1 \langle\!\langle 0, \tau(T_{q_1})(\xi), \ldots, \tau(T_{q_k})(\xi)\rangle\!\rangle. \end{aligned}$$

This proves the induction base.

Now let $n \in \mathbb{N}$ such that for all derivations of length $1 \le m \le n$ the claim (7) holds. Let $\zeta \in T_V$, $\xi \in T_\Delta$, $l \in \mathbb{N}_0$, $\zeta_1, \ldots, \zeta_l \in T_V$, $A \in V^{(l)}$ such that

$$\zeta = A(\zeta_1, \ldots, \zeta_l)$$

Let $\zeta \Longrightarrow_G{}^{n+1} \xi$ be true. There exists $\xi' \in T_{\Delta \cup V}$ such that

$$\zeta \Longrightarrow_G{}^1 \xi' \Longrightarrow_G{}^n \xi.$$

Note that the derivation mode is OI. Thus ξ' derives from ζ by applying a rule to the root of ζ.

Case 1: The first applied rule is of the form $(A(x_1, \ldots, x_l) \longrightarrow \tilde{\zeta}) \in P$ for some $\tilde{\zeta} \in T_V(X_l)$. Thus $\xi' = \tilde{\zeta} \cdot \langle 0, \zeta_1, \ldots, \zeta_l\rangle$ and by construction we can also derive

$$\zeta \Longrightarrow_{G'}{}^1 \xi' \in T_V.$$

By induction assumption we have

$$(\xi' \Longrightarrow_G^n \xi) \Longrightarrow (\xi' \Longrightarrow_{G'}^n \langle\!\langle 0, \tau(T_{q_1}(\xi), \ldots, \tau(T_{q_k})(\xi)\rangle\!\rangle)$$

which implies the derivation $\zeta \Longrightarrow_{G'}^{n+1} \langle\!\langle 0, \tau(T_{q_1}(\xi), \ldots, \tau(T_{q_k})(\xi)\rangle\!\rangle$.

Case 2: The first applied rule is of the form $(A(x_1, \ldots, x_l) \longrightarrow \sigma(x_1, \ldots, x_l)) \in P$ for some $\sigma \in \Delta$. Thus there exist $\nu_1, \ldots, \nu_l \in T_\Delta$ such that

$$\zeta \Longrightarrow_G^1 \sigma(\zeta_1, \ldots, \zeta_l) \Longrightarrow_G^n \sigma(\nu_1, \ldots, \nu_l)$$

with $\xi' = \sigma(\zeta_1, \ldots, \zeta_l)$, $\xi = \sigma(\nu_1, \ldots, \nu_l)$ and moreover the rule

$$A \longrightarrow \underbrace{\langle\!\langle kl, \eta\big(\tau(T_{q_1})(\sigma(x_1, \ldots, x_l))\big), \ldots, \eta\big(\tau(T_{q_k})(\sigma(x_1, \ldots, x_l))\big)\rangle\!\rangle}_{=:\tilde{\sigma}}$$

is by construction in P'.

There exist $n_1, \ldots, n_l \in \mathbb{N}$ with $\sum_{j=1}^{l} n_j$ such that

$$\zeta_i \Longrightarrow_G^{n_i} \nu_i \text{ for all } i \in [l],$$

thus by (strong) induction assumption we have that

$$\zeta_i \Longrightarrow_{G'}^{n_i} \langle\!\langle 0, \tau(T_{q_1})(\nu_i), \ldots, \tau(T_{q_k})(\nu_i)\rangle\!\rangle \text{ for all } i \in [l].$$

Together with the first derivation step we get

$$\begin{aligned}
\zeta \Longrightarrow_{G'}^{n+1} & \tilde{\sigma}(\langle\!\langle 0, \tau(T_{q_1})(\nu_1), \ldots, \tau(T_{q_k})(\nu_1)\rangle\!\rangle, \ldots, \langle\!\langle 0, \tau(T_{q_1})(\nu_l), \ldots, \tau(T_{q_k})(\nu_l)\rangle\!\rangle) \\
= & \tilde{\sigma} \cdot \langle\!\langle 0, \tau(T_{q_1})(\nu_1), \ldots, \tau(T_{q_k})(\nu_1), \ldots, \tau(T_{q_1})(\nu_l), \ldots, \tau(T_{q_k})(\nu_l)\rangle\!\rangle \\
= & \langle\!\langle 0, \tau(T_{q_1})(\sigma)[q_i(x_j)/\tau(T_{q_i})(\nu_j)], \ldots, \tau(T_{q_k})(\sigma)[q_i(x_j)/\tau(T_{q_i})(\nu_j)]\rangle\!\rangle \\
= & \langle\!\langle 0, \tau(T_{q_1})(\sigma(\nu_1, \ldots, \nu_l)), \ldots, \tau(T_{q_k})(\sigma(\nu_1, \ldots, \nu_l))\rangle\!\rangle.
\end{aligned}$$

Thus $\zeta \Longrightarrow_{G'}^{n+1} \langle\!\langle 0, \tau(T_{q_1})(\xi), \ldots, \tau(T_{q_k})(\xi)\rangle\!\rangle$. This proves the induction step for the claim (7).

Next we show the following claim:

$$\begin{aligned}
&\forall n \in \mathbb{N}, \zeta \in T_V, \nu \in k\text{-dil}(T(\Sigma))_0^1 : \\
&\quad (\zeta \Longrightarrow_{G'}^n \nu) \Longrightarrow (\exists \xi \in T_\Delta : \zeta \Longrightarrow_G^n \xi \wedge \nu = \langle\!\langle 0, \tau(T_{q_1})(\xi), \ldots, \tau(T_{q_k})(\xi)\rangle\!\rangle)
\end{aligned} \tag{8}$$

by (strong) induction on n.

Let $\zeta \in T_V$, $\nu \in k\text{-dil}(T(\Sigma))_0^1$ and $\zeta \Longrightarrow_{G'}^1 \nu$. Thus we have $\zeta \in V^{(0)}$ and the production $(\zeta \longrightarrow \nu)$ is in P'. By construction of P' there exist $\alpha \in \Delta^{(0)}$ and $(\zeta \longrightarrow \alpha) \in P$ such that

$$\nu = \langle\!\langle 0, \tau(T_{q_1})(\alpha), \ldots, \tau(T_{q_k})(\alpha)\rangle\!\rangle.$$

Therefore $\xi := \alpha$ satisfies the claimed properties.

Now let $n \in \mathbb{N}$ such that for all $1 \leq m \leq n$ the claim (8) holds. Let $\zeta \in T_V$, $\nu \in k\text{-dil}(T(\Sigma))_0^1$, $l \in \mathbb{N}_0$, $\zeta_1, \ldots, \zeta_l \in T_V$, $A \in V^{(l)}$ such that

$$\zeta = A(\zeta_1, \ldots, \zeta_l),$$

and let $\zeta \Longrightarrow_{G'}^{n+1} \nu$ be true. There exists $\xi' \in T_k(\Sigma, V)_0^1$ such that

$$\zeta \Longrightarrow_{G'}^{1} \xi' \Longrightarrow_{G'}^{n} \nu.$$

Case 1: The first applied rule is of the form $(A(x_1, \ldots, x_l) \longrightarrow \tilde{\zeta}) \in P'$ for some $\tilde{\zeta} \in T_V(X_l)$. Thus $\xi' = \tilde{\zeta} \cdot \langle 0, \zeta_1, \ldots, \zeta_l \rangle$ and by construction we can also derive

$$\zeta \Longrightarrow_{G}^{1} \xi' \in T_V.$$

By induction assumption we have

$$\begin{aligned}
&(\xi' \Longrightarrow_{G'}^{n} \nu) \\
\Longrightarrow &(\exists \xi \in T_\Delta : \xi' \Longrightarrow_{G}^{n} \xi \wedge \nu = \langle\!\langle 0, \tau(T_{q_1}(\xi), \ldots, \tau(T_{q_k})(\xi) \rangle\!\rangle)
\end{aligned}$$

which implies the claimed derivation $\zeta \Longrightarrow_{G}^{n+1} \xi$.

Case 2: The first applied rule is of the form

$$(A \longrightarrow \underbrace{\langle\!\langle kl, \eta\big(\tau(T_{q_1})(\sigma(x_1, \ldots, x_l))\big), \ldots, \eta\big(\tau(T_{q_k})(\sigma(x_1, \ldots, x_l))\big) \rangle\!\rangle}_{=:\tilde{\sigma}}) \in P'$$

for some $\sigma \in \Delta^{(l)}$ such that $(A(x_1, \ldots, x_l) \longrightarrow \sigma(x_1, \ldots, x_l)) \in P$. Thus $\xi' = \tilde{\sigma}(\zeta_1, \ldots, \zeta_l)$ and $\zeta \Longrightarrow_{G}^{1} \sigma(\zeta_1, \ldots, \zeta_l)$.

There exist $n_1, \ldots, n_l \in \mathbb{N}, \nu_1, \ldots, \nu_l \in k\text{-dil}(T(\Sigma))_0^1$ such that $\sum_{j=1}^{l} n_j = n$, $\zeta_i \Longrightarrow_{G'}^{n_i} \nu_i$ for all $i \in [l]$ and $\tilde{\sigma}(\nu_1, \ldots, \nu_l) \asymp \nu$.

By (strong) induction assumption we have that for any $i \in [l]$ there exists $\xi_i \in T_\Delta$ such that

$$\zeta_i \Longrightarrow_{G}^{n_i} \xi_i \wedge \nu_i = \langle\!\langle 0, \tau(T_{q_1}(\xi_i), \ldots, \tau(T_{q_k})(\xi_i) \rangle\!\rangle.$$

Thus we can derive $\zeta \Longrightarrow_{G}^{n+1} \xi$ where $\xi := \sigma(\xi_1, \ldots, \xi_l)$.

Since by definition of η it holds that

$$\begin{aligned}
\nu &= \tilde{\sigma}(\nu_1, \ldots, \nu_l) \\
&= \tilde{\sigma} \cdot \langle\!\langle 0, \tau(T_{q_1})(\xi_1), \ldots, \tau(T_{q_k})(\xi_1), \ldots, \tau(T_{q_1})(\xi_l), \ldots, \tau(T_{q_k})(\xi_l) \rangle\!\rangle \\
&= \langle\!\langle 0, \tau(T_{q_1})(\sigma(x_1, \ldots, x_l))[q_i(x_j)/\tau(T_{q_i})(\xi_j)], \ldots, \tau(T_{q_k})(\sigma(x_1, \ldots, x_l))[q_i(x_j)/\tau(T_{q_i})(\xi_j)] \rangle\!\rangle \\
&= \langle\!\langle 0, \tau(T_{q_1})(\sigma(\xi_1, \ldots, \xi_l)), \ldots, \tau(T_{q_k})(\sigma(\xi_1, \ldots, \xi_l)) \rangle\!\rangle \\
&= \langle\!\langle 0, \tau(T_{q_1})(\xi), \ldots, \tau(T_{q_k})(\xi) \rangle\!\rangle,
\end{aligned}$$

the claimed properties are satisfied for ξ. This especially proves the induction step for claim (8). Using (7) and (8) we can now prove $L(G') = \tau(T)(L(G))$.

Let $t \in T_\Sigma$. We have

$$\begin{aligned} t \in \tau(T)(L(G)) &\overset{\text{def.}}{\Longrightarrow} \exists \xi \in T_\Delta : Z \Longrightarrow_G{}^+ \xi \wedge \tau(T)(\xi) = t \\ &\overset{(7)}{\Longrightarrow} \exists \xi \in T_\Delta : Z \Longrightarrow_{G'}{}^+ \langle\!\langle 0, \underbrace{\tau(T_{q_1})(\xi)}_{=\tau(T)(\xi)}, \ldots, \tau(T_{q_k})(\xi) \rangle\!\rangle \wedge \tau(T)(\xi) = t \\ &\overset{\text{def.}}{\Longrightarrow} t \in L(G') \end{aligned}$$

and moreover we have

$$\begin{aligned} t \in L(G') &\overset{\text{def.}}{\Longrightarrow} \exists \nu \in k\text{-dil}(T(\Sigma))_0^1 : Z \Longrightarrow_{G'}{}^+ \nu \wedge \pi_k^1 \cdot \nu = t \\ &\overset{(8)}{\Longrightarrow} \exists \xi \in T_\Delta : \zeta \Longrightarrow_G{}^+ \xi \wedge \tau(T_{q_1})(\xi) = t \\ &\overset{\text{def.}}{\Longrightarrow} t \in L(G'). \end{aligned}$$

This proves Proposition 28. □

Example 29. Recall Σ and L from Example 27. We construct a nf-CFTG G and a td-td-tt T such that $\tau(T)(L(G)) = L$ and apply the construction from Proposition (28).

Let $V := \{Z^{(0)}, A_\sigma^{(2)}, A_\gamma^{(1)}\}$, $G := (V, \Sigma, Z, P)$ and $T := (\{q_1, q_2\}, \Sigma, \Sigma, \{q_1\}, R)$ where P consists of the following productions:

$$\begin{aligned} Z &\longrightarrow A_\sigma(Z, Z) \mid A_\gamma(Z) \mid \beta \mid \alpha \\ A_\sigma &\longrightarrow \sigma(x_1, x_2) \\ A_\gamma &\longrightarrow \gamma(x_1) \end{aligned}$$

and R consists of the following rules:

$$\begin{aligned} q_1(\sigma(x_1, x_2)) &\longrightarrow \sigma(\sigma(q_2(x_1), q_2(x_1)), \sigma(q_2(x_1), q_2(x_1))) \\ q_1(\gamma(x_1)) &\longrightarrow \sigma(\gamma(q_2(x_1)), \gamma(q_2(x_1))) \\ q_1(\beta) &\longrightarrow \sigma(\beta, \beta) \\ q_1(\alpha) &\longrightarrow \sigma(\alpha, \alpha) \\ q_2(\sigma(x_1, x_2)) &\longrightarrow \sigma(q_2(x_1), q_2(x_1)) \\ q_2(\gamma(x_1)) &\longrightarrow \gamma(q_2(x_1)) \\ q_2(\beta) &\longrightarrow \beta \\ q_2(\alpha) &\longrightarrow \alpha. \end{aligned}$$

Thus, $L(G) = T_\Sigma$ and $\tau(T)(\xi) = \sigma(\xi, \xi)$ for every $\xi \in T_\Sigma$.

The CFMG $G' := (V, \Sigma, Z, P')$ with productions

$$\begin{aligned} Z &\longrightarrow A_\sigma(Z, Z) \mid A_\gamma(Z) \mid \langle\!\langle 0, \sigma(\beta, \beta), \beta \rangle\!\rangle \mid \langle\!\langle 0, \sigma(\alpha, \alpha), \alpha \rangle\!\rangle \\ A_\sigma &\longrightarrow \langle\!\langle 4, \sigma(\sigma(x_2, x_4), \sigma(x_2, x_4)), \sigma(x_2, x_4) \rangle\!\rangle \\ A_\gamma &\longrightarrow \langle\!\langle 2, \sigma(\gamma(x_2), \gamma(x_2)), \gamma(x_2) \rangle\!\rangle \end{aligned}$$

is the result of the construction from Proposition (28).

Theorem 30. The following relations hold:

$$\mathscr{L}(\text{CFMG}) = \text{td-TOP}(\mathscr{L}(\text{CFTG})),$$
$$\mathscr{L}(\text{l-CFMG}) = \text{td-TOP}(\mathscr{L}(\text{l-CFTG})).$$

Proof. This immediately follows from propositions 26 and 28. □

Corollary 31. It holds that

$$\mathscr{L}(\text{CFMG}_1) = \text{h-TOP}(\mathscr{L}(\text{CFTG})),$$
$$\mathscr{L}(\text{CFMG}_1) = \text{h-TOP}(\mathscr{L}(\text{l-CFTG})).$$

We especially deduce that the class of 1-dilated CFMGs and the class of 1-dilated l-CFMGs are closed under homomorphisms.

Proof. The equations immediately follow from the constructions from propositions 26 and 28. The closedness under homomorphisms follows from the fact that homomorphisms are closed under composition as proven in [24]. □

Corollary 32. It holds that

$$\mathscr{L}(\text{CFMG}) = \text{td-TOP}(\mathscr{L}(\text{CFMG})),$$
$$\mathscr{L}(\text{l-CFMG}) = \text{td-TOP}(\mathscr{L}(\text{l-CFMG})).$$

Proof. As Baker has proven in [8] (there: Corollary 2), the class of tree transformations induced by total deterministic top-down tree transducers is closed under composition: $\text{td-TOP} \circ \text{td-TOP} = \text{td-TOP}$. Thus we have

$$\begin{aligned}\text{td-TOP}(\mathscr{L}(\text{CFMG})) &\overset{Th.30}{=} \text{td-TOP}(\text{td-TOP}(\mathscr{L}(\text{CFTG}))) \\ &\overset{Baker}{=} \text{td-TOP}(\mathscr{L}(\text{CFTG})) \\ &\overset{Th.30}{=} \mathscr{L}(\text{CFMG}).\end{aligned}$$

This obviously also holds for linear CFMGs. □

Corollary 33. Let $k \geq 1$. It holds that

$$\mathscr{L}(\text{CFMG}_k) = \text{h-TOP}(\mathscr{L}(\text{CFMG}_k)),$$
$$\mathscr{L}(\text{l-CFMG}_k) = \text{h-TOP}(\mathscr{L}(\text{l-CFMG}_k).$$

Proof. Let td-TOP_k be the class of tree transformations induced by total deterministic top-down tree transducers with k states.

It is $\text{h-TOP} \subseteq \text{td-TOP}$ by definition of homomorphisms. Moreover Baker has by construction proven in [8] that the composition of a homomorphism and a total deterministic top-down tree transducer with k states can be expressed by a total deterministic top-down treee transducer with k states: $\text{h-TOP} \circ \text{td-TOP}_k = \text{td-TOP}_k$. Thus we have

$$\text{h-TOP}(\mathscr{L}(\text{CFMG}_k) = \text{h-TOP}(\text{td-TOP}_k(\mathscr{L}(\text{CFTG})))$$

$$
\begin{aligned}
&\overset{Baker}{=} \text{td-TOP}_k(\mathscr{L}(\text{CFTG})) \\
&= \mathscr{L}(\text{CFMG}_k).
\end{aligned}
$$

This obviously also holds for linear CFMG. $\square$

Corollary 34. The following relation holds:

$$
\begin{aligned}
\mathscr{L}(\text{CFMG}) &= \text{d-TOP}(\mathscr{L}(\text{CFTG})), \\
\mathscr{L}(\text{l-CFMG}) &= \text{d-TOP}(\mathscr{L}(\text{l-CFTG})).
\end{aligned}
$$

Proof. "$\subseteq$": This follows from Theorem 30.

"$\supseteq$": Let D be a deterministic td-tt and G a context-free tree grammar. As proven in [14] (there: Chapter 4, Corollary 3.17), there exist a budet-fta B and a total, deterministic td-tt T, such that

$$\tau(D) = \tau(T) \circ \tau(B).$$

Moreover as proven in [23] and [22], we know that there exists a context-free tree grammar H, such that

$$\tau(B)(L(G)) = L(H).$$

Thus we have that

$$\tau(D)(L(G)) = \tau(T)(L(H)),$$

where the right hand side is an element of td-TOP($\mathscr{L}$(CFTG)) $\overset{Thm.30}{\subseteq}$ $\mathscr{L}$(CFMG). Therefore we have that $\tau(D)(L(G)) \in \mathscr{L}(\text{CFMG})$.

If G is linear, the proof is analogous except for the construction of H. In this case, the results in [21] imply that there exists a linear context-free tree grammar H, such that

$$\tau(B)(L(G)) = L(H).$$

This proves the second equation. $\square$

Chapter 5: Intersection of CFMG-Languages with Recognizable Tree Languages

5.1 A Construction for the Intersection

As proven in [1], the intersection of a language from $\mathscr{L}(\text{CFMG})$ with a recognizable language is again in $\mathscr{L}(\text{CFMG})$. In this chapter we give an alternative *constructive* proof of this claim which we derived from the construction given in [23] and [22]. In contrast to Rounds, who used top-down transducers, we will require the use of bottom-up transducers. The main reason for this difference lies within the particular normal forms of CFMGs and CFTGs. Lavas within a derivation tree of a CFMG in normal form can very well copy and delete variables, which is not the case for a CFTG in normal form. Therefore we ensure the identical processing of copies of trees by using deterministic bottom-up transducers. Since Rounds' construction is correct for a larger class of transducers than just top-down ftas, namely linear top-down transducers, and budet-ftas can be expressed by linear top-down transducers (see e.g. [24]), we can apply the given construction mutatis mutandis.

For a CFMG in normal form, G, that has V as the set of nonterminals and a budet-fta, B, that has Q as the set of states, the main idea for the construction is to blow up nonterminal productions in G. To achieve this, we define the set $V_{(Q)}$ to be the elements of V indexed by k-tuples of states from Q with increased ranks. Whenever a nonterminal production is in G, we horizontally dilate the right hand side to state-indexed trees that "carry" every possible state-labeling. The processing of B is now combined with the terminal productions in G by using only those successors of a derived symbol whose corresponding state-indices coincide with a possible derivation of the generated lava.

For the sake of straightforward proof structure, we first examine the aforementioned processing of budet-ftas on tuples of trees and continue by proving our construction.

Definition 35. Let $B = (Q, \Sigma, \Sigma, F, R)$ be a budet-fta, $k \geq 1$. Define the mapping

$$\Upsilon_k(B) : \{(t, q_1, \ldots, q_n) \in k\text{-dil}(T(\Sigma))^1_n \times (Q^k)^n \mid n \in \mathbb{N}_0\} \longrightarrow Q^k$$

where for any $n \in \mathbb{N}_0$, $t = \langle\!\langle kn, t_1, \ldots, t_k \rangle\!\rangle \in k\text{-dil}(T(\Sigma))^1_n$, $(q^1_1, \ldots, q^1_k)$, $\ldots$, $(q^n_1, \ldots, q^n_k)$, $(q_1, \ldots, q_k) \in Q^k$ the equivalence

$$\begin{aligned} &\Upsilon_k(B)(t, (q^1_1, \ldots, q^1_k), \ldots, (q^n_1, \ldots, q^n_k)) = (q_1, \ldots, q_k) \\ :\Longleftrightarrow\ &t_i[q^1_1(x_1), \ldots, q^1_k(x_k), \ldots, q^n_1(x_{(n-1)k+1}), \ldots, q^n_k(x_{nk})] \Longrightarrow^*_B q_i(t_i) \text{ for all } i \in [k] \end{aligned}$$

holds.

Note that because B is a budet-fta, this is well-defined. Thus for some $t \in k\text{-dil}(T(\Sigma))^1_0$, $\Upsilon_k(B)(t)$ is the unique tuple of states derived by B on the components of t.

Moreover it is correct that we can decompose such a tuple t and successively calculate the resulting states as Lemma 36 proves. ■

Lemma 36. Let $B = (Q, \Sigma, \Sigma, F, R)$ be a budet-fta, $k \geq 1$, $n \geq 0$, $u, u_1, \ldots, u_n \in k\text{-dil}(T(\Sigma))^1_0$ and $\tilde{u} \in k\text{-dil}(T(\Sigma))^1_n$. Whenever $u = \tilde{u} \cdot (u_1 \oplus \cdots \oplus u_n)$ holds, we have that

$$\Upsilon_k(B)(u) = \Upsilon_k(B)(\tilde{u}, \Upsilon_k(B)(u_1), \ldots, \Upsilon_k(B)(u_n)). \tag{9}$$

Proof. We assume the following decomposition:

$$\begin{aligned} u &= \langle\!\langle 0, v_1, \ldots, v_k \rangle\!\rangle, \\ u_i &= \langle\!\langle 0, w^i_1, \ldots, w^i_k \rangle\!\rangle \text{ for any } i \in [n], \\ \tilde{u} &= \langle\!\langle kn, \tilde{v}_1, \ldots, \tilde{v}_k \rangle\!\rangle. \end{aligned}$$

Moreover for any $q \in Q^k$, $i \in [k]$, we denote the i-th component of q by q_i.

We know that

$$v_i = \tilde{v}_i[w^1_1, \ldots, w^1_k, \ldots, w^n_1, \ldots, w^n_k],$$

which B derives to

$$\tilde{v}_i[\Upsilon_k(B)(u_1)_1(w^1_1), \ldots, \Upsilon_k(B)(u_1)_k(w^1_k), \ldots, \Upsilon_k(B)(u_n)_1(w^n_1), \ldots, \Upsilon_k(B)(u_n)_k(w^n_k)]$$

by definition of $\Upsilon_k(B)$ applied to $u_1, \ldots, u_n$. Note that any variable in $\tilde{v}_i$ may be copied or deleted. Whilst the latter case is trivial, the property of $\Upsilon_k(B)$ to be a mapping gives that every copy of a specific tree derives to the same state. Thus again by definition of $\Upsilon_k(B)$ applied to $\tilde{u}$ we get that

$$\begin{aligned} v_i &\Longrightarrow^*_B \Upsilon_k(B)(\tilde{u}, \Upsilon_k(B)(u_1), \ldots, \Upsilon_k(B)(u_n))_i(\tilde{v}_i[w^1_1, \ldots, w^1_k, \ldots, w^n_1, \ldots, w^n_k]) \\ &= \Upsilon_k(B)(\tilde{u}, \Upsilon_k(B)(u_1), \ldots, \Upsilon_k(B)(u_n))_i(v_i) \end{aligned}$$

which, since $\Upsilon_k(B)(u)$ is a mapping, proves the claim. ☐

Proposition 37. Let $k \geq 1$, $G = (V, \Sigma, Z, P)$ be a k-dilated CFMG in normal form, $B = (Q, \Sigma, \Sigma, F, R)$ a budet-fta. There exists a CFMG, $\tilde{G}$, such that

$$L(\tilde{G}) = \tau(B)(L(G)).$$

Proof. Let $m := \#Q^k$ and $(\bar{q}_1, \ldots, \bar{q}_m)$ be an enumeration of Q^k.

Define the set $V_{(Q)} := \{A^{(lm)}_{(q)} \mid A \in V^{(l)}, q \in Q^k, l \in \mathbb{N}_0\}$ and the mapping

$$\pi : Q^k \times T_V(X) \longrightarrow T_{V_{(Q)}}(X)$$

by

$$\pi(\bar{q}_i, x_j) := x^{(\bar{q}_i)}_j := x_{(j-1)m+i}$$

for any $i \in [m]$ and $j \in \mathbb{N}$ and

$$\pi(q, A(t_1, \ldots, t_l)) := A_{(q)}(\pi(\bar{q}_1, t_1), \ldots, \pi(\bar{q}_m, t_1), \ldots, \pi(\bar{q}_1, t_l), \ldots, \pi(\bar{q}_m, t_l))$$

for any $q \in Q^k$, $l \geq 0$, $A \in V^{(l)}$, $t_1, \ldots, t_l \in T_V(X)$. Note that the definition of π depends on the particular enumeration of Q^k.

Construct the k-dilated CFMG $\tilde{G} := (V_{(Q)} \cup \{Z\}, \Sigma, Z, \tilde{P})$ where $\tilde{P}$ consists of the following productions:

For any $(f, q) \in F \times Q^{k-1}$ the production

$$Z \longrightarrow Z_{(f,q)}$$

is in $\tilde{P}$ (called **type-1-production**). Moreover for any $q \in Q^k$, $u \in T_V(X)$ and $(A \longrightarrow u) \in P$, the production

$$A_{(q)} \longrightarrow \pi(q, u)$$

is in $\tilde{P}$ (called **type-2-production**). Finally for any $q, q_1, \ldots, q_l \in Q^k$, $\sigma \in k\text{-dil}(T(\Sigma))^1_l$ and $(A \longrightarrow \sigma(x_1, \ldots, x_l)) \in P$ such that $\Upsilon_k(B)(\sigma, q_1, \ldots, q_l) = q$, we have that

$$A_{(q)} \longrightarrow \sigma(x_1^{(q_1)}, \ldots, x_l^{(q_l)})$$

is a production in $\tilde{P}$ (called **type-3-production**). Note that Z does not occur on a right hand side of a production in $\tilde{P}$.

To prove that $\tilde{G}$ satisfies the claimed property, we first show that the statement

$$\begin{gathered} \forall l \in \mathbb{N}_0, w \in T_V(X_l), u_1, \ldots, u_l \in T_V, q \in Q^k : \\ \pi(q, w[u_1, \ldots, u_l]) = \pi(q, w)[\pi(\bar{q}_1, u_1), \ldots, \pi(\bar{q}_m, u_1), \ldots, \pi(\bar{q}_1, u_l), \ldots, \pi(\bar{q}_m, u_l)] \end{gathered} \tag{10}$$

holds by structural induction on w.

Let $l \in \mathbb{N}$ (for $l = 0$ the claim is trivially correct), $w \in T_V(X_l)$, $u_1, \ldots, u_l \in T_V$, $q \in Q^k$.

If $w = x_j$ for some $j \in [l]$, then we deduce that $\pi(q, w) = x_j^{(q)}$ and $w[u_1, \ldots, u_l] = u_j$. The definition of $x_j^{(q)}$ now gives that

$$\begin{aligned} \pi(q, w[u_1, \ldots, u_l]) &= \pi(q, u_j) \\ &= x_j^{(q)}[\pi(\bar{q}_1, u_1), \ldots, \pi(\bar{q}_m, u_1), \ldots, \pi(\bar{q}_1, u_l), \ldots, \pi(\bar{q}_m, u_l)], \end{aligned}$$

which proves the claim.

If $w = B(w_1, \ldots, w_p)$ for some $p \in \mathbb{N}_0$, $B \in V^{(p)}$, $w_1, \ldots, w_p \in T_V(X_l)$ such that (10) holds for $w_1, \ldots, w_p$, then we have that

$$\pi(q, w) = B_{(q)}(\pi(\bar{q}_1, w_1), \ldots, \pi(\bar{q}_m, w_1), \ldots, \pi(\bar{q}_1, w_p), \ldots, \pi(\bar{q}_m, w_p)).$$

Moreover by definition of substitution and induction assumption we know that

$$\begin{aligned} &\pi(q, w)[\pi(\bar{q}_1, u_1), \ldots, \pi(\bar{q}_m, u_1), \ldots, \pi(\bar{q}_1, u_l), \ldots, \pi(\bar{q}_m, u_l)] \\ =&B_{(q)}(\pi(\bar{q}_1, w_1[u_1, \ldots, u_l]), \ldots, \pi(\bar{q}_m, w_1[u_1, \ldots, u_l]), \ldots \\ &\ldots, \pi(\bar{q}_1, w_p[u_1, \ldots, u_l]), \ldots, \pi(\bar{q}_m, w_p[u_1, \ldots, u_l])) \end{aligned}$$

$$
\begin{aligned}
&= \pi(q, B(w_1[u_1, \dots, u_l], \dots, w_p[u_1, \dots, u_l]))\\
&= \pi(q, B(w_1, \dots, w_p)[u_1, \dots, u_l])\\
&= \pi(q, w[u_1, \dots, u_l]).
\end{aligned}
$$

This proves (10).

Next we prove that the statement

$$
\begin{gathered}
\forall n \in \mathbb{N}, u \in T_V, q \in Q^k, \zeta \in k\text{-dil}(T(\Sigma))_0^1 :\\
(\pi(q,u) \Longrightarrow_{\tilde{G}}{}^n \zeta) \iff (u \Longrightarrow_G{}^n \zeta \wedge \Upsilon_k(B)(\zeta) = q)
\end{gathered} \tag{11}
$$

holds by (strong) complete induction on n.

Let $u \in T_V, q \in Q^k, \zeta \in k\text{-dil}(T(\Sigma))_0^1$ such that $\pi(q,u) \Longrightarrow_{\tilde{G}}{}^1 \zeta$. By construction of type-3-productions, $\pi(q,u)$ consists of a single nonterminal $A_{(q)}$. Thus $u = A \in V$, there exists $(A \longrightarrow \zeta) \in P$ and $\Upsilon_k(B)(\zeta) = q$. This moreover implies $A \Longrightarrow_G{}^1 \zeta$.

Let $u \in T_V, q \in Q^k, \zeta \in k\text{-dil}(T(\Sigma))_0^1$ such that $u \Longrightarrow_G{}^1 \zeta$ and $\Upsilon_k(B)(\zeta) = q$. Since G is in normal form, u consists of a single nonterminal A and by construction of type-3-productions, $(A_{(q)} \longrightarrow \zeta) \in \tilde{P}$. Since $\pi(q, A) = A_{(q)}$, this implies $\pi(q,u) \Longrightarrow_{\tilde{G}}{}^1 \zeta$.

This proves the induction base.

Let $n \in \mathbb{N}$ such that for each $m \in [n]$ the claim (11) holds. We will continue by deducing that (11) also holds for $n+1$.

"$\Longrightarrow$": Let $u \in T_V, q \in Q^k, \zeta \in k\text{-dil}(T(\Sigma))_0^1$ such that $\pi(q,u) \Longrightarrow_{\tilde{G}}{}^{n+1} \zeta$. The definition of $\Longrightarrow_{\tilde{G}}{}^*$ implies the existence of $\xi \in T_k(\Sigma, V_{(Q)})_0^1$ such that

$$
\pi(q,u) \Longrightarrow_{\tilde{G}}{}^1 \xi \Longrightarrow_{\tilde{G}}{}^n \zeta.
$$

Let $u = A(u_1, \dots, u_l)$ for some $l \geq 0$, $A \in V^{(l)}$, $u_1, \dots, u_l \in T_V$. We deduce that

$$
\pi(q,u) = A_{(q)}(\pi(\bar{q}_1, u_1), \dots, \pi(\bar{q}_m, u_1), \dots, \pi(\bar{q}_1, u_l), \dots, \pi(\bar{q}_m, u_l)).
$$

Note that we use OI-derivations, thus the first derived nonterminal is $A_{(q)}$. We differentiate between the possible first applied productions.

<u>Case 1</u>: The first applied production is a type-2-production $A_{(q)} \longrightarrow \pi(q,w)$ for some $w \in T_V(X_l)$. The construction implies

$$
(A \longrightarrow w) \in P
$$

and thus we have that

$$
\begin{aligned}
\xi &= \pi(q,w)[\pi(\bar{q}_1, u_1), \dots, \pi(\bar{q}_m, u_1), \dots, \pi(\bar{q}_1, u_l), \dots, \pi(\bar{q}_m, u_l)]\\
&\overset{(10)}{=} \pi(q, w[u_1, \dots, u_l]).
\end{aligned}
$$

Moreover we deduce that

$$
\pi(q, w[u_1, \dots, u_l]) \Longrightarrow_{\tilde{G}}{}^n \zeta,
$$

which by induction assumption implies

$$w[u_1, \dots, u_l] \Longrightarrow_G^n \zeta \text{ and } \Upsilon_k(B)(\zeta) = q.$$

Therefore it holds that

$$u \Longrightarrow_G^{n+1} \zeta,$$

which proves one direction of the the induction step for type-2-productions.

Case 2: The first applied production is a type-3-production $A_{(q)} \longrightarrow \sigma(x_1^{(q_1)}, \dots, x_l^{(q_l)})$ for some $q_1, \dots, q_k \in Q^k$ and $\sigma \in k\text{-dil}(T(\Sigma))_l^1$.

The construction of type-3-production implies

$$(A \longrightarrow \sigma(x_1, \dots, x_l)) \in P \text{ and } \Upsilon_k(B)(\sigma, q_1, \dots, q_l) = q.$$

We moreover deduce that

$$\begin{aligned} \xi &= \sigma(x_1^{(q_1)}, \dots, x_l^{(q_l)})[\pi(\bar{q}_1, u_1), \dots, \pi(\bar{q}_m, u_1), \dots, \pi(\bar{q}_1, u_l), \dots, \pi(\bar{q}_m, u_l)] \\ &= \sigma(\pi(q_1, u_1), \dots, \pi(q_l, u_l)). \end{aligned}$$

Since ξ derives in n steps to ζ, we know that there exist $n_1, \dots, n_l \in \mathbb{N}$ and $\zeta_1, \dots, \zeta_l \in k\text{-dil}(T(\Sigma))_0^1$ such that $\sum_{i=1}^l n_i = n$ and

$$\pi(q_i, u_i) \Longrightarrow_G^{n_i} \zeta_i \text{ for each } i \in [l].$$

We therefore get the decomposition $\zeta = \sigma \cdot (\zeta_1 \oplus \dots \oplus \zeta_l)$ and by induction assumption we know that

$$u_i \Longrightarrow_G^{n_i} \zeta_i \text{ and } \Upsilon_k(B)(\zeta_i) = q_i \text{ for each } i \in [l].$$

Finally we deduce that

$$u \Longrightarrow_G^1 \sigma(u_1, \dots, u_l) \Longrightarrow_G^n \sigma(\zeta_1, \dots, \zeta_l) \asymp \zeta$$

and by Lemma 36, equation (9) we have that

$$\begin{aligned} \Upsilon_k(B)(\zeta) &= \Upsilon_k(B)(\sigma, \Upsilon_k(B)(\zeta_1), \dots, \Upsilon_k(B)(\zeta_l)) \\ &= \Upsilon_k(B)(\sigma, q_1, \dots, q_l) = q. \end{aligned}$$

This proves one direction of the induction step for type-3-productions. Since the first applied production is not of type 1, this also proves the whole induction step for this direction.

"$\Longleftarrow$": Let $u \in T_V, q \in Q^k, \zeta \in k\text{-dil}(T(\Sigma))_0^1$ such that $u \Longrightarrow_G^{n+1} \zeta$ and $\Upsilon_k(B)(\zeta) = q$. By definition of $\Longrightarrow_G^*$ it holds that there exists $\xi \in T_k(\Sigma, V)_0^1$ such that

$$u \Longrightarrow_G^1 \xi \Longrightarrow_G^n \zeta.$$

Let $u = A(u_1, \ldots, u_l)$ for some $l \geq 0$, $A \in V^{(l)}$, $u_1, \ldots, u_l \in T_V$. We deduce that

$$\pi(q, u) = A_{(q)}(\pi(\bar{q}_1, u_1), \ldots, \pi(\bar{q}_m, u_1), \ldots, \pi(\bar{q}_1, u_l), \ldots, \pi(\bar{q}_m, u_l)).$$

Note that we use OI-derivations, thus the first derived nonterminal is $A_{(q)}$. We differentiate between the possible first applied productions.

Case 1: The first applied rule is of the form $A \longrightarrow w$ for some $w \in T_V(X_l)$. The construction of type-2-productions implies the existence of

$$(A_{(q)} \longrightarrow \pi(q, w)) \in \tilde{P}.$$

Therefore since $\xi = w[u_1, \ldots, u_l]$, we have that

$$\begin{aligned} \pi(q, u) &\Longrightarrow_{\tilde{G}}^{1} \pi(q, w)[\pi(\bar{q}_1, u_1), \ldots, \pi(\bar{q}_m, u_1), \ldots, \pi(\bar{q}_1, u_l), \ldots, \pi(\bar{q}_m, u_l)] \\ &\overset{(10)}{=} \pi(q, w[u_1, \ldots, u_l]) = \pi(q, \xi). \end{aligned}$$

By induction assumption we know that $\pi(q, \xi) \Longrightarrow_{\tilde{G}}^{n} \zeta$, which implies

$$\pi(q, u) \Longrightarrow_{\tilde{G}}^{n+1} \zeta.$$

Case 2: The first applied rule is of the form $A \longrightarrow \sigma(x_1, \ldots, x_l)$ for some $\sigma \in k\text{-dil}(T(\Sigma))^1_l$. This implies $\xi = \sigma(u_1, \ldots, u_l)$. Furthermore there exist $n_1, \ldots, n_l \in \mathbb{N}$ and $\zeta_1, \ldots, \zeta_l \in k\text{-dil}(T(\Sigma))^1_0$ such that $\sum_{i=1}^{l} n_i = n$, $\zeta = \sigma \cdot (\zeta_1 \oplus \cdots \oplus \zeta_l)$ and

$$u_i \Longrightarrow_{G}^{n_i} \zeta_i \text{ for each } i \in [l].$$

We define for any $i \in [l]$ the state $q_i := \Upsilon_k(B)(\zeta_i)$. By Lemma 36, equation (9) we have that

$$\begin{aligned} q &= \Upsilon_k(B)(\zeta) \\ &= \Upsilon_k(B)(\sigma, \Upsilon_k(B)(\zeta_1), \ldots, \Upsilon_k(B)(\zeta_l)) \\ &= \Upsilon_k(B)(\sigma, q_1, \ldots, q_l). \end{aligned}$$

Moreover by induction assumption we know that

$$\pi(q_i, u_i) \Longrightarrow_{\tilde{G}}^{n_i} \zeta_i \text{ for each } i \in [l]$$

and

$$(A_{(q)} \longrightarrow \sigma(x_1^{(q_1)}, \ldots, x_l^{(q_l)})) \in \tilde{P}).$$

Thus because of

$$\begin{aligned} &\sigma(x_1^{(q_1)}, \ldots, x_l^{(q_l)})[\pi(\bar{q}_1, u_1), \ldots, \pi(\bar{q}_m, u_1), \ldots, \pi(\bar{q}_1, u_l), \ldots, \pi(\bar{q}_m, u_l)] \\ &= \sigma(\pi(q_1, u_1), \ldots, \pi(q_l, u_l)), \end{aligned}$$

we can derive

$$\pi(q,u) \Longrightarrow_{\tilde{G}}^{1} \sigma(\pi(q_1,u_1),\ldots,\pi(q_l,u_l)) \Longrightarrow_{\tilde{G}}^{n} \sigma(\zeta_1,\ldots,\zeta_l) \asymp \zeta.$$

This proves the second direction of the the induction step. Thus claim (11) is proven.

We continue by showing that $L(\tilde{G}) = \tau(B)(L(G))$.
Let $t \in T_\Sigma$. It holds that

$$\begin{aligned}
t \in L(\tilde{G}) &\overset{\text{def.}}{\Longleftrightarrow} && \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1 : Z \Longrightarrow_{\tilde{G}}^{+} \zeta \wedge \pi_k^1 \cdot \zeta = t \\
&\overset{\text{constr.}}{\Longleftrightarrow} && \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1, (f,q) \in F \times Q^{k-1} : \\
& && Z \Longrightarrow_{\tilde{G}}^{1} Z_{(f,q)} \wedge Z_{(f,q)} \Longrightarrow_{\tilde{G}}^{+} \zeta \wedge \pi_k^1 \cdot \zeta = t \\
&\overset{(11)}{\Longleftrightarrow} && \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1, (f,q) \in F \times Q^{k-1} : \\
& && Z \Longrightarrow_{G}^{+} \zeta \wedge \Upsilon_k(B)(\zeta) = (f,q) \wedge \pi_k^1 \cdot \zeta = t \\
&\overset{\text{def. } \Upsilon_k(B)}{\Longleftrightarrow} && \exists \zeta \in k\text{-dil}(T(\Sigma))_0^1, f \in F : Z \Longrightarrow_{G}^{+} \zeta \wedge \pi_k^1 \cdot \zeta = t \wedge t \Longrightarrow_{B}^{*} f(t) \\
&\overset{\text{def.}}{\Longleftrightarrow} && t \in L(G) \wedge (t,t) \in \tau(B) \\
&\overset{\text{def.}}{\Longleftrightarrow} && t \in \tau(B)(L(G)).
\end{aligned}$$

This proves the proposition. □

Example 38. Consider the ranked alphabet Σ and the tree language L where

$$\begin{aligned}
\Sigma &:= \{\sigma^{(2)}, \beta^{(0)}, \alpha^{(0)}\} \\
L &:= \{\sigma(t,t) \mid t \in T_\Sigma\}.
\end{aligned}$$

A 1-dilated CFMG in normal form that generates L is $G := (V, \Sigma, Z, P)$ where $V := \{Z^{(0)}, A^{(0)}, S^{(1)}, B^{(2)}\}$ and P consists of the rules

$$\begin{aligned}
Z &\xrightarrow{r_1} S(A) \\
S &\xrightarrow{r_2} \langle\!\langle 1, \sigma(x_1,x_1)\rangle\!\rangle(x_1) \\
A &\xrightarrow{r_3} B(A,A) \\
A &\xrightarrow{r_4} \langle\!\langle 0, \beta\rangle\!\rangle \\
A &\xrightarrow{r_5} \langle\!\langle 0, \alpha\rangle\!\rangle \\
B &\xrightarrow{r_6} \langle\!\langle 2, \sigma(x_1,x_2)\rangle\!\rangle(x_1,x_2).
\end{aligned}$$

Let moreover $B := (Q, \Sigma, \Sigma, F, R)$ be the budet-fta with

$$\begin{aligned}
Q &:= \{0,1,2\}, \\
F &:= \{0\}
\end{aligned}$$

and rules

$$\begin{array}{lll} \sigma(n(x_1), m(x_2)) & \longrightarrow ((n+m) \bmod 3)(\sigma(x_1, x_2)) & \forall n, m \in Q \\ \alpha & \longrightarrow 1(\alpha) & \\ \beta & \longrightarrow 0(\beta). & \end{array}$$

It is fairly easy to see that B accepts exactly the trees over Σ that contain 0 'α's modulo 3. Thus we deduce that

$$\tau(B)(L(G)) = \{\sigma(t,t) \mid t \in T_\Sigma, (t,t) \in \tau(B)\}.$$

We construct the CFMG $\tilde{G}$ as in the proof of Proposition 37. The new set of nonterminals is given by

$$V_{(Q)} = \{Z_{(0)}, Z_{(1)}, Z_{(2)}, A_{(0)}, A_{(1)}, A_{(2)}, S_{(0)}, S_{(1)}, S_{(2)}, B_{(0)}, B_{(1)}, B_{(2)}\}.$$

The only constructed type-1-production is

$$Z \longrightarrow Z_{(0)},$$

whilst the type-2-productions constructed from production r_1 are

$$\begin{array}{ll} Z_{(0)} \longrightarrow & \begin{array}{c} S_{(0)} \\ A_{(0)} \quad A_{(1)} \quad A_{(2)} \end{array} \\ Z_{(1)} \longrightarrow & \begin{array}{c} S_{(1)} \\ A_{(0)} \quad A_{(1)} \quad A_{(2)} \end{array} \\ Z_{(2)} \longrightarrow & \begin{array}{c} S_{(2)} \\ A_{(0)} \quad A_{(1)} \quad A_{(2)} \end{array} \end{array}.$$

The production r_3 is analogously translated.

To correctly construct type-3-productions we have to consider the processing of B on right hand sides of the corresponding productions in G. This coincides with the mapping $\Upsilon_k(B)$ from Definition 35.

Production r_2 is thus translated to the following productions:

$$\begin{aligned} S_{(0)} &\longrightarrow \langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle(x_1^{(0)}) \\ S_{(2)} &\longrightarrow \langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle(x_1^{(1)}) \\ S_{(1)} &\longrightarrow \langle\!\langle 1, \sigma(x_1, x_1)\rangle\!\rangle(x_1^{(2)}) \end{aligned}$$

and production r_6 is translated to the productions

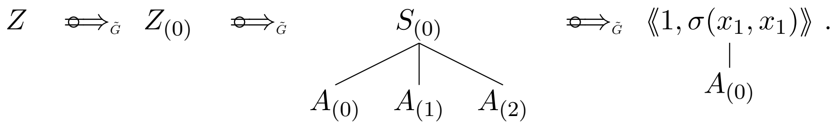
$$B_{(0)} \longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(0)}, x_2^{(0)})$$

$$
\begin{aligned}
B_{(1)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(0)}, x_2^{(1)})\\
B_{(2)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(0)}, x_2^{(2)})\\
B_{(1)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(1)}, x_2^{(0)})\\
B_{(2)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(1)}, x_2^{(1)})\\
B_{(0)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(1)}, x_2^{(2)})\\
B_{(2)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(2)}, x_2^{(0)})\\
B_{(0)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(2)}, x_2^{(1)})\\
B_{(1)} &\longrightarrow \langle\!\langle 2, \sigma(x_1, x_2)\rangle\!\rangle(x_1^{(2)}, x_2^{(2)}).
\end{aligned}
$$

The construction on productions r_4 and r_5 is analogous.

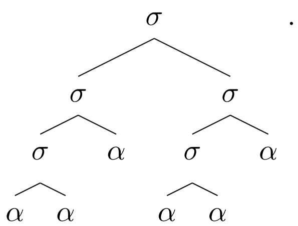
Now the CFMG $\tilde{G} := (V_{(Q)} \cup \{Z\}, \Sigma, Z, \tilde{P})$ (where $\tilde{P}$ consists of all the constructed productions) has $L(\tilde{G}) = \tau(B)(L(G))$ as Proposition 37 proves.

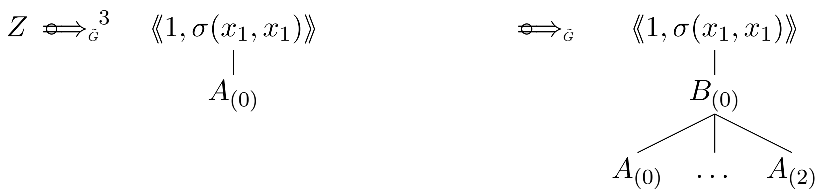
To illustrate the language generated by $\tilde{G}$, we show that we can not derive the tree $\sigma(\alpha, \alpha)$ but the tree $\sigma(\beta, \beta)$. Therefore we consider the unique OI-derivation of Z to the following tree:

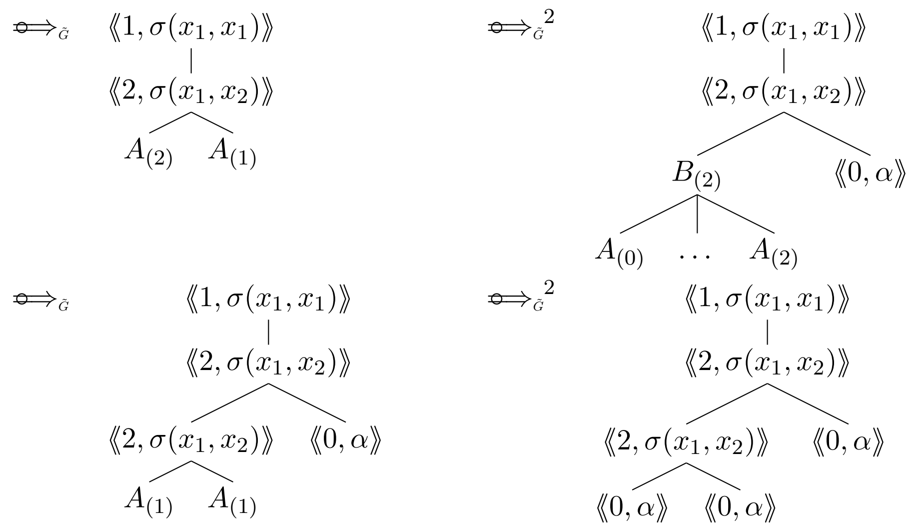


Equation (11) from the proof of Proposition 37 tells us that $A_{(0)}$ derives exactly the trees that A derives in G and that are processed into state 0 by the fta B. Thus we can not derive a single occurrence of α (since α is processed into state 1 by B) but a single occurrence of β from A, which we can verify with the given constructed productions.

The following derivation shows that we can derive the tree







where the last tree is equivalent to $\langle\!\langle 0, \sigma(\sigma(\sigma(\alpha,\alpha),\alpha),\sigma(\sigma(\alpha,\alpha),\alpha))\rangle\!\rangle$, which contains the tree we wanted to derive. Note how the derivation trees are derived in such a way, that the corresponding states that B would derive are propagated through the nonterminals. ■

Theorem 39. Let Σ be a ranked alphabet, $\Lambda_1 \in \mathscr{L}(\mathrm{CFMG}(\Sigma))$, $\Lambda_2 \in REC(\Sigma)$. It holds that

$$\Lambda_1 \cap \Lambda_2 \in \mathscr{L}(\mathrm{CFMG}(\Sigma)).$$

Proof. This follows directly from Proposition 37. □

We can moreover deduce that the dilatation index is not changed by our construction. Thus we have the following Corollary.

Corollary 40. Let Σ be a ranked alphabet, $\Lambda_1 \in \mathscr{L}(\mathrm{CFMG}_k(\Sigma))$, $\Lambda_2 \in REC(\Sigma)$. It holds that

$$\Lambda_1 \cap \Lambda_2 \in \mathscr{L}(\mathrm{CFMG}_k(\Sigma)).$$

Proof. This follows from the construction given in the proof of Proposition 37. □

Theorem 41. The emptiness and membership problems for context-free magmoid grammars are solvable.

Proof. For a CFMG G there exist a td-td-tt T and a CFTG G' such that

$$L(G) = \tau(T)(L(G'))$$

by Theorem 30. Thus since T is total and deterministic, $L(G)$ is empty if and only if $L(G')$ is empty. As Maibaum has proven in [20], the emptiness problem for context-free tree grammars is solvable. Therefore the emptiness problem for context-free magmoid grammars is solvable.

Let G be a CFMG over a ranked alphabet Σ and $\xi \in T_\Sigma$. It is

$$\xi \in L(G) \Longleftrightarrow \{\xi\} \cap L(G) \neq \emptyset.$$

Since $\{\xi\}$ is a regular tree language, by Theorem 39 we have that

$$\{\xi\} \cap L(G) \in \mathscr{L}(\mathrm{CFMG}(\Sigma)).$$

Therefore the membership problem is solvabe for context-free magmoid grammars. $\square$

Chapter 6: Conclusion

6.1 Prospectus

In this thesis we proved that context-free magmoid grammars differ from context-free tree grammars in the application of total and deterministic top-down tree transducers. We additionally proved that this connection holds for *linear* CFMGs and *linear* CFTGs. By applying known results, we deduced for example the closure of CFMGs under application of homomorphisms as well as total and deterministic top-down tree transducers.

Furthermore we gave a constructive proof of closure of languages generated by CFMGs under intersection with recognizable tree languages and proved our construction. This was done by adapting Rounds' solution given in [23] and [22] to the case of magmoid grammars and bottom-up deterministic finite-state tree automata.

An important implication of our results was that the emptiness and membership problems are solvable for context-free magmoid grammars.

The results proven in this thesis were based on a thorough formulation of the structures of $T(\Sigma)$, $T_k(\Sigma, V)$ and context-free magmoid grammars which we gave beforehand.

Thus all in all we portrayed the concept of context-free magmoid grammars and proved two important results for the theory of the corresponding tree languages. To our knowledge, one of these results was not known in current research and for the second one we gave a constructive proof in contrast to the known algebraic proof.

6.2 Future Work

A possible task for the future is to generalize different types of tree automata to the structure of magmoids, for example *pushdown tree automata* as seen in [17].

Moreover a *Greibach normal form* (as in [6] or [13]) might be generalized for context-free magmoid grammars. For this, the first major result of this thesis could be applied to constructions for the case of context-free tree grammars. Since Greibach normal forms play an important role for research in properties of CFTGs, a generalization might be useful for a deeper understanding of CFMGs.

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